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On a New Family of Degenerate Parabolic Equations

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In this paper we consider the equation

$$h_t + (p-1)\left[h^n\left[\left(h_x^2\right)^{\frac{p}{2}-1} h_{xx}\right]_x\right]_x = 0,$$

which was first derived in (Ulusoy, *Nonlinearity* 20 (2007): 685–712). We prove results on the regularity of non-negative solutions. In Ulusoy, an entropy dissipation–entropy estimate was provided for the $p=3$ and $n=2$ case using the energy functional $K_q := \int \frac{h_x^2}{h^q}\,\mathrm{d}x$. Here, we extend our calculations to include various other p and n values. After establishing some results on the support properties of solutions, we finally complete the analysis of the long-time behavior of non-negative weak solutions.

1 Introduction

Diffusion processes are modeled by a parabolic evolution equation of the form

$$h_t = \left(M(h)\left(\frac{\delta H}{\delta h}\right)_x\right)_x. \tag{1.1}$$

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Here $M(h)$ is called the *mobility term* and H is an *energy functional*, so that $\frac{\delta H}{\delta h}$ is the *chemical potential*.

Recently, the first-order energy functional of the form

$$H_2(h) := \frac{1}{2} \int h_x^2 \, \mathrm{d}x \tag{1.2}$$

was employed. For example, the following equation

$$h_t + (M(h)h_{xxx})_x = 0, \ \ -a \le x \le a, \tag{1.3}$$

with either periodic or "no flux" boundary conditions; i.e. $h_x(t, \pm a) = h_{xxx}(t, \pm a) = 0$; is called the *thin-film equation* and here the mobility term is given by $M(h) = h^n$ and the energy functional is given by (1.2).

Different mobility terms represent different physical situations. For instance in (1.3), when $n = 1$, or equivalently $M(h) = h$, the equation describes the evolution of the thickness of a thin bridge between two masses of fluid in a Hele–Show cell. $M(h) = h^3$ case is used in the modeling of capillary-driven flow. More precisely, here h is the thickness of a fluid film on a substrate where the film is evolving under the influence of the surface tension, but not gravity. Finally, when $M(h) = h^2$, it is used for the presence of the slip length to allow the contact line to move at the fluid–substrate interface. See [6, 7, 9, 19, 21, 22] for more information and derivations for the thin-film equation.

We also remark here that the thin-film equation is a special case (the $m = 2$ case) of the so-called "doubly nonlinear thin film equation" considered in [1]. The equation reads as follows:

$$h_t + [|h|^n |h_{xxx}|^{m-2} h_{xxx}]_x = 0, \tag{1.4}$$

where $n > 0$ and $m \ge 2$ are real constants. Equation (1.4) describes the evolution of the height $h(t, x)$ of a surface-tension-driven thin liquid film on a solid surface in lubrication approximation [1, 17, 21, 27]. The $m = 2$ case in (1.4) corresponds to a Newtonian fluid, and $m \ne 2$ occurs when considering "power-law" liquids. In [1] the authors prove the existence of solutions to the problem (1.4), and obtain sharp upper bounds for the propagation of their support. They also derive a necessary condition for the occurrence of waiting-time phenomenon.

On the other hand, another example is the problem of relaxation of axisymmetric crystal surfaces with a single facet below the roughening transition. In [20] (and also the references therein) this problem is analyzed via a continuum approach that accounts for step energy g_1 and step–step interaction energy $g_2 > 0$. We point out that the evolution of the surface morphology here is caused by the motion of steps. The energy functional used for this problem is

$$H_3(h) := \int \left(g_0 + g_1|\nabla h| + \tfrac{1}{3}g_2|\nabla h|^3\right) \mathrm{d}x, \tag{1.5}$$

where the g_0 term represents the surface free energy of the reference plane, g_1 is the step energy, and g_2 includes entropic repulsions due to fluctuations at the step edges and pairwise energetic interactions between adjacent steps. We will omit details and, moreover, we will not analyze the equation obtained closely. We mention this problem to show that there are situations in which different power law surface energy functionals are used. Readers who are interested in this problem may see [20] and the references therein.

Motivated by recent investigations, different energy functionals of the form (1.6) as given by

$$H_p(h(t,x)) := \frac{1}{p} \int |h_x(t,x)|^p \, \mathrm{d}x, \tag{1.6}$$

have been employed very recently [20, 25, 26]. In [26], we derived the following initial boundary value problem:

$$h_t + (p-1)\left[h^n\left[\left(h_x^2\right)^{\frac{p}{2}-1} h_{xx}\right]_x\right]_x = 0 \tag{1.7}$$

in $Q_T := (0,T) \times \Omega$, where $T > 0$ and Ω is the bounded interval

$$\Omega = \{-a < x < a\},$$

with initial conditions

$$h(0,x) = h_0(x), \quad h_0 \in H^p(\Omega) \tag{1.8}$$

and with *no-flux* boundary conditions

$$h_x = h^n\left(\left(h_x^2\right)^{\frac{p}{2}-1} h_{xx}\right)_x = 0 \ \text{ for } x \in \{-a, a\}. \tag{1.9}$$

Remark. The boundary conditions considered in [26] should be replaced by (1.9). We thank the referees for pointing this out.

Our first result is on the regularity and long-time behavior of solutions. As pointed out in [26], we were unable to deduce a result related to higher regularity of solutions. Here, by the help of integral estimates of [26], we prove in particular that $h(t,\cdot) \in C^1([-a,a])$ for almost every $t > 0$. Actually, we obtain a sharper result. Setting

$$b_n = \begin{cases} \frac{p}{(p-1)} & \text{if } 0 < n \leq 3\frac{(p-1)}{p} \\ \frac{3}{n} & \text{if } 3\frac{(p-1)}{p} \leq n < 3, \end{cases}$$

we show that $h^{1/b}(t,\cdot) \in C^1([-a,a])$ for almost every $t > 0$, for any $b \in (0, b_n)$. We show that the solution eventually relaxes to a constant and as a consequence there exists a time after which the solution stays positive. For $n \geq 2 + \frac{p}{p-1}$ we deduce also that the support of the solution is a constant. In [26] an entropy dissipation–entropy estimate, using the functional K_q, was provided for the special case $p = 3, n = 2$. Here we extend our calculations to include various other values of p and n. The case $1 \leq n \leq 2$ in the proof of long-time behavior of non-negative weak solutions was postponed in [26]. Here we complete that missing part.

To facilitate the reading of this paper, we briefly introduce our main results and their main proof techniques in the following.

1.1 Regularity and large-time behavior

We prove the following result related to the regularity properties of solutions.

Theorem 1 (regularity). Assume $0 < n < 3$ and let h_0 satisfy

$$n \in (0,\infty), \;\; h_0 \in H^p((-a,a)), \quad h_0 \geq 0, \;\; h_0 \not\equiv 0 \;\text{ in }\; [-a,a]. \tag{1.10}$$

Let h_ϵ be the solution of the problem:

$$h_t + (p-1)\big[P_\epsilon(h)\big((h_x^2)^{p/2-1} h_{xx}\big)_x\big]_x = 0, \;\text{ in } \Omega := (0,\infty) \times (-a,a), \tag{1.11}$$

with initial condition

$$h_\epsilon(0,x) = h_{0\epsilon}(x) \geq h_0(x) + \epsilon^\theta, \;\; \theta \in (0, 2/5],$$

where $h_{0\epsilon}$ satisfies

$$h_{0\epsilon} \in C^{\infty}([-a,a]),\ \ h_{0\epsilon} > 0,\ \text{ for } x \in [-a,a], h_{0\epsilon} \to h_0 \ \text{ in } H^p((-a,a)) \ \text{ as } \epsilon \to 0 \tag{1.12}$$

and boundary conditions (1.9), for $P_\epsilon(s)$ given by

$$P_\epsilon(s) := \frac{s^{2+p/(p-1)} s^n}{\epsilon s^n + s^{2+p/(p-1)}}. \tag{1.13}$$

Let h be a solution of Equation (1.7) with initial and boundary conditions (1.8) and (1.9) obtained by

$$h_{\epsilon_k} \to h \ \text{ in } \ C_{loc}(\bar{Q}) \ \text{ as } \epsilon_k \to 0. \tag{1.14}$$

Set

$$b_n = \begin{cases} \frac{p}{(p-1)} & \text{if } 0 < n \le 3\frac{(p-1)}{p} \\ \frac{3}{n} & \text{if } 3\frac{(p-1)}{p} \le n < 3. \end{cases}$$

Then, for any $b \in (0, b_n)$,

$$h^{1/b}(t,\cdot) \in C^1([-a,a]), \text{for almost every } t > 0. \tag{1.15}$$

□

Remark. Since $b_n > 1$ for $0 < n < 3$, we may substitute $b_n = 1$ in (1.15) and obtain that $h(t,\cdot) \in C^1([-a,a])$ for almost every $t > 0$.

Theorem 2 (large-time behavior). Let h and h_0 be as in Theorem 1, then the following convergence result holds:

$$h(t,\cdot) \to \frac{1}{2a}\int_{-a}^{a} h_0(x)\,\mathrm{d}x \text{ uniformly in } [-a,a] \text{ as } t \to \infty. \tag{1.16}$$

□

1.2 Support properties

For solutions of Equation (1.7) with initial and boundary conditions (1.8) and (1.9), we prove that for $n \ge 2 + \frac{p}{p-1}$ the support of the solution remains constant. In the proof of this result, we employ some of the integral estimates derived in [26]. On the other hand, as a consequence of (1.16) we deduce that there exists a time, depending on the initial data, such that for later times the solution stays positive.

Theorem 3 (support properties)

Let h and h_0 be as in Theorem 1, then the following results hold.

(i) If $0 < n < 3$ and $h_{0\epsilon}$ satisfies (1.12), then there exists $T = T_{h_0} \geq 0$ such that

$$h(t,x) > 0 \ \text{ for } |x| \leq a, \ \ t > T. \tag{1.17}$$

(ii) If $n \geq 2 + \frac{p}{(p-1)}$, then

$$\operatorname{supp} h(t,\cdot) = \operatorname{supp} h_0, \ \text{ for } t > 0. \tag{1.18}$$

□

1.3 Entropy dissipation–entropy estimate

We prove that the functional $K_q(h(t,x)) := \int_\Omega \frac{h_x^2}{h^q}\,\mathrm{d}x$ is an *entropy functional* for positive smooth solutions of (1.1) for various physical p, q, and n values. That is, we prove that we can bound the rate of decrease of K_q in terms of itself along any smooth positive solution for (1.1). More precisely, we prove that there exists a constant $C > 0$ such that

$$K_q(t) \leq \left[\frac{2}{5\big(Ct + \frac{2}{5}[K_q(0)]^{-5/2}\big)}\right]^{2/5}, \tag{1.19}$$

where for the sake of completeness we also include the $p = 2$ case.

This clearly gives an initial polynomial decay (like $t^{-2/5}$) of positive smooth solutions to the equilibrium.

1.4 Long-time behavior of non-negative weak solutions

In [26], we used the energy functional (1.6) appropriately to deduce the long-time behavior of non-negative weak solutions in the case $0 < n < 1, n > 2$ and we postponed the remaining case. Here we complete the missing part. The idea of the method, similar to [24], is to bound the rate of change of the derivative of the energy functional (for the solution of the approximate problems) from below in terms of itself. Another useful fact is the finiteness of a space–time integral, which is obtained by the dissipation of an entropy functional. Finally, we let $\epsilon \to 0$ and deduce the long-time behavior of non-negative weak solutions.

2 Preliminaries

In [26] the existence of non-negative weak solutions of the problem (1.7), (1.8), and (1.9) has been established.

Definition: A function $h \in C(\bar{Q}) \cap L^{\infty}_{loc}([0,\infty); H^{-p}((-a,a)))$ is a weak solution of the problem (1.7), (1.8), and (1.9) if

$$h \in C^{1,4}(P), \quad h^{n/2}\big((h_x^2)^{p/2-1} h_{xx}\big)_x \in L^2(P), \tag{2.1}$$

where P denotes the positivity set

$$P = \{(t,x) \in \mathbb{R}^+ \times [-a,a] : h(t,x) > 0\},$$

and h satisfies the integral identity

$$\iint_Q h\psi_t \,\mathrm{d}x\,\mathrm{d}t + \iint_P h^n\big((h_x^2)^{p/2-1} h_{xx}\big)_x \psi_x \,\mathrm{d}x\,\mathrm{d}t = 0, \tag{2.2}$$

for all $\psi \in Lip(\bar{Q})$ with compact support in $\mathbb{R}^+ \times [-a,a]$, and h satisfies the initial and boundary conditions

$$h(0,x) = h_0(x), \quad x \in (-a,a),$$
$$h_x(t,\pm a) = 0 \ \text{ if } h(t,\pm a) > 0 \ \text{ for } t > 0.$$

Following the ideas of Bernis and Friedman [6]. I introduced a procedure to construct a non-negative solution of the problem (1.7), (1.8), and (1.9) in [26]. I also use this construction here so we briefly recall its main steps.

If $n \geq 2 + \frac{p}{p-1}$ and $h_0 > 0$ in $[-a,a]$, then the problem (1.7), (1.8), and (1.9) has a unique positive and classical solution, and this suggests the following approximation for h^n

$$P_\epsilon(s) := \frac{s^{2+p/(p-1)}s^n}{\epsilon s^n + s^{2+p/(p-1)}}. \tag{2.3}$$

Then, for any fixed $\epsilon > 0$

$$\frac{P_{\epsilon(s)}}{s^{2+p/(p-1)}} = O(1), \text{ as } s \to 0,$$

and there exists a unique, positive smooth solution h_ϵ of the following equation:

$$h_t + (p-1)\left[P_\epsilon(h)\left(\left(h_x^2\right)^{p/2-1}h_{xx}\right)_x\right]_x = 0, \text{ in } Q, \tag{2.4}$$

combined with the no-flux boundary conditions (1.9) and

$$h(0,x) = h_{0\epsilon}(x), \text{ for } x \in (-a,a), \tag{2.5}$$

where $h_{0\epsilon}$ satisfies

$$h_{0\epsilon} \in C^\infty([-a,a]), \; h_{0\epsilon} > 0, \text{ for } x \in [-a,a], \; h_{0\epsilon} \to h_0 \text{ in } H^p((-a,a)) \text{ as } \epsilon \to 0. \tag{2.6}$$

Formally, one has that

$$\frac{1}{p}\int_{-a}^{a} |h_{\epsilon x}|^p(T,x)\,\mathrm{d}x + (p-1)^2 \iint_{Q_T} P_\epsilon(h_\epsilon)\left(\left(h_{\epsilon x}^2\right)^{p/2-1}h_{\epsilon xx}\right)_x^2 \mathrm{d}x\,\mathrm{d}t = \frac{1}{p}\int_{-a}^{a} |h_{0\epsilon x}|^p(x)\,\mathrm{d}x, \tag{2.7}$$

where $Q_T := (0,T) \times (-a,a)$. Combining (2.7) with

$$\int_{-a}^{a} |h_{0\epsilon x}|^p\,\mathrm{d}x \le C < \infty, \tag{2.8}$$

we deduce that

$$|h_\epsilon(t,x) - h_\epsilon(t,y)| \le K|x-y|^{(p-1)/p}, \tag{2.9}$$

in Q for some constant K independent of ϵ and

$$\iint_Q P_\epsilon(h_\epsilon)\left(\left(h_{\epsilon x}^2\right)^{p/2-1}h_{\epsilon xx}\right)_x^2 \mathrm{d}x\,\mathrm{d}t \le C < \infty.$$

One can also show that [25]

$$|h_\epsilon(t_1,x) - h_\epsilon(t_2,x)| \le M|t_1 - t_2|^\tau, \tag{2.10}$$

where $\tau = \frac{p-1}{5p-2}$ and M is some constant independent of ϵ. Integrating the equation (2.4) over Q_T we find, by employing the initial and boundary conditions too, also that

$$\int_{-a}^{a} h_\epsilon(t,x)\,\mathrm{d}x = \int_{-a}^{a} h_{0\epsilon}(x)\,\mathrm{d}x. \tag{2.11}$$

Using (2.11), (2.8), and Poincaré inequality [14], one obtains a uniform bound for the L^∞ bound of h_ϵ. Hence, we obtain an upper bound on the $C_{t,x}^{\tau,\frac{p-1}{p}}(\bar{Q}_T)$-norm of h_ϵ that is independent of ϵ and T. By the Arzela–Ascoli theorem [16] there exists a function $h \in C(\bar{Q})$ and a sequence ϵ_k such that

$$h_{\epsilon_k} \to h \;\text{ in }\; C_{loc}(\bar{Q}) \;\text{ as }\; k \to \infty, \tag{2.12}$$

and as in [26] any limit function h obtained by (2.12) is a weak solution of the problem (1.7), (1.8), and (1.9).

Proposition. Let h_0 satisfy

$$n \in (0,\infty),\;\; h_0 \in H^p((-a,a)),\;\; h_0 \ge 0,\;\; h_0 \not\equiv 0 \;\text{ in } [-a,a]. \tag{2.13}$$

Then, the function h defined by (2.12) is a non-negative solution of the problem (1.7), (1.8), and (1.9). In addition, $h \in C_{t,x}^{\tau,(p-1)/p}(\bar{Q})$ and h satisfies

$$\int_{-a}^{a} h(t,x)\,\mathrm{d}x = \int_{-a}^{a} h_0(x)\,\mathrm{d}x \;\text{ for } t > 0. \tag{2.14}$$

□

Remark. In the $p=2$ case the uniqueness of positive weak solutions was proved in [6]. It turns out that when $p \ne 2$ the method used in [6] does not work anymore. On the other hand, we conjecture that such a solution is unique and leave the proof of this claim to an upcoming paper.

We also recall here some of the useful integral estimates derived in [26] for a non-negative solution of the problem (1.7), (1.8), and (1.9). These estimates will be useful in proving results on regularity and support properties of solutions. Let $\alpha \ne 0$ and $n > 0$

satisfy

$$\frac{p-1}{p} < \alpha + n < 2.$$

Then,

$$\int_0^\infty \int_\Omega h^{\alpha+n-3} h_x^4 \big(h_x^2\big)^{p/2-1}\,\mathrm{d}x\,\mathrm{d}t < \infty, \tag{2.15}$$

and for almost every $t > 0$ there exists a constant $C(t) < \infty$ such that

$$\begin{aligned} &\text{if } h(t,y) = 0 \text{ for some } y \in [-a,a], \text{ then} \\ &|h(t,x)| \le C(t)|x-y|^m, \ \text{ for } x \in [-a,a], \end{aligned} \tag{2.16}$$

where

$$m := \frac{p+1}{\alpha + n + p - 1}.$$

We now provide the details of our analysis on the problem (1.7), (1.8), and (1.9). We restate the theorems so that one can easily follow the proofs. □

3 Regularity and Large-Time Behavior of Solutions

Theorem 1. Assume $0 < n < 3$ and let h_0 satisfy

$$n \in (0,\infty), \ \ h_0 \in H^p((-a,a)), \ \ h_0 \ge 0, \ \ h_0 \not\equiv 0 \ \text{ in } [-a,a]. \tag{3.1}$$

Let h_ϵ be the solution of the problem

$$h_t + (p-1)\big[P_\epsilon(h)\big(\big(h_x^2\big)^{p/2-1} h_{xx}\big)_x\big]_x = 0, \ \text{ in } Q := (0,\infty)\times(-a,a), \tag{3.2}$$

with initial condition

$$h_\epsilon(0,x) = h_{0\epsilon}(x) \ge h_0(x) + \epsilon^\theta, \ \ \theta \in (0, 2/5],$$

where $h_{0\epsilon}$ satisfies

$$h_{0\epsilon} \in C^{\infty}([-a,a]), \;\; h_{0\epsilon} > 0, \;\; \text{for } x \in [-a,a], \;\; h_{0\epsilon} \to h_0 \;\text{ in }\; H^p((-a,a)) \;\text{ as } \epsilon \to 0 \tag{3.3}$$

and boundary conditions (1.9), for $P_\epsilon(s)$ given by

$$P_\epsilon(s) := \frac{s^{2+p/(p-1)} s^n}{\epsilon s^n + s^{2+p/(p-1)}}. \tag{3.4}$$

Let h be a solution of Equation (1.7) with initial and boundary conditions (1.8) and (1.9) obtained by

$$h_{\epsilon_k} \to h \;\text{ in }\; C_{loc}(\bar{Q}) \;\text{ as } \epsilon_k \to 0. \tag{3.5}$$

Set

$$b_n = \begin{cases} \frac{p}{(p-1)} & \text{if } 0 < n \le 3\frac{(p-1)}{p} \\ \frac{3}{n} & \text{if } 3\frac{(p-1)}{p} \le n < 3. \end{cases}$$

Then, for any $b \in (0, b_n)$,

$$h^{1/b}(t,\cdot) \in C^1([-a,a]), \;\; \text{for almost every } t > 0. \tag{3.6}$$

□

Proof. We note that since $0 < n < 3$ we may choose $\alpha > -1 (\alpha \neq 0)$ satisfying the conditions necessary for the integral estimates proved in [26]. Moreover, it is enough to show that if $0 < b < b_n$, then there is a constant $\tau > 0$ such that for almost every $t > 0$ there exists $C(t) < \infty$ such that if $h(t,y) = 0$, then

$$\left|\left(h^{1/b}\right)_x(t,x)\right| \le C(t)|x-y|^{\tau}, \;\; x \in \Omega. \tag{3.7}$$

By the integral estimates of [26] for the solution h of the problem (1.7), (1.8), and (1.9) we deduce that if $h(t,y) = 0$, then for some finite constant C, we have that

$$|h(t,x)| \le C(t)|x-y|^{\frac{p+1}{\alpha+n+p-1}}, \tag{3.8}$$

where $\frac{p-1}{p} < \alpha + n < 2$. We also have from the proof of Lemma 6.1 of [26] that if $h(t, y) = 0$, then

$$|(h^r)_x| \leq C(t)|x - y|^{\frac{q-1}{bq}}, \tag{3.9}$$

where

$$b = \frac{p+2-q}{q}, \quad r = 1 + \frac{(1 - \frac{1}{\gamma})(\alpha + n + 1)}{p + 2 - q}.$$

Here γ is a positive constant, used in [26], satisfying

$$\gamma_1 \leq \gamma \leq \gamma_2, \tag{3.10}$$

where

$$\gamma_1 := \frac{(\alpha + n + p - 1) - \sqrt{(\alpha + n - 2)(p - 1 - p(\alpha + n))}}{(p+1)}, \tag{3.11}$$

and

$$\gamma_2 := \frac{(\alpha + n + p - 1) + \sqrt{(\alpha + n - 2)(p - 1 - p(\alpha + n))}}{(p+1)}, \tag{3.12}$$

and it will be chosen below. q is given by

$$q = \frac{4\gamma - (\alpha + n + 1)}{\gamma} \in (1, 2). \tag{3.13}$$

Combining (3.8) and (3.9) and assuming $br < 1$, we deduce that

$$\begin{aligned} \left|\left(h^{1/b}\right)\right| &\leq C(t)|x - y|^{(\frac{1}{br} - 1)\frac{r(p+1)}{\alpha+n+p-1}} |x - y|^{\frac{q-1}{p+2-q}} \\ &\leq C(t)|x - y|^{\frac{1}{b}(\frac{p+1}{\alpha+n+p-1}) - 1}. \end{aligned} \tag{3.14}$$

Therefore, to proceed, we have to prove that

Given $0 < n < 3$ *and* $0 < b < b_n$, *we can choose* $\alpha > -1 (\alpha \neq 0)$ *and* γ *satisfying*

(i) $br < 1$,

(ii) $\frac{p-1}{p} < \alpha + n < 2$,

(iii) $\alpha + n + p - 1 < 3\gamma < \alpha + n + p - 1 + \sqrt{(\alpha + n - 2)(p - 1 - p(\alpha + n))}$.

Notice that once we prove these, then (3.7) follows with $\tau = \frac{1}{b}\frac{p+1}{\alpha+n+p-1} - 1$. We begin by choosing γ by

$$\gamma = \frac{1}{b_n} + \nu = \begin{cases} \frac{p-1}{p} + \nu & \text{if } 0 < n \le 3\frac{(p-1)}{p} \\ \frac{n}{3} + \nu & \text{if } 3\frac{(p-1)}{p} \le n < 3. \end{cases}$$

Note that (i) is satisfied if

$$0 < \nu < d_n := \min(\nu_1, \nu_2),$$

where

$$\nu_1 := \frac{\alpha + n + 1}{(\alpha + n + 1)b - p + 2} - \frac{p-1}{p}, \quad \nu_2 := \frac{\alpha + n + 1}{(\alpha + n + 1)b - p + 2} - \frac{n}{3}.$$

We now fix $\nu \in (0, d_n)$ and choose α by

$$\alpha = \frac{3}{b_n} - n - 1 + \mu = \begin{cases} 3\frac{p-1}{p} - n - 1 + \mu & \text{if } 0 < n \le 3\frac{(p-1)}{p} \\ -1 + \mu & \text{if } 3\frac{(p-1)}{p} \le n < 3, \end{cases}$$

where $\mu > 0$ so that $\alpha \neq 0$. Clearly, $\alpha > -1$. $\alpha + n > \frac{p-1}{p}$ is satisfied as $p \ge 2$. On the other hand, $\alpha + n + p - 1 < 3\gamma$ is satisfied if $\mu < 3\nu + 2 - p$. As $p \ge 2$ this says $\mu < 3\nu = 3(1 - \frac{1}{b_n})$. Thus, $\alpha + n < 2$. Finally, we note that the last inequality in (iii) is valid if $0 < \mu < 3\nu + 2 - p$ such that $3\nu + 2 - p - \mu$ is small enough.

This completes the proof. ■

Theorem 2. Let h and h_0 be as in Theorem 1, then the following convergence result holds.

$$h(t, \cdot) \to \frac{1}{2a}\int_{-a}^{a} h_0(x)\,\mathrm{d}x \text{ uniformly in } [-a, a] \text{ as } t \to \infty. \tag{3.15}$$

□

Proof. From the integral estimate (2.15), we deduce that

$$\int_0^T \int_{-a}^{a} |(h^M)_x|^R \,\mathrm{d}x\,\mathrm{d}t < \infty, \tag{3.16}$$

where

$$M := \frac{\alpha + n - 3}{p + 2} + 1, \quad R := p + 2.$$

Using this, if we define

$$K(t) := \max_{[-a,a]} h^{\frac{\alpha+n+p-1}{p+2}}(t,\cdot) - \min_{[-a,a]} h^{\frac{\alpha+n+p-1}{p+2}}(t,\cdot),$$

we deduce that $K \in L^1(\mathbb{R})$. Thus, $K(t) \to 0$ as $t \to \infty$ because $K(t)$ is uniformly continuous in $\mathbb{R}^+$ (by uniform Hölder continuity of h). Hence, we conclude that

$$\max_{[-a,a]} h(t,\cdot) - \min_{[-a,a]} h(t,\cdot) \to 0 \ \text{ as } t \to \infty. \tag{3.17}$$

Combining (3.17) with the mass conservation, we finish the proof. ■

4 Support Properties

Theorem 1. Let h and h_0 be as in Theorem 1, then the following results hold.

(i) If $0 < n < 3$ and $h_{0\epsilon}$ satisfies (1.12), then there exists $T = T_{h_0} \geq 0$ such that

$$h(t,x) > 0 \ \text{ for } |x| \leq a, \ \ t > T. \tag{4.1}$$

(ii) If $n \geq 2 + \frac{p}{(p-1)}$, then

$$\text{supp } h(t,\cdot) = \text{supp } h_0, \ \ \text{for } t > 0. \tag{4.2}$$

□

Proof.
(i)—This follows from (1.16).
(ii)—In order to prove (4.2) we need to show that for $n \geq 2 + p/(p-1)$, one has

$$\text{supp } h(t,\cdot) \subseteq \text{supp } h_0, \ \ t > 0, \tag{4.3}$$

as the other side of the inclusion follows from (i) of Theorem 2 in [26].

To prove (4.3) we suppose on the contrary that there exists a time $t > 0$, a constant $\delta > 0$ and a smooth function ϕ with support in Ω satisfying

$$h(t,x) > \delta > 0 \ \text{ for } x \in \operatorname{supp} \phi,$$

$$\operatorname{supp} \phi \cap \operatorname{supp} h_0 = \emptyset.$$

Let $c > 0$ be a constant. By (2.1) and (2.2) we may take, as in [2], $\psi = \frac{\phi}{h+c}$ as a test function in (2.2). This gives us

$$\int_\Omega \phi(x)\ln(h(t,x)+c) - \int_\Omega \phi(x)\ln(h_0(x)+c)$$
$$= \iint_{P\cap Q_t} \left((h_x^2)^{p/2-1}h_{xx}\right)_x \frac{\phi' h^n}{h+c}\,\mathrm{d}x\,\mathrm{d}t - \iint_{P\cap Q_t} \left((h_x^2)^{p/2-1}h_{xx}\right)_x \frac{\phi h_x h^n}{(h+c)^2}\,\mathrm{d}x\,\mathrm{d}t =: L_1 + L_2. \tag{4.4}$$

By the choice of ϕ, we know that

$$\int_\Omega \phi(x)\ln(h(t,x)+c) - \int_\Omega \phi(x)\ln(h_0(x)+c) \to \infty \ \text{ as } c \to 0. \tag{4.5}$$

Since $n \ge 2 + p/(p-1)$, $h^{n/2}((h_x^2)^{p/2-1}h_{xx})_x \in L^2(P\cap Q_t), h_x \in L^2(Q_t)$ and h is bounded in Q_t. Now, to get a contradiction, we will try to bound the last two terms in (4.4) uniformly. Let $P_t := P \cap Q_t$. By Cauchy–Schwarz inequality [23] we have

$$|L_1| \le \left(\iint_{P_t} h^n\left((h_x^2)^{p/2-1}h_{xx}\right)_x^2 \mathrm{d}x\,\mathrm{d}t\right)^{1/2}\left(\iint_{P_t} h^{n-2}\left(\frac{\phi' h}{h+c}\right)^2\right)^{1/2} \le C_1. \tag{4.6}$$

$$|L_2| \le \left(\iint_{P_t} h^n\left((h_x^2)^{p/2-1}h_{xx}\right)_x^2 \mathrm{d}x\,\mathrm{d}t\right)^{1/2}\left(\iint_{P_t} h^{n-(2+p/(p-1))}\left(\frac{\phi^2 h_x^2 h^{2+p/(p-1)}}{h+c}\right)^4\right)^{1/2} \le C_2. \tag{4.7}$$

Since C_1 and C_2 are constants independent of c, we get a contradiction. ■

5 An Entropy Dissipation–Entropy Estimate for (1.7)

The term "entropy" is used frequently for a Lyapunov functional whose rate of decrease can be bounded in terms of itself. That is, if $H(f)$ is some functional of f, and along the flow of some evolution we have

$$\frac{\mathrm{d}}{\mathrm{d}t}H(f) \leq -\Phi(H(f)), \tag{5.1}$$

with Φ some continuous strictly monotone increasing function on $\mathbb{R}_+$, then the functional $H(f)$ is called an entropy, and the inequality (5.1) is called an entropy dissipation–entropy inequality. The point is that (5.1) can be used to quantitatively estimate the rate of decay of $H(f)$.

Consider again a smooth solution of (1.7) and define the functional K_q by,

$$K_q(h(t,x)) := \int_\Omega \frac{h_x^2}{h^q}\,\mathrm{d}x. \tag{5.2}$$

We note that this functional has been discovered by Laugesen [18] for the thin-film equation. Laugesen showed that K_q is a Lyapunov functional for the thin-film equation provided that $q \in [0, 1/2]$. Moreover, K_q was used in [10] to prove an entropy dissipation–entropy estimate for a thin-film-type equation. Recently, the special case $n = 2$ and $p = 3$ has been considered in [26], where it was noted that the same kind of calculations work for a wider range of p and n values.

Differentiating K_q along a smooth, positive solution of (1.7) yields that (integrals below are over the set Ω)

$$\begin{aligned}\frac{\mathrm{d}K_q(h)}{\mathrm{d}t} &= -2\int \frac{h_x}{h^q}\Big[h^n\big((p-1)(p-2)\big(h_x^2\big)^{\frac{p}{2}-2}h_x h_{xx}^2 + (p-1)\big(h_x^2\big)^{\frac{p}{2}-1}h_{xxx}\big)\Big]_{xx}\,\mathrm{d}x \\ &\quad + q\int \frac{h_x^2}{h^{q+1}}\Big[h^n\big((p-1)(p-2)\big(h_x^2\big)^{\frac{p}{2}-2}h_x h_{xx}^2 + (p-1)\big(u_x^2\big)^{\frac{p}{2}-1}h_{xxx}\big)\Big]_x\,\mathrm{d}x. \end{aligned} \tag{5.3}$$

We apply integration by parts twice to the first term and once to the second term, so that h_{xxx} is the highest order derivative appearing in the calculations.

$$\begin{aligned}\frac{\mathrm{d}K_q(h)}{\mathrm{d}t} &= \tfrac{2}{3}(p-1)(p-2)(p-3)\int \frac{\big(h_x^2\big)^{\frac{p}{2}-2}h_{xx}^4}{h^{q-n}}\,\mathrm{d}x \\ &\quad + \Big[\tfrac{2}{3}(p-2)(5q+n)(q-n+1) - q(q+1)(p-1)(p-2)\Big]\int \frac{\big(h_x^2\big)^{\frac{p}{2}-1}h_x^2 h_{xx}^2}{u^{q-n+2}}\,\mathrm{d}x \\ &\quad + 4\Big[q(p-1) - \tfrac{1}{3}(p-2)(5q+n)\Big]\int \frac{\big(h_x^2\big)^{\frac{p}{2}-1}h_x h_{xx} h_{xxx}}{h^{q-n+1}}\,\mathrm{d}x \\ &\quad - 2(p-1)\int \frac{\big(h_x^2\big)^{\frac{p}{2}-1}h_{xxx}^2}{h^{q-n}}\,\mathrm{d}x - q(q+1)(p-1)\int \frac{\big(h_x^2\big)^{\frac{p}{2}-1}h_x^3 h_{xxx}}{h^{q-n+2}}\,\mathrm{d}x. \end{aligned} \tag{5.4}$$

For future reference we define

$$I_1 = \int (h_x^2)^{p/2-1} \frac{h_{xxx}^2}{h^{q-n}}\, \mathrm{d}x, \quad I_2 = \int (h_x^2)^{p/2-1} \frac{h_x^2 h_{xx}^2}{h^{q-n+2}}\, \mathrm{d}x, \quad I_3 = \int (h_x^2)^{p/2-1} \frac{h_x^6}{h^{q-n+4}}\, \mathrm{d}x; \tag{5.5}$$

$$J_{12} = \int (h_x^2)^{p/2-1} \frac{h_x h_{xx} h_{xxx}}{h^{q-n+1}}\, \mathrm{d}x, \quad J_{13} = \int (h_x^2)^{p/2-1} \frac{h_x^3 h_{xxx}}{h^{q-n+2}}\, \mathrm{d}x, \quad J_{23} = \int (h_x^2)^{p/2-1} \frac{h_x^4 h_{xx}}{h^{q-n+3}}\, \mathrm{d}x. \tag{5.6}$$

Keeping the same order and defining the coefficients of the integrals accordingly, we rewrite (5.4) as

$$\frac{\mathrm{d}K_q(h)}{\mathrm{d}t} =: c_0 T_1 + c_1 I_2 + c_2 J_{12} + c_3 J_{13} + c_4 I_1. \tag{5.7}$$

We focus on the case $p \geq 2$ in this paper, which is realistic as the first nonconstant term appearing in the Taylor polynomial approximation for $\sqrt{1+x^2}$ is $\frac{1}{2}x^2$. Note that for $2 \leq p \leq 3$ the first term, $c_0 T_1$, in (5.4) is nonpositive so that it can be neglected in the procedure. In the case $p > 3$, the first term becomes non-negative and it does not appear in (5.5) or (5.6). Thus, to proceed further in this case, one needs to bound this term in terms of the integrals in the lists (5.5) and (5.6). For the moment such a bound is not available to us, so we focus on the case $2 \leq p \leq 3$ here and leave the other cases for a forthcoming paper of mine. In this case we have

$$\frac{\mathrm{d}}{\mathrm{d}t} K_q(h) \leq c_1 I_2 + c_2 J_{12} + c_3 J_{13} + c_4 I_1, \tag{5.8}$$

where $c_i, i = 1, 2, 3, 4$ are the coefficients of the integrals in (5.4) and I_1, J_{12}, J_{13}, and I_2 are given in (5.5) and (5.6).

Step 1: *We show that*

$$\frac{\mathrm{d}K_q(h)}{\mathrm{d}t} \leq -C_{pqn} I_3, \tag{5.9}$$

where C_{pqn} is a positive constant, which depends on p, q, and n, and I_3 is given in (5.5).

Proof of Step 1. To show that the right-hand side of (5.4) is negative, we will try to write it as a sum of negative squares. To do this, define the non-negative quantity A by,

$$A := \int \left[\alpha h_{xxx} + \beta \frac{h_x h_{xx}}{h} + \gamma \frac{h_x^3}{h^2} \right]^2 (h_x^2)^{\frac{p}{2}-1} h^{n-q}\, \mathrm{d}x, \tag{5.10}$$

where the numbers α, β, and γ will be chosen. (5.10) can be written as

$$A = \alpha^2 I_1 + \beta^2 I_2 + \gamma^2 I_3 + 2\alpha\beta J_{12} + 2\alpha\gamma J_{13} + 2\beta\gamma J_{13}, \tag{5.11}$$

of which $I_1, I_2, I_3, J_{12}, J_{13}, J_{23}$ are given in (5.5) and (5.6).

Lemma 1. Integration by parts yields the following relations:

$$I_2 = -\frac{1}{(p+1)} J_{13} + \frac{q-n+2}{(p+1)} J_{23} \tag{5.12}$$

$$J_{23} = \frac{(q-n+3)}{p+3} I_3. \tag{5.13}$$

□

Proof. This is straightforward computation. ■

Since there are no useful integration by parts identities for I_1 and J_{12}, we use the definition of A appropriately to eliminate these terms.

$$-\alpha^2 I_1 - 2\alpha\beta J_{12} = -A + \beta^2 I_2 + \gamma^2 I_3 + 2\alpha\gamma J_{13} + 2\beta\gamma J_{23}. \tag{5.14}$$

We use (5.14) in (5.7) appropriately.

For $p = 2$ or $p = 3$: In this case we have to choose

$$\alpha := \sqrt{-c_4}, \quad \beta = -\frac{c_2}{2\sqrt{-c_4}}. \tag{5.15}$$

Using (5.15) in (5.14) and plugging this into (5.4), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} K_q(h) \leq \left(c_1 - \frac{c_2^2}{4c_4}\right) I_2 + \gamma^2 I_3 + (c_3 + 2\sqrt{-c_4}\gamma) J_{13} - \frac{c_2}{\sqrt{-c_4}} \gamma J_{23}. \tag{5.16}$$

Using the integration by parts relation (5.12) to eliminate I_2 term in (5.16), we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} K_q(h) \leq & \left[c_3 + 2\sqrt{-c_4}\gamma - \frac{1}{(p+1)}\left(c_1 - \frac{c_2^2}{4c_4}\right)\right] J_{13} \\ & + \left[\left(\frac{q-n+2}{p+1}\right)\left(c_1 - \frac{c_2^2}{4c_4}\right) - \frac{c_2}{\sqrt{-c_4}}\gamma\right] J_{23} + \gamma^2 I_3. \end{aligned} \tag{5.17}$$

Note that J_{13} can have either sign. Thus, we choose γ so that the multiple of it vanishes. This leads to the following choice of γ.

$$\gamma := \frac{\frac{1}{(p+1)}\left(c_1 - \frac{c_2^2}{4c_4}\right) - c_3}{2\sqrt{-c_4}}. \tag{5.18}$$

Plugging this choice of γ in (5.17) and also using the integration by parts identity (5.13), to eliminate J_{23} term, we finally have

$$\frac{\mathrm{d}}{\mathrm{d}t} K_q(h) \leq C(p,q,n) I_3, \tag{5.19}$$

where $C(p,q,n)$ is a constant defined by

$$\begin{aligned} C(p,q,n) := &\left(\left(\frac{q-n+2}{p+1}\left(c_1 - \frac{c_2^2}{4c_4}\right)\right) - \frac{c_2}{\sqrt{-c_4}}\left(\frac{q-n+2}{p+1}\left(c_1 - \frac{c_2^2}{4c_4}\right)\right)\right)\left(\frac{q-n+3}{p+3}\right) \\ &+ \left(\frac{\frac{1}{(p+1)}\left(c_1 - \frac{c_2^2}{4c_4}\right) - c_3}{2\sqrt{-c_4}}\right)^2. \end{aligned} \tag{5.20}$$

A simple calculation yields that if $p = 2$ and $n = 1$ then

$$C_1 := C(2,q,1) = -\frac{q^2}{360}(3 + 18q - 53q^2),$$

which exactly obeys the calculations in [10]. On the other hand, for $p = 2, n = 2$ we have

$$C_2 := C(2,q,2) = -\frac{q^2}{360}(18 - 6q - 53q^2),$$

which works well. If $0 \leq q < \frac{9+4\sqrt{15}}{53}$ then $C_1 \leq 0$ and for $0 \leq q < \frac{3\sqrt{107}-3}{53}$ then $C_2 < 0$. Hence, for the thin film equation, i.e. $p = 2$ case in (1.7), we can show an entropy dissipation–entropy estimate for the physical cases $n = 1$ and $n = 2$. On the other hand, we note that numerical calculations suggest that this can be done for a wider range of noninteger values too. For the physical case $n = 3$ we do not expect to have such an estimate by the results of [18].

Unfortunately for $p = 3$ case, it seems that we can include only the physical case $n = 2$ and we slightly miss the $n = 1$ case. We note that noninteger values can be included in this case too but we do not analyze these at the moment. I have given the calculations in [26] for $p = 3, n = 2$ case. In this case

$$C_3 := C(3, q, 2) = \frac{2371}{6912}q^4 + \frac{77}{432}q^3 - \frac{11}{144}q^2 - \frac{1}{54}q - \frac{1}{108}.$$

In this case for a critical value $q^* \in (0.4, 0.5)$, we have $C_3 \leq 0$ for $q \in [0, q^*]$.

Step 2: *Now, we show that*

$$I_3 \geq N_q K_q(h), \tag{5.21}$$

where N_q is a positive constant.

Proof of Step 2. Notice that

$$I_3 \geq \int \frac{|h_x|^7}{h^{q-n+4}} \mathrm{d}x = \int \left(\frac{h_x^2}{h^q}\right)^{7/2} \frac{1}{h^r} \mathrm{d}x.$$

Letting $z = \frac{h_x^2}{h^q}$, and letting $v = h$, we have that

$$I_3 \geq \int z^{7/2} v^{-r} \, \mathrm{d}x. \tag{5.22}$$

The function $(v, s) \to v^{7/2} s^{-r}$ is jointly convex if $r \leq 5/2$, so that by Jensen's inequality [23],

$$\begin{aligned} \frac{1}{2a} \int_{-a}^{a} z^{7/2} v^{-r} \mathrm{d}x &\geq \left(\frac{1}{2a} \int_{-a}^{a} h \, \mathrm{d}x\right)^{7/2} \left(\frac{1}{2a} \int_{-a}^{a} v \, \mathrm{d}x\right)^{-r} \\ &= \frac{1}{2a \left(\int_{-a}^{a} h_0(x) \, \mathrm{d}x\right)^r} (K_q(h))^{7/2}. \end{aligned} \tag{5.23}$$

Combination of (5.22) and (5.23) gives the result. Note that we have the following restriction on n

$$r = -\frac{5}{2}q - n + 4 \leq \frac{5}{2} \iff n \geq \frac{3}{2} - \frac{5}{2}q. \tag{5.24}$$

Step 3: *Consequence*

For the physical cases $(p, n) \in \{(2,1),(2,2),(3,2)\}$ one can deduce using the preceding calculations that

$$K_q(t) \le \left[\frac{2}{5\left(Ct + \frac{2}{5}[K_q(0)]^{-5/2}\right)} \right]^{2/5}. \tag{5.25}$$

This clearly gives an initial polynomial decay (like $t^{-2/5}$) of positive smooth solutions to the equilibrium, and once $K_q(h)$ is small enough, we can use linearization to obtain an exponential decay. Note that such an explicit linearization has been employed in [10]. We do not give the details here.

6 Long-Time Behavior of Weak Solutions: The Case 1 ≤ n ≤ 2

As pointed out in Section 1, here we employ the energy functional H_p to deduce the long-time behavior of non-negative weak solutions. The idea, same as the one in [24], is to bound the time derivative of the energy from below in terms of itself. This can rigorously be done for the approximate solutions h_ϵ, as they are smooth positive solutions. At the end, one should show that the result also holds for the limit function, which we call the weak solution. Here is the main result of this section.

Theorem 4. Assume $0 < n \le 2$ and $h_0 \in H^p(\Omega)$ has finite mass and satisfies $J[h_0] < \infty$. Then, there exists a constant $C > 0$ such that

$$J[h(t,\cdot)] \le J[h_0(\cdot)] \exp(-Ct), \tag{6.1}$$

where $J[h(t,x)] := \frac{1}{p}\int_\Omega |h_x|^p \, \mathrm{d}x$. □

Proof. Now we shall introduce the notation. For $t \ge 0$ we define

$$J_\epsilon(t) := \frac{1}{p}\int_\Omega |h_{\epsilon x}|^p \, \mathrm{d}x,$$

and

$$I_\epsilon(t) := (p-1)\int_\Omega P_\epsilon(h_\epsilon)\left[\left(h_{\epsilon x}^2\right)^{p/2-1} h_{\epsilon xx}\right]_x^2 \mathrm{d}x,$$

where P_ϵ is given in (2.3). It is easy to see that

$$-\int_\Omega P_\epsilon^{1/2}(h_\epsilon)\big[\big(h_{\epsilon x}^2\big)^{p/2-1}h_{\epsilon xx}\big]_x h_{\epsilon x}\,\mathrm{d}x \le (C\,I_\epsilon\,J_\epsilon)^{1/2},$$

where C is a positive constant depending on $p, |\Omega|, ||h_{0x}||_{L^p(\Omega)}$. We also use the following notation $g_\epsilon(s) := P_\epsilon^{1/2}(s)$. For future reference we compute

$$g_\epsilon(s) = s^{(2+p/(p-1))/2}\big(\epsilon + s^{2+p/(p-1)-n}\big)^{-1/2}$$

and from this, one easily gets

$$g_\epsilon''(s) = (\epsilon + s^m)^{-5/2}[C_1\epsilon s^{\alpha+m-2} + C_2\epsilon^2 s^{\alpha-2} + C_3 s^{2m+\alpha-2}],$$

where

$$\begin{aligned} \alpha &:= \frac{p}{2(p-1)} + 1, \quad m := 2 + \frac{p}{p-1} - n, \\ C_1 &:= \big(-\tfrac{3}{2}\alpha m + \alpha(\alpha-1) + (\alpha - \tfrac{1}{2}m)(\alpha+m-1)\big), \\ C_2 &:= \alpha(\alpha-1), \\ C_3 &:= \big(\alpha - \tfrac{1}{2}m\big)(\alpha+m-1) - \tfrac{3}{2}m\big(\alpha - \tfrac{1}{2}m\big). \end{aligned} \tag{6.2}$$

Note that for $n \in [1,2]$

$$\frac{p}{(p-1)} \le m \le 1 + \frac{p}{(p-1)} < 2 + \frac{p}{(p-1)}$$

which is the analogous inequality for our case. Note that this actually works for $0 < n < 2 + \frac{p}{(p-1)}$. We observe that the coefficients C_1 and C_3 are non-negative and hence we can eliminate the terms involving C_1 and C_3. This leads to

$$\begin{aligned} (C\,I_\epsilon\,J_\epsilon)^{1/2} \ \ge\ & \int_\Omega g_\epsilon(h_\epsilon)\big(h_{\epsilon x}^2\big)^{p/2-1}h_{\epsilon xx}^2\,\mathrm{d}x - \epsilon^2\frac{\alpha(\alpha-1)}{(p+1)}\int_\Omega \big(\epsilon + h_\epsilon^m\big)^{-5/2}h_\epsilon^{\alpha-2}\big(h_{\epsilon x}^2\big)^{p/2-1}h_{\epsilon x}^4\,\mathrm{d}x \\ =:\ & I_1 - \frac{1}{(p+1)}C_2\epsilon^2 I_{2,2}. \end{aligned} \tag{6.3}$$

It is also not difficult to obtain an upper bound for the L^∞ norm of $h_\epsilon(t,\cdot)$ uniformly in t. Let M_ϵ be the upper bound for $|h_\epsilon|_{L^\infty(\Omega)}$. Also, we can easily deduce

$$g_\epsilon(h_\epsilon) \geq \left(\epsilon + M_\epsilon^m\right)^{-1/2}(\epsilon + M_\epsilon)^{-(p-2)/(2(p-1))}h_\epsilon^2.$$

Using this, we have

$$\begin{aligned}(C I_\epsilon J_\epsilon)^{1/2} &\geq \left(\epsilon + M_\epsilon^m\right)^{-1/2}(\epsilon + M_\epsilon)^{-(p-2)/(2(p-1))}\int_\Omega h_\epsilon^2\left(h_{\epsilon x}^2\right)^{p/2-1}h_{\epsilon xx}^2\,\mathrm{d}x \\ &\quad - \frac{1}{(p+1)}C_2\epsilon^2 I_{2,2} =: L_\epsilon J_1 - \frac{1}{(p+1)}C_2\epsilon^2 I_{2,2}. \qquad (6.4)\end{aligned}$$

For the second term in (6.4), using the idea of Lemma 2 in [24], we obtain

$$\begin{aligned}(C I_\epsilon J_\epsilon)^{1/2} &\geq L_\epsilon J_1 - S_\epsilon \epsilon^w \int_\Omega h_\epsilon^{-2}\left(h_{\epsilon x}^2\right)^{p/2-1}h_{\epsilon x}^4\,\mathrm{d}x \\ &= L_\epsilon J_1 - S_\epsilon \epsilon^w J_2. \qquad (6.5)\end{aligned}$$

We note that the constants can be determined explicitly and, moreover, S_ϵ is a finite constant with $S_\epsilon \to S$ as $\epsilon \to 0$ and S is finite. On the other hand, $L_\epsilon \to L$, where L is finite constant, as $\epsilon \to 0$.

To proceed further we recall the following lemma from [26].

Lemma 2. One has the following inequality for $0 \leq h \in H^3(\Omega)$ and $h_x(\pm a) = 0$;

$$\int_\Omega h^\beta\left(h_x^2\right)^{p/2-1}h_{xx}^2\,\mathrm{d}x \geq \frac{(1-\beta)^2}{(p+1)^2}\int_\Omega h^{\beta-}\left(h_x^2\right)^{p/2-1}h_x^4\,\mathrm{d}x. \qquad (6.6)$$

For our purposes we also need the following result, analogous to Lemma 4 of [24].

Lemma 3. Let h_ϵ be the solution of (2.4). Then there exists constants C and α such that

$$\int_\Omega h_\epsilon^2\left(h_{\epsilon x}^2\right)^{p/2-1}h_{\epsilon xx}^2\,\mathrm{d}x \geq C r^\alpha \int_\Omega |h_{\epsilon x}|^p\,\mathrm{d}x, \qquad (6.7)$$

where $r := \int_\Omega h_\epsilon\,\mathrm{d}x > 0$. □

Proof of Lemma 3. Note that an integration by parts yields

$$\int_\Omega |h_{\epsilon x}|^p\,\mathrm{d}x = -(p-3)\int_\Omega h_\epsilon\left(h_{\epsilon x}^2\right)^{p/2-1}h_{\epsilon xx}\,\mathrm{d}x.$$

Note also that by Cauchy–Schwarz inequality [23]

$$J_\epsilon(h_\epsilon) \le C(M_\epsilon, m)\left(\int_\Omega h_\epsilon^{1+p/(2(p-1))}\left(h_{\epsilon x}^2\right)^{p/2-1} h_{\epsilon xx}^2\,\mathrm{d}x\right)^{1/2} J_\epsilon(h_\epsilon)^{(p-2)/p}|\Omega|^{4/p}.$$

Letting $J_1^* := \int_\Omega h_\epsilon^{1+p/(2(p-1))}(h_{\epsilon x}^2)^{p/2-1}h_{\epsilon xx}^2\,\mathrm{d}x$ we get

$$J_1^* \ge A(M_\epsilon, m, p, |\Omega|)J_\epsilon^{4/p}, \tag{6.8}$$

where A is a finite constant. For any $0 < \lambda < (\frac{r}{|\Omega|C_1})^p$, where C_1 is a constant given below, we have $J_1^* \ge C\lambda^{4/p-1}(\int_\Omega |h_{\epsilon x}|^p dx)$, if $\int_\Omega |h_{\epsilon x}|^p\,\mathrm{d}x \ge \lambda$. On the other hand, by using Sobolev and Poincaré inequalities, we also have

$$J_1^* \ge C\left(\frac{r}{|\Omega|} - C_1\lambda^{1/p}\right)^p\left(\int_\Omega |h_{\epsilon x}|^p\,\mathrm{d}x\right),$$

whenever $\int_\Omega |h_{\epsilon x}|^p dx \le \lambda$. It follows, as in [24], that

$$J_1^* \ge \min\left\{C\lambda^{4/p-1},\ C'\left(\frac{r}{|\Omega|} - C_1\lambda^{1/p}\right)^p\right\}\int_\Omega |h_{\epsilon x}|^p\,\mathrm{d}x$$

for $0 < \lambda < (\frac{r}{|\Omega|C_1})^p$. It follows that

$$J_1^* \ge C J_\epsilon, \tag{6.9}$$

where the finite constant C can be obtained explicitly, and it depends on m, p, $|\Omega|$, $\int_\Omega h_\epsilon\,\mathrm{d}x$ and constants of the Sobolev and Poincaré inequalities [14]. ■

Combining what we have so far, we deduce that

$$C_\epsilon J_\epsilon - (C I_\epsilon J_\epsilon)^{1/2} \le C_p C_\epsilon \epsilon^w \int_\Omega \left(h_{\epsilon x}^2\right)^{p/2-1} h_{\epsilon xx}^2\,\mathrm{d}x, \tag{6.10}$$

from which we obtain

$$C_{p\epsilon}^2 J_\epsilon \le I_\epsilon + C_p C_\epsilon \epsilon^w \int_\Omega \left(h_{\epsilon x}^2\right)^{p/2-1} h_{\epsilon xx}^2\,\mathrm{d}x. \tag{6.11}$$

Noting that $I_\epsilon = \frac{\mathrm{d}}{\mathrm{d}t} J_\epsilon$, we obtain from (6.11)

$$\frac{\mathrm{d}}{\mathrm{d}t} J_\epsilon \le -C_p^* C_\epsilon^2 J_\epsilon + C_p C_\epsilon \epsilon^w \int_\Omega \left(h_{\epsilon x}^2\right)^{p/2-1} h_{\epsilon xx}^2 \,\mathrm{d}x, \tag{6.12}$$

where $C_p^*, C_\epsilon, C_p, w$ are all positive. Applying a version of the Gronwall's inequality [12], we deduce that

$$\begin{aligned} J_\epsilon(t) &\le e^{-C_p C_\epsilon^2 t} \left[J_\epsilon(0) + C_p C_\epsilon \epsilon^w \int_0^t \int_\Omega \left(h_{\epsilon x}^2\right)^{p/2-1} h_{\epsilon xx}^2 \,\mathrm{d}x \,\mathrm{d}t \right] \\ &\le e^{-C_p C_\epsilon^2 t} J_\epsilon(0) + C_p C_\epsilon \epsilon^w \int_0^t \int_\Omega \left(h_{\epsilon x}^2\right)^{p/2-1} h_{\epsilon xx}^2 \,\mathrm{d}x \,\mathrm{d}t. \end{aligned} \tag{6.13}$$

Noting that $w > 0$, $\int_0^t \int_\Omega (h_{\epsilon x}^2)^{p/2-1} h_{\epsilon xx}^2 \,\mathrm{d}x \,\mathrm{d}t < \infty$ by the energy dissipation and $C_\epsilon \to C_0 < \infty$ as $\epsilon \to 0$ and

$$\int_\Omega |h_x|^p \,\mathrm{d}x \le \liminf_{\epsilon \searrow 0} \int_\Omega |h_{\epsilon x}|^p \,\mathrm{d}x, \quad \forall\, t > 0,$$

we pass to the limit as $\epsilon \searrow 0$ and deduce finally that

$$J[h(t,\cdot)] \le J[h_0(\cdot)] \exp(-Ct),$$

where C is a finite positive constant. This concludes the proof of Theorem 4. ■

Remark. The approach of [24, 26] can be employed to similar equations. Consider the so called "modified thin film equation" [3–5, 8] given by

$$h_t = -h^n h_{xxxx}, \quad x \in \Omega = (-a, a), a > 0 \tag{6.14}$$

under periodic or no-flux boundary conditions. Here if one considers the energy functional $E[h(t,x)] := \int_\Omega h_{xx}^2 \,\mathrm{d}x$ then the following dissipation result holds for positive smooth solutions of (6.14).

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega h_{xx}^2 \,\mathrm{d}x = -\int_\Omega h^n h_{xxxx}^2 \,\mathrm{d}x.$$

On the other hand, we also have by Cauchy–Schwarz inequality that

$$\int_\Omega h h_{xxxx}\,\mathrm{d}x \le \left(\int_\Omega h^n h_{xxxx}^2\right)^{1/2} \left(\int_\Omega h^{2-n}\right)^{1/2}.$$

This yields after integration by parts, and using the boundary conditions that

$$D[h(t,x)] := \int_\Omega h^n h_{xxxx}^2\,\mathrm{d}x \ge \frac{E^2[h(t,x)]}{\int_\Omega h^{2-n}\,\mathrm{d}x}. \tag{6.15}$$

Thus, if $n \le 2$ then we have

$$D[h(t,x)] \ge C\,E^2[h(t,x)].$$

Finally, we deduce that

$$E[h(t,x)] \le \frac{E[h_0]}{1 + C\,E[h_0]t}, \tag{6.16}$$

where $h_0(x) = h(0,x)$. Note that (6.16) gives a polynomial decay of positive smooth solutions of (6.14).

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