DISTRIBUTIONAL AND CLASSICAL SOLUTIONS TO THE CAUCHY BOLTZMANN PROBLEM FOR SOFT POTENTIALS WITH INTEGRABLE ANGULAR CROSS SECTION

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Abstract. This paper focus on the study of existence and uniqueness of distributional and classical solutions to the Cauchy Boltzmann problem for the soft potential case assuming S^{n-1} integrability of the angular part of the collision kernel (Grad cut-off assumption). For this purpose we revisit the Kaniel–Shinbrot iteration technique to present elementary proofs of existence when the initial data is near vacuum and near a local Maxwellian. In the latter case we allow initial data with infinite mass. We study the propagation of regularity using a recent estimate for the positive collision operator given in [3], by E. Carneiro and the authors, that permits to study such propagation without additional conditions on the collision kernel. Finally, an L^p-stability result (with $1 < p < \infty$) is presented for this case assuming the aforementioned condition.

1. Introduction

Consider the Cauchy Boltzmann problem: Find a function $f(t,x,v) \geq 0$ that solves the equation

$$\frac{\partial f}{\partial t} + v \cdot \nabla f = Q(f,f) \text{ in } (0,+\infty) \times \mathbb{R}^{2n} \tag{1.1}$$

and solves the initial condition $f(0,x,v) = f_0(x,v)$. Equation (1.1) is known as Boltzmann equation and it is used as standard model in the kinetic theory of gases. Much of the difficulty in solving this problem comes from the complexity of the collision operator Q which is defined for any two (suitable) functions f and g by the expression

$$Q(f,g) := \int_{\mathbb{R}^n} \int_{S^{n-1}} \left\{ f('v)g('v_*) - f(v)g(v_*) \right\} \, B(u, \hat{u} \cdot \sigma) \, d\sigma dv_*, \tag{1.2}$$

where the symbols $'v$, $'v_*$, u are defined by

$$'v = v - (u \cdot \sigma)\, \sigma \ , \quad 'v_* = v_* + (u \cdot \sigma)\, \sigma \ \text{ and } \quad u = v - v_*.$$

The function $B \geq 0$ is known as cross section kernel and depends on the type of interaction between the particles of the gas. For a detailed presentation on the physical meaning of the equation (1.1) see the reference [7].

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Solutions for this problem are known to exists in the *renormalized sense* for initial data with finite mass, energy and entropy. We refer to [8] for the study of renormalized solutions and to [1], [16], [19] and [25] for further developments in the theory of very weak solutions. It is also known the existence of distributional solutions (i.e. mild or weak solutions) when the initial data is "small" in some sense, or when is locally "close" to the equilibrium (Maxwell distribution). The theory of distributional solutions for the inhomogeneous Boltzmann case for small initial data started in the early 80's with the work of Illner and Shinbrot [17] who adapted the technique proposed in [18] to produce global solutions in the hard potential case. More recent developments in this theory, including the treatment of the soft potential case, can be found in [4], [15] and [22].

In the near local equilibrium case (i.e. near a local Maxwellian distribution) there are several manuscripts as well. In the reference [21], existence and uniqueness of distributional solutions for hard spheres is treated, meanwhile in the references [10], [23] the soft potential case is discussed including the trend to equilibrium.

In the near global equilibrium case (i.e near a global Maxwellian distribution), the existent results are focused on the existence and uniqueness of classical solutions in the soft potential case. The near global equilibrium theory greatly differs from the near vacuum and the near local Maxwellian theories and started with independent works of Caflisch [6] and Ukai-Asano [24] in early 80's. We refer to [11] for recent developments using energy methods to prove existence and uniqueness of classical solutions in the full soft potential range. It is worth to mention that the existence of distributional (mild or weak) solutions for this case is an open problem.

The main purpose of this paper is to treat both the distributional and classical theory in a more unified way for the cases of near vacuum and near local Maxwellian distributions. We will use techniques already present in the aforementioned references simplifying the existent proofs and generalizing the assumptions on the collision kernel, i.e. avoiding pointwise cut-off or regularization, especially when obtaining classical solutions. We reduce the smoothness assumptions on the initial data in order to obtain smooth classical solutions in both the small data and near local equilibrium cases maintaining minimal regularity on the collision kernel.

This paper is organized as follows: In section 2, distributional solutions for the Cauchy Boltzmann problem are constructed under the condition of "small" initial data. A brief discussion of Kaniel-Shinbrot iteration is presented before the main result is proved. This result of existence and uniqueness applies for both soft and hard potentials. For the remainder of the paper the proof will be focused only in the soft potential case, thus, in section 3 solutions for the Boltzmann Cauchy problem are constructed for the near equilibrium case. Specifically, they are build under the assumption that the initial data is "close" to the local Maxwellian $(0 < \alpha,\ 0 \leq \beta)$

$$\exp\left(-\alpha\left|x-v\right|^{2}-\beta\left|v\right|^{2}\right). \tag{1.3}$$

The main idea of this discussion is taken from reference [23] and, as in section 2, uses the Kaniel-Shinbrot iteration approach. We construct a lower and upper

Maxwellian distribution barriers to implement the iteration. Although our methodology is similar to that of [23], we have a more direct approach that leads to a relaxation on the conditions imposed to the barriers, in particular, we permit lower and upper barrier with different decay toward infinity. We note that reference [10] also allows different decay for the lower and upper Maxwellian distribution barriers using a functional fix point argument. In this manuscript we chose the methodology given in [23] since it is more direct and no additional proof is required to obtain uniqueness of solutions.

In section 4, we study the L^p-propagation of the solution's gradient which permits to construct classical solutions of equation (1.1) with natural regularity assumptions on the initial datum, in particular, we assume $\nabla_x f_0 \in L^p(\mathbb{R}^{2n})$ for some $1 < p < \infty$. We use a new estimate presented in Carneiro and the authors [3] that allows to study the propagation of regularity without pointwise cut-off or regularization in the collision kernel, and to produce a global in time estimate on the spatial gradient of solutions. In this way we generalize some of the aspects treated in [5]. Furthermore, we address the propagation of regularity in the velocity variable by giving a local in time estimate in the velocity gradient.
Finally, two elementary results on stability are shown: stability of solutions in the space of functions uniformly bounded by Maxwellian distributions for the near vacuum case, and a L^p-stability of solutions ($1 < p < \infty$) in both cases. The latter is proved using the techniques previously exposed in this section. This result complements the L^1-stability theorem proved by Ha for soft potentials in [13] and [14]. We point out, however, that the results in these references are presented under the rather restrictive assumption $b(z) \leq K \cos(z)$.

1.1. **Assumption on the model.** Assume that the collision kernel $B(u, \hat{u} \cdot \sigma)$ satisfies

(i) $B(u, \hat{u} \cdot \sigma) = |u|^{-\lambda}\ b(\hat{u} \cdot \sigma)$ with $0 \leq \lambda < n - 1$.
(ii) Grad's assumption: $b(\hat{u} \cdot \sigma) \in L^1(S^{n-1})$. We will denote

$$\|b\|_{L^1(S^{n-1})} = \int_{S^{n-1}} b(\hat{u} \cdot \sigma) d\sigma.$$

Grad's assumption allows to split the collision operator in a gain and a loss part, namely,

$$Q(f, g) = Q_+(f, g) - Q_-(f, g)$$

with obvious definitions for each part. Moreover, the negative part can be expressed as

$$Q_-(f, g) = f\ R(g),$$

where

$$\begin{aligned} R(g) &= \int_{\mathbb{R}^n} \int_{S^{n-1}} g(v_*) |u|^{-\lambda} b(\hat{u} \cdot \sigma) d\sigma dv_* \\ &= \|b\|_{L^1(S^{n-1})} \int_{\mathbb{R}^n} g(v_*) |u|^{-\lambda} dv_* = \|b\|_{L^1(S^{n-1})}\ g * |v|^{-\lambda}. \end{aligned} \quad (1.4)$$

1.2. **Notation and spaces.** Before starting with the proof let us introduce some space functions where we seek solutions.

(i) Let $M_{\alpha,\beta}(x,v) := \exp\left(-\alpha|x|^2 - \beta|v|^2\right)$. We define the space of functions bounded by a space-velocity Maxwellian, denoted by $\mathcal{M}_{\alpha,\,\beta}$, as

$$\mathcal{M}_{\alpha,\beta} = L^\infty(\mathbb{R}^{2n},\ M^{-1}_{\alpha,\,\beta})\,.$$

This space will be endowed with norm

$$\|f\|_{\alpha,\beta} = \left\| f\ M^{-1}_{\alpha,\beta} \right\|_{L^\infty(\mathbb{R}^{2n})}.$$

(ii) Let X a Banach space. Define $W^{1,1}(0,T;X)$ as those functions $f \in L^1(0,T;X)$ such that its time derivative f_t exists in the weak sense and belongs to $L^1(0,T;X)$. The norm used for this space is

$$\|f\|_{W^{1,1}(0,T;X)} = \|f\|_{L^1(0,T;X)} + \|f_t\|_{L^1(0,T;X)}\,.$$

2. Distributional solutions for small initial data

In order to apply the Kaniel and Shinbrot iteration it is convenient to introduce the (well known) trajectory operator #

$$f^{\#}(t,x,v) := f(t,x+tv,v).$$

This operator is the evaluation along the trajectories of the transport operator $\partial_t + v\cdot\nabla$. Hence, equation (1.1) reduces to

$$\frac{df^{\#}}{dt}(t) = Q^{\#}(f,f)(t) \text{ with } f(0) = f_0. \tag{2.1}$$

Definition. A *distributional solution* in $[0,T]$ of problem (1.1) is a function $f \in W^{1,1}(0,T;L^\infty(\mathbb{R}^{2n}))$ that solves (2.1) a.e. in $(0,T]\times\mathbb{R}^{2n}$.

Equation (2.1) is a good base point to define the concept of solution because it does not demand the differentiability in the x-variable for f, equation (1.1) does. Moreover, if f is smooth in the x-variable equations (1.1) and (2.1) are equivalent in the sense that f is a solution of the former if and only if is a solution of the later. In other words, equation (2.1) is a relaxed version of equation (1.1).

The concept of distributional (or *mild*) solution is suited to apply a technique introduced at the end of the 70's by Kaniel and Shinbrot [18]. This technique was first applied for these authors to find local in time mild solution for the Boltzmann equation. Later, it has been used with success to find global distributional solutions in the context of small initial data for the cases of elastic hard spheres in 3-dimension [17], relativistic Boltzmann [9], and recently the inelastic Boltzmann [2].

Kaniel and Shinbrot iteration: In order to present Kaniel and Shinbrot technique we define the sequences $\{l_n(t)\}$ and $\{u_n(t)\}$ as the mild solutions of the linear problems

$$\begin{aligned} &\frac{dl^{\#}_n}{dt}(t) + Q^{\#}_{-}(l_n,u_{n-1})(t) = Q^{\#}_{+}(l_{n-1},l_{n-1})(t) \text{ and}\\ &\frac{du^{\#}_n}{dt}(t) + Q^{\#}_{-}(u_n,l_{n-1})(t) = Q^{\#}_{+}(u_{n-1},u_{n-1})(t), \end{aligned} \tag{2.2}$$

with $0 \le l_n(0) \le f_0 \le u_n(0)$. The construction begins by choosing a pair of functions (l_0, u_0) satisfying what Kaniel and Shinbrot called *the beginning condition* in $[0,T]$, i.e. $u_0^\# \in L^\infty(0,T;\mathcal{M}_{\alpha,\beta})$ and

$$0 \le l_0^\#(t) \le l_1^\#(t) \le u_1^\#(t) \le u_0^\#(t) \;\text{ a.e. in }\; 0 \le t \le T. \tag{2.3}$$

We summarize the final result in [18] with the following theorem (see also [2])

Theorem 2.1. *Let $\{l_n(t)\}$ and $\{u_n(t)\}$ the sequences defined by the mild solutions of the linear problems (2.2). In addition, assume that the beginning condition (2.3) is satisfied in $[0,T]$, then*

(i) The sequences $\{l_n(t)\}$ and $\{u_n(t)\}$ are well defined for $n \ge 1$. In addition, $\{l_n(t)\}$, $\{u_n(t)\}$ are increasing and decreasing sequences respectively, and

$$l_n^\#(t) \le u_n^\#(t) \;\text{ a.e. in }\; 0 \le t \le T.$$

(ii) If $0 \le l_n(0) = f_0 = u_n(0)$ for $n \ge 1$, then

$$\lim_{n\to\infty} l_n(t) = \lim_{n\to\infty} u_n(t) = f(t) \quad \text{a.e. in } [0,T].$$

The limit $f(t) \in C(0,T;\mathcal{M}_{\alpha,\beta})$ is the unique distributional solution of the Boltzmann equation in $[0,T]$ and fulfills

$$0 \le l_0^\#(t) \le f^\#(t) \le u_0^\#(t) \;\text{ a.e. in } [0,T].$$

The following lemma, which holds for soft and hard potentials, provides the essential estimate used in the existence of distributional solutions in the near vacuum case.

Lemma 2.2. *Assume $-1 \le \lambda < n-1$. Then, for any $0 \le s \le t \le T$ and functions $f^\#$, $g^\#$ that lie in $L^\infty(0,T;\mathcal{M}_{\alpha,\beta})$ the following inequality holds*

$$\begin{aligned}\int_s^t \left| Q_+^\#(f,g)(\tau) \right| d\tau \le & \\ & k_{\alpha,\beta} \exp\left(-\alpha|x|^2 - \beta|v|^2\right) \left\| f^\# \right\|_{L^\infty(0,T;\mathcal{M}_{\alpha,\beta})} \left\| g^\# \right\|_{L^\infty(0,T;\mathcal{M}_{\alpha,\beta})},\end{aligned} \tag{2.4}$$

where

$$k_{\alpha,\beta} = \sqrt{\pi}\, \alpha^{-1/2}\, \|b\|_{L^1(S^{n-1})} \left(\frac{|S^{n-1}|}{n-\lambda-1} + C_n\, \beta^{-n/2} \right),$$

with the constant C_n depending only on the dimension. In other words,

$$\int_0^T \left| Q_+^\#(f,g)(\tau) \right| d\tau \in L^\infty(0,T;\mathcal{M}_{\alpha,\beta}).$$

Proof. An explicit calculation produces the inequality,

$$\begin{aligned}&\left| Q_+^\#(f,f)(\tau,x,v) \right| \\ &\qquad \le e^{-\beta|v|^2} \left\| f^\# \right\|_{L^\infty(0,T;\mathcal{M}_{\alpha,\beta})} \left\| g^\# \right\|_{L^\infty(0,T;\mathcal{M}_{\alpha,\beta})} \\ &\int_{\mathbb{R}^n} e^{-\beta|v_*|^2} \int_{S^{n-1}} e^{-\alpha|x+\tau(v-'v)|^2 - \alpha|x+\tau(v-'v_*)|^2} b(\hat{u}\cdot\sigma) d\sigma\; |u|^{-\lambda} dv_*.\end{aligned} \tag{2.5}$$

Note that

$$|x+\tau(v-'v)|^2 + |x+\tau(v-'v_*)|^2 = |x|^2 + |x+\tau u|^2\,,$$

and,

$$\int_s^t e^{-\alpha|x+\tau u|^2} d\tau \le \int_{-\infty}^{\infty} e^{-\alpha|\tau u|^2} d\tau \le \frac{\sqrt{\pi}}{\alpha^{1/2}}\; |u|^{-1}.$$

Therefore, integrating (2.5) in $[s,t]$

$$\int_s^t \left|Q_+^{\#}(f,f)(\tau,x,v)\right| d\tau \leq \frac{\sqrt{\pi}}{\alpha^{1/2}} \|b\|_{L^1(S^{n-1})} \exp\left(-\alpha|x|^2 - \beta|v|^2\right)$$
$$\|f^{\#}\|_{L^\infty(0,T;\mathcal{M}^{\alpha,\beta})} \|g^{\#}\|_{L^\infty(0,T;\mathcal{M}^{\alpha,\beta})} \int_{\mathbb{R}^n} \exp\left(-\beta|v_*|^2\right) \ |u|^{-(\lambda+1)} dv_*.$$

The proof is completed by observing that,

$$\int_{\mathbb{R}^n} \exp\left(-\beta|v_*|^2\right) \ |u|^{-(\lambda+1)} dv_*$$
$$\leq \int_{\{|v_*|<1\}} |u|^{-(\lambda+1)} dv_* + \int_{\{|v_*|\geq 1\}} \exp\left(-\beta|v_*|^2\right) dv_*$$
$$\leq \frac{|S^{n-1}|}{n-\lambda-1} + C_n \ \beta^{-n/2}.$$

□

We are now in condition to prove theorem 2.3 for the global existence of distributional solutions for soft potentials. As we previously commented, this proof is valid for both soft and hard potentials as it relies solely on lemma 2.2 and theorem 2.1. The key step to apply theorem 2.1 is to find suitable functions that satisfy the *beginning condition* globally. The most natural (and simplest) choice for the first members is

$$l_0^{\#} = 0 \quad \text{and} \quad u_0^{\#} = C\exp\left(-\alpha|x|^2 - \beta|v|^2\right).$$

Now compute the following two members

$$l_1^{\#}(t) = f_0 \exp\left(-\int_0^t R^{\#}(u_0)(\tau)d\tau\right) \quad \text{and} \quad u_1^{\#}(t) = f_0 + \int_0^t Q_+^{\#}(u_0,u_0)(\tau)d\tau.$$

Clearly $0 \leq l_0^{\#} \leq l_1^{\#} \leq u_1^{\#}$. In addition, using previous expression and lemma 2.2 we conclude that for all $t \geq 0$,

$$u_1^{\#}(t) \leq \left(\|f_0\|_{\alpha,\beta} + k_{\alpha,\beta}\left\|u_0^{\#}\right\|_{\alpha,\beta}^2\right) \exp\left(-\alpha|x|^2 - \beta|v|^2\right).$$

Note that $\left\|u_0^{\#}\right\|_{\alpha,\beta} = C$, therefore it suffices to choose C such that

$$\|f_0\|_{\alpha,\beta} + k_{\alpha,\beta}C^2 = C$$

to satisfy the beginning condition globally. This is possible as long as

$$\|f_0\|_{\alpha,\beta} \leq \frac{1}{4k_{\alpha,\beta}}\ .$$

This proves the following theorem.

Theorem 2.3. *Let $B(u,\hat{u}\cdot\sigma)$ satisfying the conditions (i) and (ii) with condition (i) relaxed to $-1 \leq \lambda < n-1$. Then, the Boltzmann equation has a unique global distributional solution if*

$$\|f_0\|_{\alpha,\beta} \leq \frac{1}{4k_{\alpha,\beta}}\ ,$$

where the constant $k_{\alpha,\beta}$ is given in lemma 2.2. Moreover, the distributional solution satisfies for any $T \geq 0$

$$\left\| f^{\#} \right\|_{L^\infty(0,T;\mathcal{M}_{\alpha,\beta})} \leq C := \frac{1 - \sqrt{1 - 4k_{\alpha,\beta} \left\| f_0 \right\|_{\alpha,\beta}}}{2k_{\alpha,\beta}}. \tag{2.6}$$

As a consequence of theorem 2.3, one concludes that the distributional solution f is controlled by a traveling Maxwellian, and

$$\lim_{t\to\infty} f(t,x,\xi) \to 0 \text{ a.e. in } \mathbb{R}^{2n}.$$

3. Distributional solutions near local Maxwellian

The aim of this section is to use theorem 2.1 to construct solutions for the Cauchy Boltzmann problem in the soft potential case when the initial data is close locally to the equilibrium, more specifically near to the local Maxwellian distribution given by (1.3). In contrast to the construction for small data made in the previous section, the negative part of the collision operator will be essential for this derivation. The main idea of this construction is taken from [23], additionally, we refer to [20] that worked the Maxwellian case with infinite energy.

Let us introduce some convenient notation and terminology that will help to maintain the proof short and clear. First, we define the *distance* between two Maxwellian distributions $M_i = C_i M_{\alpha_i,\beta_i}$ for $i = 1, 2$ as

$$d(M_1, M_2) := |C_2 - C_1| + |\alpha_2 - \alpha_1| + |\beta_2 - \beta_1|.$$

Second, we say that f is ϵ*–close* to the Maxwellian distribution $M = C\ M_{\alpha,\beta}$ if there exist Maxwellian distributions M_i $(i = 1, 2)$ such that $d(M_i, M) < \epsilon$ for some small $\epsilon > 0$, and

$$M_1 \leq f \leq M_2.$$

It will also be convenient to define the function

$$\phi_{\alpha,\beta}(t,x,v) := \|b\|_{L^1(S^{n-1})} \int_{\mathbb{R}^n} \exp\left(-\alpha\,|x+u|^2 - \beta\,|v - u/t|^2\right) |u|^{-\lambda} du.$$

A simple analysis argument shows that for $-n < \lambda \leq 0$ the function $\phi_{\alpha,\beta}$ is bounded and

$$\|\phi_{\alpha_2,\beta_2} - \phi_{\alpha_1,\beta_1}\|_{L^\infty} \leq C(\min \alpha_i, \min \beta_i)\ d(M_1, M_2), \tag{3.1}$$

for $0 < \alpha_i$ and $0 < \beta_i$. Similar control holds for $\beta_1 = \beta_2 = 0$ with constant depending only on $\min \alpha_i$.

Theorem 3.1. *Let $B(u, \hat{u}\cdot\sigma)$ satisfying the conditions (i) and (ii). In addition, assume that f_0 is ϵ–close to the local Maxwellian distribution $M(x,v) = C\ M_{\alpha,\beta}(x - v, v)$ $(0 < \alpha,\ 0 < \beta)$. Then, for sufficiently small ϵ the Boltzmann equation has a unique solution satisfying*

$$C_1(t)\ M_{\alpha_1,\beta_1}(x - (t+1)v, v) \leq f(t) \leq C_2(t)\ M_{\alpha_2,\beta_2}(x - (t+1)v, v). \tag{3.2}$$

for some positive functions $0 < C_1(t) \leq C \leq C_2(t) < \infty$, and parameters $0 < \alpha_2 \leq \alpha \leq \alpha_1$ and $0 < \beta_2 \leq \beta \leq \beta_1$. Moreover, the case $\beta = 0$ (infinite mass) is permitted as long as $\beta_1 = \beta_2 = 0$.

Proof. Since f_0 is ϵ*–close* to $M(x,v)$ there exists two local Maxwellian distributions such that $d(M_i, M) < \epsilon \ \ (i = 1, 2)$ and

$$C_1 \ M_{\alpha_1,\beta_1}(x - v, v) \leq f_0(x, v) \leq C_2 \ M_{\alpha_2,\beta_2}(x - v, v). \tag{3.3}$$

Since ϵ is small, we may assume that

$$\tfrac{1}{2}\ C \leq C_1 \leq C_2 \leq 2\ C\ , \quad \tfrac{1}{2}\ \alpha \leq \alpha_2 \leq \alpha_1 \leq 2\ \alpha \quad \text{and} \quad \tfrac{1}{2}\ \beta \leq \beta_2 \leq \beta_1 \leq 2\ \beta. \tag{3.4}$$

We solve the Cauchy Boltzmann problem for $t \geq 1$, with initial data f_0 given at $t = 1$. Let us first build a lower and upper barriers of the sequences as follows: Fix $0 \leq l_0^{\#}(t) = C_1(t) M_{\alpha_1,\beta_1}$ and $u_0^{\#}(t) = C_2(t) M_{\alpha_2,\beta_2}$ with $0 \leq C_1(t) \leq C_2(t)$. These functions $C_1(t)$ and $C_2(t)$ will be chosen such that the inequalities

$$\begin{aligned} &\frac{dl_0^{\#}}{dt}(t) + Q_-^{\#}(l_0, u_0)(t) \leq Q_+^{\#}(l_0, l_0)(t) \ \ \text{and} \\ &\frac{du_0^{\#}}{dt}(t) + Q_-^{\#}(u_0, l_0)(t) \geq Q_+^{\#}(u_0, u_0)(t), \end{aligned} \tag{3.5}$$

with $C_1(1) = C_1$ and $C_2(1) = C_2$ are satisfied. A simple evaluation shows that

$$\begin{aligned} &Q_+^{\#}(l_0, l_0)(t) \\ &\quad = C_1^2(t)\ M_{\alpha_1,\beta_1}\ \|b\|_{L^1(S^{n-1})} \int_{\mathbb{R}^n} \exp\left(-\alpha_1 \left|x + tu\right|^2 - \beta_1 \left|v - u\right|^2\right) |u|^{-\lambda} du \\ &\quad = \frac{C_1^2(t)}{t^{n-\lambda}} M_{\alpha_1,\beta_1}\ \phi_1, \end{aligned}$$

where $\phi_1 = \phi_{\alpha_1,\beta_1}$. Similarly

$$\begin{aligned} &Q_+^{\#}(l_0, u_0)(t) \\ &\quad = C_1(t)\ C_2(t)\ M_{\alpha_1,\beta_1}\ \|b\|_{L^1(S^{n-1})} \int_{\mathbb{R}^n} \exp\left(-\alpha_2 \left|x + tu\right|^2 - \beta_2 \left|v - u\right|^2\right) |u|^{-\lambda} du \\ &\quad = \frac{C_1(t)\ C_2(t)}{t^{n-\lambda}} M_{\alpha_1,\beta_1}\ \phi_2, \end{aligned}$$

where $\phi_2 = \phi_{\alpha_2,\beta_2}$. A similar calculation holds for the rest of the terms, thus, inequalities (3.5) are satisfied if

$$\begin{aligned} C_1'(t) + \frac{C_1(t)\ C_2(t)}{t^{n-\lambda}}\ \phi_2 &\leq \frac{C_1^2(t)}{t^{n-\lambda}}\ \phi_1 \\ C_2'(t) + \frac{C_1(t)\ C_2(t)}{t^{n-\lambda}}\ \phi_1 &\geq \frac{C_2^2(t)}{t^{n-\lambda}}\ \phi_2\ . \end{aligned} \tag{3.6}$$

Notice that

$$\begin{aligned} &\frac{\phi_1\ C_1^2(t) - \phi_2\ C_1(t)\ C_2(t)}{t^{n-\lambda}} = \\ &\quad = \frac{C_1^2(t) - C_1(t)\ C_2(t)}{2\ t^{n-\lambda}}\ (\phi_1 + \phi_2) + \frac{C_1^2(t) + C_1(t)\ C_2(t)}{2\ t^{n-\lambda}}\ (\phi_1 - \phi_2) \\ &\quad \geq \frac{C_1^2(t) - C_1(t)\ C_2(t)}{2\ t^{n-\lambda}}\ \ \|\phi_1 + \phi_2\|_{L^\infty} - \frac{C_1^2(t) + C_1(t)\ C_2(t)}{2\ t^{n-\lambda}}\ \ \|\phi_1 - \phi_2\|_{L^\infty}\ . \end{aligned}$$

Similar computations leads to

$$\frac{\phi_2\ C_2^2(t) - \phi_1\ C_1(t)\ C_2(t)}{t^{n-\lambda}} \leq$$

$$\leq \frac{C_2^2(t) - C_1(t)\, C_2(t)}{2\, t^{n-\lambda}} \;\; \|\phi_1 + \phi_2\|_{L^\infty} + \frac{C_2^2(t) + C_1(t)\, C_2(t)}{2\, t^{n-\lambda}} \;\; \|\phi_1 - \phi_2\|_{L^\infty} \,.$$

Therefore inequalities (3.6) (and thus (3.5)) are satisfied if we choose $C_1(t)$ and $C_2(t)$ such that

$$\begin{aligned} C_1'(t) &= \frac{C_1^2(t) - C_1(t)\, C_2(t)}{2\, t^{n-\lambda}} \;\; \|\phi_1 + \phi_2\|_{L^\infty} - \frac{C_1^2(t) + C_1(t)\, C_2(t)}{2\, t^{n-\lambda}} \;\; \|\phi_1 - \phi_2\|_{L^\infty} \\ C_2'(t) &= \frac{C_2^2(t) - C_1(t)\, C_2(t)}{2\, t^{n-\lambda}} \;\; \|\phi_1 + \phi_2\|_{L^\infty} + \frac{C_2^2(t) + C_1(t)\, C_2(t)}{2\, t^{n-\lambda}} \;\; \|\phi_1 - \phi_2\|_{L^\infty} \,. \end{aligned} \tag{3.7}$$

In [20] the reader can find a proof, for $\lambda = 0$ and $\|\phi_1 - \phi_2\|_{L^\infty} = 0$ (i.e. $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$), that such $C_1(t)$ and $C_2(t)$ exist for $t \geq 1$ and are uniformly bounded on time provided that the parameter ϵ is sufficiently small. Our set of equations seems substantially more complicated, however, note that equations (3.7) imply the differential relation

$$\frac{C_1'(t)}{C_1(t)} = -\frac{C_2'(t)}{C_2(t)} \,,$$

which implies the algebraic relation for any $t_0,\ t \geq 1$

$$\frac{C_1(t)}{C_1(t_0)} = \frac{C_2(t_0)}{C_2(t)} \,.$$

Hence, $C_2(t)$ obeys the equation

$$C_2'(t) + \left(\|\phi_1 + \phi_2\|_{L^\infty} - \|\phi_1 - \phi_2\|_{L^\infty}\right) \frac{C_1(1)\, C_2(1)}{2\, t^{n-\lambda}} = \\ = \left(\|\phi_1 + \phi_2\|_{L^\infty} + \|\phi_1 - \phi_2\|_{L^\infty}\right) \frac{C_2^2(t)}{2\, t^{n-\lambda}} \,,$$

which has explicit solution. Indeed, let

$$k^2 = \frac{\|\phi_1 + \phi_2\|_{L^\infty} - \|\phi_1 - \phi_2\|_{L^\infty}}{\|\phi_1 + \phi_2\|_{L^\infty} + \|\phi_1 - \phi_2\|_{L^\infty}}\; C_1(1)\, C_2(1) \,,$$

then

$$\frac{C_2(1) + k}{C_2(1) - k}\; \frac{C_2(t) - k}{C_2(t) + k} = \exp\left(k\; \frac{\|\phi_1 + \phi_2\|_{L^\infty} + \|\phi_1 - \phi_2\|_{L^\infty}}{n - \lambda - 1} \left(1 - \frac{1}{t^{n-\lambda-1}}\right)\right) \,.$$

Therefore, $C_2(t)$ will be uniformly bounded for $t \geq 1$ as long as

$$\exp\left(k\; \frac{\|\phi_1 + \phi_2\|_{L^\infty} + \|\phi_1 - \phi_2\|_{L^\infty}}{n - \lambda - 1}\right) < \frac{C_2(1) + k}{C_2(1) - k} \,. \tag{3.8}$$

Using (3.1) and (3.4), an elementary calculation shows that

$$\begin{aligned} |C_2(1) - k| &\leq K_1(C, \alpha, \beta)\; d(M_1, M_2) \leq 2\; K_1(C, \alpha, \beta)\; \epsilon \,, \\ \exp\left(k\; \frac{\|\phi_1 + \phi_2\|_{L^\infty} + \|\phi_1 - \phi_2\|_{L^\infty}}{n - \lambda - 1}\right) &\leq K_2(C, \alpha, \beta) \,, \\ \text{and} \quad C_2(1) + k &\geq K_3(C, \alpha, \beta) \,. \end{aligned}$$

Hence, inequality (3.8) is easily obtained for sufficiently small $\epsilon > 0$.

Using (3.3) we can use a comparison principle between $l_0^{\#}(t)$ and $l_1^{\#}(t)$ and between $u_0^{\#}(t)$ and $u_1^{\#}(t)$ to conclude that the *beginning condition* holds for $t \geq 1$. An application of theorem 2.1 in the interval $[1, \infty)$ produces the desired solution.

Finally, note that the norm $\|\phi_{\alpha,\beta}\|_{L^\infty}$ is, in general, controlled by a constant independent of β, hence by using (3.1) the case $\beta_1 = \beta_2 = 0$ of infinite mass and energy is included in the result. $\square$

Remark: In fact *item (ii)* of the original version of theorem 2.1 was proved by Kaniel and Shinbrot assuming that $u_0^{\#}(t)$ was summable, however, this is not a major restriction for the result to be true as the authors in [20] proved.

4. Classical solutions

In this section we prove the existence of classical solutions for the cases presented in the two previous sections. For this we assume basic regularity in the initial data and prove that such regularity is propagated through time. This methodology is quite different to the existent one for the soft potential case, see reference [11], where high regularity in the initial data is essential for the argument (based in Sobolev embeddings) to work. Essentially we follow the scheme presented in [5] adding some new features as needed.

Definition. A *classical solution* in $[0,T]$ of problem (1.1) is a function such that

(i) $f(t) \in W^{1,1}(0,T;L^\infty(\mathbb{R}^{2n}))$

(ii) $\nabla f \in L^1(0,T;L^p(\mathbb{R}^{2n}))$ for some $1 \le p$,

which solves equation (1.1) a.e. in $[0,T) \times \mathbb{R}^{2n}$.

Before presenting the proof, we need the following estimate.

Theorem 4.1. *Let the collision kernel satisfying assumptions* (i) *and* (ii)*. Then for* $1 < p < \infty$,

$$\begin{aligned} \|Q_+(f,g)\|_{L^p_v(\mathbb{R}^n)} &\le C_1 \|f\|_{L^p_v(\mathbb{R}^n)} \ \|g\|_{L^\gamma_v(\mathbb{R}^n)} \ , \\ \|Q_+(f,g)\|_{L^p_v(\mathbb{R}^n)} &\le C_2 \|g\|_{L^p_v(\mathbb{R}^n)} \ \|f\|_{L^\gamma_v(\mathbb{R}^n)} \ \text{ and} \\ \|Q_-(g,f)\|_{L^p_v(\mathbb{R}^n)} &\le C_3 \|f\|_{L^p_v(\mathbb{R}^n)} \|g\|_{L^\gamma_v(\mathbb{R}^n)} \ , \end{aligned} \tag{4.1}$$

where $\gamma = \frac{n}{n-\lambda}$ *and* $C_i = C(n,\lambda,p,\|b\|_{L^1(S^{n-1})})$ *for* $i = 1,2,3$.

Proof. The proof of the first two estimates can be found in [3]. In fact, this is a particular case of a Hardy-Littlewood-Sobolev type of inequality that holds for the gain part of the collision operator in the soft potential case. The constants can be explicitly computed and are proportional to

$$C_i \propto \left|S^{n-2}\right| \int_{-1}^{1} \left(\frac{2}{1-s}\right)^{\frac{n-\lambda}{2q}} \ b(s)(1-s^2)^{\frac{n-3}{2}} ds, \quad \text{with} \quad i = 1,2 \, ,$$

and parameter $1 < q = q(n,\lambda,p) < \infty$. Observe that there is a singularity at $s = 1$ in the integrand, nevertheless, this is not a problem since the collision operator can be redefined (by using symmetry) with collision angular kernel

$$\overline{b}(s) := \left(b(s) + b(-s)\right)\chi\left\{s \le 0\right\}.$$

This gives the dependence of the constants on $\|b\|_{L^1(S^{n-1})}$ for $n \ge 2$.

The estimate for Q_- can be proved noticing that for any test function ψ one has

$$\int_{\mathbb{R}^n} Q_-(g,f)(v)\psi(v)dv = \|b\|_{L^1} \int_{\mathbb{R}^n} f(v) \left(\int_{\mathbb{R}^n} g(u)\psi(u)|v-u|^{-\lambda} du \right) dv$$

$$\leq \|b\|_{L^1} \int_{\mathbb{R}^n} f(v) \left[\left(|\psi|^a * |u|^{-\lambda}\right)(v)\right]^{1/a} \left[\left(|g|^{a'} * |u|^{-\lambda}\right)(v)\right]^{1/a'} dv,$$

where $1/a + 1/a' = 1$. This is precisely inequality (4.2) in the proof of Theorem 2 (HLS inequality for soft potentials) in [3]. The rest of the proof continues as given in this reference by taking $r = p$ and $q = \gamma$. □

In order to study the propagation of regularity it is convenient to define for $h > 0$ and $\hat{x} \in S^{n-1}$ the finite difference operator

$$\left(D_{h,\hat{x}} f\right)(x) := \frac{f(x + h\hat{x}) - f(x)}{h} \ ,$$

also, define the translation operator

$$\left(\tau_{h,\hat{x}} f\right)(x) := f(x + h\hat{x}).$$

For notation simplicity we write these operators as D and τ respectively. Let f be the distributional solution of the Cauchy Boltzmann problem for either small data or near local Maxwellian case. Fix $h > 0$ and $\hat{x} \in S^{n-1}$ and apply the operator D to both sides of equation (2.1) to obtain

$$\frac{d(Df)^{\#}}{dt}(t) = (DQ(f,f))^{\#}(t) = Q^{\#}(Df, f)(t) + Q^{\#}(\tau f, Df)(t). \tag{4.2}$$

Multiplying this equation by

$$p \ |(Df)^{\#}|^{p-1} \ \mathrm{sgn}((Df)^{\#}) \ ,$$

and integrating in $\mathbb{R}^{2n}$ we are led to

$$\frac{d \, \|Df\|^p_{L^p}}{dt} = p \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |Df|^{p-1} \mathrm{sgn}(Df) \left(Q(Df, f) + Q(\tau f, Df)\right) dv dx. \tag{4.3}$$

Note that $\mathrm{sgn}(Df) \ Q_-(Df, f) \leq 0$. Therefore, using equation (4.3) we obtain that for $1 < p < \infty$,

$$\begin{aligned} \frac{d \, \|Df\|^p_{L^p}}{dt} &\leq p \int_{\mathbb{R}^n} \|Df\|^{p-1}_{L^p_v(\mathbb{R}^n)} \Big(\|Q_+(Df, f)\|_{L^p_v(\mathbb{R}^n)} + \|Q_+(\tau f, Df)\|_{L^p_v(\mathbb{R}^n)} \\ &\qquad\qquad + \|Q_-(\tau f, Df)\|_{L^p_v(\mathbb{R}^n)} \Big) \, dx \\ &\leq p \ C \int_{\mathbb{R}^n} \|Df\|^p_{L^p_v(\mathbb{R}^n)} \Big(\|f\|_{L^\gamma_v(\mathbb{R}^n)} + \|\tau f\|_{L^\gamma_v(\mathbb{R}^n)} \Big) \, dx. \end{aligned} \tag{4.4}$$

In obtaining these inequalities, we have used Hölder's inequality in the inner integral with exponent p and theorem 4.1. Moreover, the distributional solution $f(t, x, v)$ is controlled by a traveling Maxwellian, then

$$\|f\|_{L^\gamma_v(\mathbb{R}^n)} \leq \frac{C}{(1+t)^{n/\gamma}} = \frac{C}{(1+t)^{n-\lambda}} \ ,$$

$$\text{and similarly} \quad \|\tau f\|_{L^\gamma_v(\mathbb{R}^n)} \leq \frac{C}{(1+t)^{n-\lambda}} \ .$$

Notice that this estimate is valid for solutions with infinite mass and energy due to estimate (3.2). Thus, a Gronwall's argument on (4.4) shows that

$$\|Df\|_{L^p(\mathbb{R}^{2n})}(t) \leq \|Df_0\|_{L^p(\mathbb{R}^{2n})} \exp\left(\int_0^t \tfrac{C}{(1+s)^{n-\lambda}} \ ds \right), \tag{4.5}$$

where $C = C(n, p, \lambda, \|b\|_{L^1(S^{n-1})})$.

Theorem 4.2. *Fix $0 \leq T \leq \infty$ and assume the conditions (i)–(ii) in the collision kernel B. Also, assume that f_0 satisfies the smallness assumption of theorem 2.3 or is near to a local Maxwellian as in theorem 3.1. In addition, assume that $\nabla f_0 \in L^p(\mathbb{R}^{2n})$ for some $1 < p < \infty$. Then, there is a unique classical solution f to problem (1.1) in the interval $[0,T]$ satisfying the estimates of these theorems, and*

$$\left\|\nabla f\right\|_{L^p(\mathbb{R}^{2n})}(t) \leq C \left\|\nabla f_0\right\|_{L^p(\mathbb{R}^{2n})} \quad \text{for all } t \in [0,T], \tag{4.6}$$

with constant $C = C(n, p, \lambda, \|b\|_{L^1(S^{n-1})})$.

Proof. Thanks to theorem 2.3 (or theorem 3.1), we have existence of a unique distributional solution f to the Cauchy Boltzmann problem. Estimate (4.6) for ∇f follows after sending $h \to 0$ in inequality (4.5). After knowing that the weak gradient exist a.e. we can use the chain rule to obtain that for a.e. $(t,x,v) \in (0,T) \times \mathbb{R}^{2n}$

$$\frac{df^{\#}}{dt}(t,x,v) = \left(\frac{\partial f}{\partial t} + v \cdot \nabla f\right)(t, x+tv, v) = Q(f,f)(t,x+tv,v).$$

Thus, f solves equation (1.1) a.e. in $(0,T) \times \mathbb{R}^{2n}$. □

Remark: We can also argue in the following way for the small initial data case: Impose Maxwellian decay on ∇f_0 and observe that multiplying equation (4.2) by $\text{sgn}((Df)^{\#}(t))$, integrating the result in $(0,t)$ and using lemma 2.2

$$\begin{aligned}
&\left\|(Df)^{\#}(t)\right\|_{\alpha,\beta} \leq \\
&\leq \|Df_0\|_{\alpha,\beta} + k_{\alpha,\beta}\left\|(Df)^{\#}\right\|_{L^\infty(0,T;\mathcal{M}_{\alpha,\beta})}\left(\left\|f^{\#}\right\|_{L^\infty(0,T;\mathcal{M}_{\alpha,\beta})} + \left\|(\tau f)^{\#}\right\|_{L^\infty(0,T;\mathcal{M}_{\alpha,\beta})}\right) \\
&\leq \|Df_0\|_{\alpha,\beta} + k_{\alpha,\beta}\left\|(Df)^{\#}\right\|_{L^\infty(0,T;\mathcal{M}_{\alpha,\beta})}\left\|f^{\#}\right\|_{L^\infty(0,T;\mathcal{M}_{2\alpha,\beta})}\left(1 + \exp\left(2\alpha\, h^2\right)\right).
\end{aligned}$$

Fix $\|f_0\|_{2\alpha,\beta} \leq \frac{3}{16 k_{2\alpha,\beta}}$ and use theorem 2.3 to conclude that the distributional solution satisfies

$$\left\|f^{\#}\right\|_{L^\infty(0,T;\mathcal{M}_{2\alpha,\beta})} \leq \frac{1}{4k_{2\alpha,\beta}}\ .$$

But $\frac{k_{\alpha,\beta}}{k_{2\alpha,\beta}} = \sqrt{2}$, therefore,

$$\left\|(Df)^{\#}\right\|_{L^\infty(0,T;\mathcal{M}_{\alpha,\beta})} \leq \frac{4\ \|Df_0\|_{\alpha,\beta}}{4 - \sqrt{2}\left(1 + \exp\left(2\alpha\, h^2\right)\right)}\ .$$

Send $h \to 0$ to conclude that the distributional solution satisfies

$$\left\|(\nabla f)^{\#}\right\|_{L^\infty(0,T;\mathcal{M}_{\alpha,\beta})} \leq \frac{2\ \|\nabla f_0\|_{\alpha,\beta}}{2 - \sqrt{2}}\ .$$

This device produces a classical solution to problem (1.1) with Maxwellian decay in its gradient provided small initial data. Thus, a reiterative use of the argument above can produce classical solutions as smooth as desired on the condition that the initial datum is smooth with its derivatives having Maxwellian decay. Observe that the process is valid for both soft and hard potentials and only uses the integrability of b.

4.1. **Velocity regularity.** In this subsection we investigate briefly the propagation of velocity smoothness for the classical solutions obtained above. To this end we use the finite difference operator for the v-variable

$$(D_{h,\hat{v}}f)(v) := \frac{f(v+h\hat{v})-f(v)}{h}\ ,$$

for a fix $h>0$ and $\hat{v}\in S^{n-1}$. Similarly for the translation operator $\tau_{h,\hat{v}}$. As above, we write such operators as D and τ for notation simplicity. Take a classical solution o f the Cauchy Boltzmann problem f and apply the finite difference operator in equation (1.1) to obtain

$$\frac{d(Df)}{dt}(t)+v\cdot\nabla(Df)(t)+\hat{v}\cdot\nabla(\tau f)(t)=DQ(f,f)(t)=Q(Df,f)(t)+Q(\tau f,Df)(t).$$

This equality follows after using the change of variables $v_*\rightarrow v_*+h\hat{v}$ in the collision operator. Moreover, multiplying this equation by

$$p\ |(Df)|^{p-1}\ \mathrm{sgn}((Df))\ ,$$

and integrating in $\mathbb{R}^{2n}$, we obtain

$$\begin{aligned}\frac{d\,\|Df\|^p_{L^p}}{dt}(t) &\leq p\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|Df|^{p-1}\mathrm{sgn}(Df)\left(Q(Df,f)+Q(\tau f,Df)\right)dvdx\\ &\qquad\qquad + p\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|\nabla(\tau f)||Df|^{p-1}dvdx\\ &\leq p\ C\int_{\mathbb{R}^n}\|Df\|^p_{L^p_v(\mathbb{R}^n)}\left(\|f\|_{L^\gamma_v(\mathbb{R}^n)}+\|\tau f\|_{L^\gamma_v(\mathbb{R}^n)}\right)dx\\ &\qquad\qquad + p\,\|Df\|^{p-1}_{L^p(\mathbb{R}^{2n})}\,\|\nabla(\tau f)\|_{L^p(\mathbb{R}^{2n})}\\ &\leq \frac{p\ C}{(1+t)^{n-\lambda}}\,\|Df\|^p_{L^p(\mathbb{R}^{2n})}+p\,\|Df\|^{p-1}_{L^p(\mathbb{R}^{2n})}\,\|\nabla f\|_{L^p(\mathbb{R}^{2n})}\,. \end{aligned}\tag{4.7}$$

In order to quantify the size of the L^p-norm of Df let $X(t):=\|Df\|^p_{L^p(\mathbb{R}^{2n})}(t)$, then inequality (4.7) takes the form

$$\frac{dX(t)}{dt}\leq a(t)X(t)+b(t)X^{\frac{p-1}{p}}(t)\ ,$$

with

$$a(t)=\frac{p\ C}{(1+t)^{n-\lambda}}\quad\text{and}\quad b(t)=p\,\|(\nabla f)(t)\|^{p-1}_{L^p(\mathbb{R}^{2n})}\,.$$

Therefore,

$$X^{\frac{1}{p}}(t)\leq X_0^{\frac{1}{p}}\exp\left(\tfrac{1}{p}\int_0^t a(s)ds\right)+\frac{1}{p}\int_0^t\exp\left(\tfrac{1}{p}\int_\sigma^t a(s)ds\right)b(\sigma)d\sigma,$$

hence, using the estimate (4.6)

$$\|Df\|_{L^p(\mathbb{R}^{2n})}(t)\leq\left(\|Df_0\|_{L^p(\mathbb{R}^{2n})}+t\,\|\nabla f_0\|_{L^p(\mathbb{R}^{2n})}\right)\exp\left(\int_0^t\tfrac{C}{1+s^{n-\lambda}}ds\right).$$

Thus, letting $h\rightarrow 0$ we conclude the following theorem.

Theorem 4.3. *Let f be a classical solution in $[0,T]$ with f_0 satisfying the condition of theorem 2.3 or theorem 3.1 and $\nabla_x f_0\in L^p(\mathbb{R}^{2n})$ for some $1<p<\infty$. In addition assume that $\nabla_v f_0\in L^p(\mathbb{R}^{2n})$. Then, f satisfies the estimate*

$$\|(\nabla_v f)(t)\|_{L^p(\mathbb{R}^{2n})}\leq C\left(\|\nabla_v f_0\|_{L^p(\mathbb{R}^{2n})}+t\,\|\nabla_x f_0\|_{L^p(\mathbb{R}^{2n})}\right),\tag{4.8}$$

with $C = C(n, p, \lambda, \|b\|_{L^1(S^{n-1})})$ *independent of the time.*

4.2. L^p **and** $\mathcal{M}_{\alpha,\beta}$ **stability.** We finish this last section by presenting a short discussion on the stability of solutions in the L^p $(1 < p < \infty)$ and $\mathcal{M}_{\alpha,\beta}$ spaces. We refer to [13] and [14] for a complete discussion on the L^1 stability for solutions near vacuum and near Maxwellian for soft and hard potentials.
First, take f and g solutions of the Boltzmann Cauchy problem associated to the initial data f_0 and g_0 respectively. These data fulfill the condition of theorem 2.3 (or theorem 3.1) so that f and g are controlled by Maxwellian distributions as described in these theorems. Thus,

$$\frac{d(f-g)^{\#}}{dt}(t) = Q^{\#}(f,f)(t) - Q^{\#}(g,g)(t) = \frac{1}{2}\left[Q^{\#}(f-g, f+g) - Q^{\#}(f+g, f-g)\right].$$

After multiplying by $|(f-g)^{\#}|^{p-1}\text{sgn}((f-g)^{\#})$ with $p > 1$ and following the usual steps we arrive to

$$\frac{d\,\|f-g\|^p_{L^p}}{dt}(t) \leq C \int_{\mathbb{R}^n} \|f-g\|^p_{L^p_v(\mathbb{R}^n)}\, \|f+g\|_{L^\gamma_v(\mathbb{R}^n)}\, dx.$$

Since f and g are controlled by traveling Maxwellian distributions we have

$$\|f+g\|_{L^\gamma_v(\mathbb{R}^n)} \leq \frac{C}{(1+t)^{n-\lambda}}\,.$$

Therefore, an application of Gronwall's lemma leads to

Theorem 4.4. *Let f and g distributional solutions of problem (1.1) associated to the initial data f_0 and g_0 respectively. Assume that these data satisfies the condition of theorem 2.3 (or theorem 3.1). Then, there exist $C > 0$ independent of time such that*

$$\|f-g\|_{L^p(\mathbb{R}^{2n})} \leq C\ \|f_0 - g_0\|_{L^p(\mathbb{R}^{2n})} \quad \text{with} \quad 1 < p < \infty. \tag{4.9}$$

Moreover, for f_0 and g_0 sufficiently small in $\mathcal{M}_{\alpha,\beta}$ it holds

$$\left\|(f-g)^{\#}\right\|_{L^\infty(0,T;\mathcal{M}_{\alpha,\beta})} \leq C\ \|f_0 - g_0\|_{L^\infty(0,T;\mathcal{M}_{\alpha,\beta})}\,. \tag{4.10}$$

Proof. It remains to prove estimate (4.10) which is a direct consequence of lemma 2.2 and estimate (2.6). □

Remark: Obviously, theorem 4.4 gives uniqueness of solutions $f^{\#} \in L^\infty(0,T;\mathcal{M}_{\alpha,\beta})$ (for $\alpha > 0$, $\beta > 0$), in particular, solutions constructed in [10] are unique.

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