CONVOLUTION INEQUALITIES FOR THE BOLTZMANN COLLISION OPERATOR

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Abstract. We study integrability properties of a general version of the Boltzmann collision operator for hard and soft potentials in n-dimensions and integrable angular cross section on the $n-1$-dimensional sphere (Grad cut-off assumption). A rearrangement of the collisional integrals allows us to write the collision operator as a weighted convolution, where the weight is given in turn by an operator invariant under rotations. Using a symmetrization technique in L^p we prove a Young's inequality for for the gain part of the collisional integral in the case of of variable hard potentials, which is optimal for Maxwellian molecule type models in L^2. Further, we find an inedited form of the Hardy-Littlewood-Sobolev inequality in the soft potentials case, which corresponds to singular collision kernels. In all cases, the inequality constants are explicitly given by formulas depending on the integral of the angular cross section. We also obtain estimates with Maxwellian weights for variable hard potentials. All these estimates are valid for conservative or dissipative interactions between particles.

1. Introduction

The nonlinear Boltzmann equation is a classical model for a gas at low or moderate densities. The gas in a spatial domain $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, is modeled by the evolution of the mass density function $f(x, v, t)$, $(x, v) \in \Omega \times \mathbb{R}^n$, modeling the probability of finding a particle at position x, with velocity v at the time $t \in \mathbb{R}$. The transport equation for f reads

$$(\partial_t + v \cdot \nabla_x) f = Q(f, f)\,, \tag{1.1}$$

where $Q(f, f)$ is a quadratic integral operator, expressing the change of f due to instantaneous binary collisions of particles. The precise form of $Q(f, f)$ will be introduced below, for both conservative (elastic) [12] and dissipative (inelastic) interactions [11]. The $Q(f, f)$ operator factorizes as the difference of two positive operators, usually denoted by the $Q^+(f, f)(x, v, t)$ rate of gain of probability due to two pre-collisional velocities for which one of them will take the direction v and the $Q^-(f, f)(x, v, t)$ rate of loss of probability due to particles that get knocked out of the direction v.

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In addition, these operators depend on the form of their collision kernels which model the collision frequency depending on the intramolecular potentials between interacting particles. More specifically, these kernels depend on functions of the relative speed and on the scattering angle, the latter modeled by an angular function referred as the angular cross section. In all the cases we assume that the angular cross section is modeled by an integrable angular function on the S^{n-1} sphere (this condition, in the theory of the Boltzmann equation, is called the *Grad cut-off assumption*). The collisional kernels are further divided into the following classes: variable hard potentials, corresponding to unbounded forms of the relative speed, modeling stronger collision rates, and soft potentials modeling weaker collision rates, both as the relative speed is larger; and Maxwell molecule type of interactions where collisional kernels are independent of the relative speed. Soft potential kernels give place to singular term in the collisional integral.

It is the purpose of this work to investigate the L^r-integrability of the gain operator as a bilinear form $Q^+(f,g)(x,v,t)$ acting on probability mass densities f and g, and to search for exact representation formulas for the inequality constants and possible optimal estimates depending on the L^p and L^q norms of f and g, respectively.

In order to achieve these results, we use a rearrangement of the collisional integrals that allows us to write the gain term of the collision operator as a weighted convolution, where the weight is given by a suitable bilinear operator invariant under rotations. This representation exhibits the convolution nature of the collisional operators. Following the initial idea developed in [2], we approach the convolution estimates by using an L^p-radial symmetrization technique. We prove a Young's inequality for variable hard potentials, which is optimal for Maxwellian molecule type models in L^2, both for conservative and dissipative interactions.

Furthermore, we find a completely new Hardy-Littlewood-Sobolev type inequality for collision kernels corresponding to soft potentials, where the weighted convolution structure contains a singular kernel. In all cases, the inequality constants are given by explicit formulas depending only on certain integrability conditions of the angular cross section. We also obtain estimates with Maxwellian weights for variable hard potentials. All these estimates are valid for elastic or inelastic interactions between particles.

We point out that our work extends and improves the work of Gustafsson [17] on finding Young's inequality for the Q^+ operator, developed by interpolation arguments (Riesz-Thorin) for the cases $(p,q,r)=(1,p,p)$ and $(p,1,p)$. In the more general case (p,q,r) Gustafsson used a nonlinear interpolation theorem whose arguments lead to rather poor and non-explicit constants in general. A crucial point in his argument is the restriction of collisions to be neither frontal nor grazing, so it uses a pointwise cut-off of the angular cross section both for angles near zero or π (i.e. the constant in his Young's inequality blows up at the endpoints). In addition he requires the standard integrability of the angular kernel. See also [21] for a different approach to the result of Gustafsson under the same restrictions.

Our work considerably improves both [17] and [21], as it removes the pointwise cut-off restriction for the angular cross section for head-on and grazing collisions. Our estimates are constructive and provide exact constants which depend only on integrability of the angular collisional cross section in both cases, hard spheres and variable hard potentials, which are sharp (best) constants for the case of Maxwell molecules type in L^2 for both conservative and dissipative interactions. The conservative interaction case of this result was first proved by two of the authors [2], and here we give a simplified argument to represent the weighted convolution structure of the gain operator. In the context of Boltzmann equation, sharp constants in these inequalities are important since one hopes to control the gain by the loss operator in order to obtain regularity.

The new type of convolution estimates treated in this paper is in the case of singular collision kernels in relative speed (soft potentials), for which a Hardy-Littlewood-Sobolev inequality is obtained with exact constant representations as well. Our work is also motivated by the ones of Beckner [5] on the sharp constants for convolution estimates, and Lieb [18] on the sharp Hardy-Littlewood-Sobolev inequality. Both of these works use radial rearrangement techniques to reduce the problem to radial functions, and this is essentially one of the core ideas of this paper as well (see Lemma 3 below).

As a consequence of these convolutions estimates of collision integrals with singular potentials, we have recently used them to obtain classical solutions and L^p-stability for the Cauchy problem associated to the Boltzmann equation for soft potentials, with integrable cross section, and initial data near vacuum or near local Maxwellian distribution [4].

In summary, the main contributions of the inequalities presented in this paper when compared to the previous literature are: simpler proofs; extensions to the full range of exponents p, q, r; extensions to dissipative (inelastic) interactions and soft potentials and exact and optimal constants depending on integral conditions on the angular cross section in the collision kernel.

1.1. **Preliminaries.** In this paper we study the integrability properties of the gain part of the Boltzmann collision operator in the case of inelastic collisions. This operator is commonly denoted by Q^+ and can be defined via duality by the formula

$$\int_{\mathbb{R}^n} Q^+(f,g)(v)\psi(v)\,\mathrm{d}v := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(v)g(v_*)\int_{S^{n-1}} \psi(v')B(|u|,\hat{u}\cdot\omega)\,\mathrm{d}\omega\,\mathrm{d}v_*\,\mathrm{d}v, \tag{1.2}$$

where the functions $f, g, \psi \in C_0(\mathbb{R}^n)$ (continuous with compact support). The symbol $\hat{u}$ represents the unitary vector in the direction of u ($\hat{u} = u/|u|$) and $\mathrm{d}\omega$ is the surface measure on the sphere S^{n-1}. The variables v, v_* (pre-collision velocities), v', v'_* (post-collision velocities) and u (relative velocity) are related by

$$u = v - v_* \quad , \quad v' = v - \frac{\beta}{2}(u - |u|\omega) \quad \text{and} \quad v + v_* = v' + v'_* \tag{1.3}$$

The inelastic properties of the collision operator are encoded in the positive scalar function $\beta : [0,\infty) \to [\frac{1}{2}, 1]$ defined by $\beta(z) := \frac{1+e(z)}{2}$, where parameter e is the so-called *restitution coefficient* which enjoys the following two properties that assure micro-reversibility of the interactions:

(i) $z \mapsto e(z)$ is absolutely continuous and non-increasing.
(ii) $z \mapsto ze(z)$ is non-decreasing.

The dependence of the restitution coefficient on the physical variables is commonly given by $z = |u|\sqrt{\frac{1-\hat{u}\cdot\omega}{2}}$, i.e. the restitution coefficient e, and thus β, depends only on the impact velocity

$$\beta\left(|u|\ \sqrt{\tfrac{1-\hat{u}\cdot\omega}{2}}\right) = \frac{1+e\left(|u|\ \sqrt{\tfrac{1-\hat{u}\cdot\omega}{2}}\right)}{2}\,. \tag{1.4}$$

We point out that the model for e could be more complex (for example assuming dependence on macroscopic variables like temperature), however this will not be the case in this paper.
The particle interaction is elastic when the parameter $\beta = 1$, and is referred as *sticky* particles when $\beta = 1/2$. A complete discussion of the physical aspects of the restitution coefficient can be found in [11]. Standard models for the restitution coefficient, for example constant restitution coefficient and viscoelastic hard spheres, satisfy the assumptions (i) and (ii) above. We refer the interested reader to [1], [6], [10], [16] and [20] for additional numerical and mathematical references that use this class of models.

The nature of the interactions modeled by Q^+ is encoded in the kernel $B(|u|,\hat{u}\cdot\omega)$ modeled by strength of intramolecular potentials, and many physical models accept the representation (henceforth assumed)

$$B(|u|,\hat{u}\cdot\omega) = |u|^{\lambda} b(\hat{u}\cdot\omega) \quad \text{with} \quad -n<\lambda.$$

Depending on the parameter λ the interaction receives different names: soft-potentials when $-n<\lambda<0$, meaning that larger relative velocity corresponds to a weaker collision frequency; Maxwell molecules type of interactions when $\lambda = 0$, of collision frequency independent of the relative velocity; variable hard-potentials when $\lambda > 0$, meaning that larger relative velocity corresponds to stronger collision frequency. The (nonnegative) angular cross section part of the collision kernel $b(\hat{u}\cdot\omega)$ is required to satisfy integrability with respect to the unit direction σ in the $n-1$dimensional sphere, where σ has the direction of the conservative post collisional velocity. This condition is called the *Grad cut-off assumption*.

$$\int_{S^{n-1}} b(\hat{u}\cdot\omega)\mathrm{d}\omega < \infty.$$

We refer to [10] and [14] for a detailed discussion on the inelastic collision operator.

1.2. **Description of the results.** In [2], Alonso and Carneiro present the L^p-analysis of the operator Q^+ in the elastic case (restitution coefficient $e\equiv 1$) for the case of Maxwell type of interactions and variable hard potentials (i.e. $0\leq\lambda\leq 1$). It is the purpose of this paper to extend the results of [2] to the more general setting of inelastic interactions, as well as to soft potentials and the case corresponding to Maxwellian weighted estimates for variable hard potentials.

Let ψ and ϕ be bounded and continuous functions. Define the bilinear operator

$$\mathcal{P}(\psi,\phi)(u) := \int_{S^{n-1}} \psi(u^-)\phi(u^+)b(\hat{u}\cdot\omega)\,\mathrm{d}\omega\,, \tag{1.5}$$

where the symbols u^+ and u^-, commonly known as Bobylev's variables, are defined by

$$u^- := \tfrac{\beta}{2}(u-|u|\omega) \quad \text{and} \quad u^+ := u-u^- = (1-\beta)u + \tfrac{\beta}{2}(u+|u|\omega). \tag{1.6}$$

The operator (1.5) was first introduced by A. V. Bobylev in a slightly different setting. Indeed, in [7] and [8] he shows that in the elastic Maxwell molecules case (i.e. $\lambda = 0$ and $\beta \equiv 1$), we have

$$\widehat{Q^+(f,g)} = \mathcal{P}(\hat{f}, \hat{g})\,. \tag{1.7}$$

Later, it was noticed that such relation was still valid for any constant $\beta \neq 1$. In [13] one can find a complete presentation of the use of the Fourier transform in the analysis of the Boltzmann collision operator, including the explicit computation of the relation (1.7).

From equations (1.2) and (1.5) we obtain the following relation between the operators Q^+ and $\mathcal{P}$

$$\int_{\mathbb{R}^n} Q^+(f,g)(v)\psi(v)\,\mathrm{d}v = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(v)g(v-u)\mathcal{P}(\tau_v\mathcal{R}\psi, 1)(u)\;|u|^\lambda\,\mathrm{d}u\,\mathrm{d}v, \tag{1.8}$$

where τ and $\mathcal{R}$ are the translation and reflection operators

$$\tau_v\psi(x) := \psi(x-v) \quad\text{and}\quad \mathcal{R}\psi(x) := \psi(-x).$$

Representation (1.8) shows that the integrability properties of the collision operator Q^+ are closely related to those of the bilinear operator $\mathcal{P}$. A similar approach was carried out in [15] which relates the operator Q^+ to a slightly different angular averaging operator.

In Section 2 we develop the L^p-analysis of the operator $\mathcal{P}$, exploiting a symmetrization method introduced in [2] that will provide sharp constants in some of our inequalities. Generally, the constants appearing in this paper will depend on (explicit) the integral conditions on the angular collision kernel.

In Section 3 we prove a full Young's inequality for hard potentials. For this, consider the weighted Lebesgue spaces $L^p_k(\mathbb{R}^n)$ ($p \geq 1$, $k \geq 0$) defined by the norm

$$\|f\|_{L^p_k(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(v)|^p\;\left(1+|v|^{pk}\right)\mathrm{d}v\right)^{1/p}.$$

We prove the following.

Theorem 1. *Let $1 \leq p,\, q,\, r \leq \infty$ with $1/p + 1/q = 1 + 1/r$. Assume that*

$$B(|u|, \hat{u}\cdot\omega) = |u|^\lambda b(\hat{u}\cdot\omega)\,,$$

with $\lambda \geq 0$. For $\alpha \geq 0$, the bilinear operator Q^+ extends to a bounded operator from $L^p_{\alpha+\lambda}(\mathbb{R}^n) \times L^q_{\alpha+\lambda}(\mathbb{R}^n) \to L^r_\alpha(\mathbb{R}^n)$ via the estimate

$$\left\|Q^+(f,g)\right\|_{L^r_\alpha(\mathbb{R}^n)} \leq C\,\|f\|_{L^p_{\alpha+\lambda}(\mathbb{R}^n)}\,\|g\|_{L^q_{\alpha+\lambda}(\mathbb{R}^n)}. \tag{1.9}$$

The constant depends on $C = C(n,\alpha,p,q,b,\beta)$ determined in Lemma 4.

Young-type inequalities reveal the convolution nature of the operator Q^+ and were first introduced in the work of Gustafsson [17] under restrictive conditions on the angular cross section, assuming pointwise cut-off away from *zero* (grazing collisions) and π (head on collisions). These estimates also appear in the work of Mouhot and Villani [21, Theorem 2.1] under the same restrictive conditions as in the work of Gustafsson [17].

In removing the restriction of pointwise cut-off away from *zero* and π for the studies of the L^p integrability of the Q^+ operator, we point out that Gamba, Panferov and Villani [15, Lemma 4.1] studied first these estimates in the case $(p,1,p)$ with polynomial weights using only the integrability of the angular cross section. Later, Bobylev, Gamba and Panferov [10] introduced an angular averaging estimate to obtain moment decay formulas to the gain operator by getting polynomial weighted estimates in $(1,1,1)$ for variable hard potentials, in the conservative (elastic) or dissipative (inelastic) interactions case (with constant restitution coefficient) for bounded angular cross section. These estimates were recently extended, in the conservative case, to singular and integrable cross sections with a given growth rate in the singularity, and they were used in the study of the regularity and asymptotic Gaussian bounds for solutions of the space homogeneous Boltzmann equation [3] and [14]. The conditions of the growth rate for the estimates in [14] are necessary in order to obtain moment decay rates of the gain operator Q^+ with respect to the same moment of the loss operator Q^-, i.e. a better control of the constants of polynomial weighted $(1,1,1)$ estimates for the Q^+.

In Section 4 we prove a Hardy-Littlewood-Sobolev inequality for the collision operator in the case of soft potentials which corresponds to a convolution structure with a singular kernel. This is an inedited result under any condition on the angular cross section, which in the most general case for our result must be integrable in the S^{n-1} dimensional sphere.

Theorem 2. *Let $1 < p,\ q,\ r < \infty$ with $-n < \lambda < 0$ and $1/p + 1/q = 1 + \lambda/n + 1/r$. For the kernel*

$$B(|u|, \hat{u}\cdot\omega) = |u|^{\lambda}\ b(\hat{u}\cdot\omega),$$

the bilinear operator Q^+ extends to a bounded operator from $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n) \to L^r(\mathbb{R}^n)$ via the estimate

$$\left\| Q^+(f,g) \right\|_{L^r(\mathbb{R}^n)} \leq C\, \|f\|_{L^p(\mathbb{R}^n)}\, \|g\|_{L^q(\mathbb{R}^n)}. \tag{1.10}$$

Theorem 2 reinforces the convolution character of $Q^+(f,g)$, basically establishing that, in the case of soft potentials, it behaves as $f * g * |u|^{\lambda}$. The constants we obtain for the two inequalities above are explicit, but generally not sharp. Only in the cases $\alpha = \lambda = 0$, $(p,q,r) = (2,1,2)$ and $(p,q,r) = (1,2,2)$ we find the sharp constant for the Young's inequality (1.9) (see the remark after Theorem 5). In fact, the quest for the sharp forms of these inequalities in the other cases, which could be seen as analogues of the remarkable works of Beckner [5] and Lieb [18], seems inaccessible at this time.

Finally, in Section 5, we apply the Young's inequality for hard potentials to obtain estimates for the collision operator with Maxwellian weights. These weighted inequalities are important tools in the study of propagation of moments [10] and $L^1 - L^\infty$ comparison principles [14].

2. Radial Symmetrization and the operator $\mathcal{P}$

Let $G = SO(n)$ be the group of rotations of $\mathbb{R}^n$ (orthonormal transformations of determinant 1), in which we will use the variable R to designate a generic rotation. We assume that the Haar measure $\mathrm{d}\mu$ of this compact topological group is normalized so that

$$\int_G \mathrm{d}\mu(R) = 1.$$

Let $f \in L^p(\mathbb{R}^n)$, $p \geq 1$. We define the radial symmetrization $f_p^\star$ by

$$f_p^\star(x) = \left(\int_G |f(Rx)|^p\, \mathrm{d}\mu(R) \right)^{\frac{1}{p}}, \quad \text{if} \ \ 1 \leq p < \infty. \tag{2.1}$$

and

$$f^\star_\infty(x) = \text{ess sup}_{|y|=|x|}|f(y)| \tag{2.2}$$

where the essential sup in (2.2) is taken over the sphere of radius $|x|$ with respect to the surface measure over this sphere. The rearrangement $f^\star_p$ defined in (2.1)-(2.2) can be seen as an L^p-average of f over all the rotations $R \in G$ and it satisfies the following properties:

(i) $f^\star_p$ is radial.
(ii) If f is continuous (or compactly supported) then $f^\star_p$ is also continuous (or compactly supported).
(iii) If g is a radial function then $(fg)^\star_p(x) = f^\star_p(x)g(x)$.
(iv) Let $\mathrm{d}\nu$ be a rotationally invariant measure on $\mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n} |f(x)|^p \,\mathrm{d}\nu(x) = \int_{\mathbb{R}^n} |f^\star_p(x)|^p \,\mathrm{d}\nu(x).$$

In particular,

$$\|f\|_{L^p(\mathbb{R}^n)} = \|f^\star_p\|_{L^p(\mathbb{R}^n)}.$$

Our first result of this section is the following.

Lemma 3. *Let $f, g, \psi \in C_0(\mathbb{R}^n)$ and $1/p + 1/q + 1/r = 1$, with $1 \le p, q, r \le \infty$. Then*

$$\left|\int_{\mathbb{R}^n} \mathcal{P}(f,g)(u)\psi(u)\,\mathrm{d}u\right| \le \int_{\mathbb{R}^n} \mathcal{P}(f^\star_p, g^\star_q)(u)\psi^\star_r(u)\,\mathrm{d}u.$$

Proof. From (1.4), (1.5) and (1.6) we observe that for any rotation R one has

$$\mathcal{P}(f,g)(Ru) = \mathcal{P}(f\circ R, g\circ R)(u).$$

Therefore,

$$\begin{aligned}\left|\int_{\mathbb{R}^n} \mathcal{P}(f,g)(u)\,\psi(u)\,\mathrm{d}u\right| &= \left|\int_{\mathbb{R}^n} \mathcal{P}(f,g)(Ru)\,\psi(Ru)\,\mathrm{d}u\right| \\ &= \left|\int_{\mathbb{R}^n} \mathcal{P}(f\circ R, g\circ R)(u)\,\psi(Ru)\,\mathrm{d}u\right| \\ &\le \int_{\mathbb{R}^n}\int_{S^{n-1}} |f(Ru^-)|\,|g(Ru^+)|\,|\psi(Ru)| b(\hat{u}\cdot\omega)\,\mathrm{d}\omega\,\mathrm{d}u.\end{aligned} \tag{2.3}$$

Note that the left hand side of (2.3) is independent of R. Thus, an integration over the group $G = SO(n)$ leads to

$$\begin{aligned}&\left|\int_{\mathbb{R}^n} \mathcal{P}(f,g)(u)\psi(u)\,\mathrm{d}u\right| \\ &\quad\le \int_{\mathbb{R}^n}\int_{S^{n-1}} \left(\int_G |f(Ru^-)||g(Ru^+)||\psi(Ru)|\,\mathrm{d}\mu(R)\right) b(\hat{u}\cdot\omega)\,\mathrm{d}\omega\,\mathrm{d}u.\end{aligned} \tag{2.4}$$

An application of Hölder's inequality with exponents p, q and r yields

$$\int_G |f(Ru^-)|\,|g(Ru^+)|\,|\psi(Ru)|\,\mathrm{d}\mu(R) \le f^\star_p(u^-)\,g^\star_q(u^+)\,\psi^\star_r(u),$$

which together with equation (2.4) proves the lemma. □

Lemma 3 shows that L^p-estimates for the operator $\mathcal{P}$ will follow by considering radial functions. If $f : \mathbb{R}^n \to \mathbb{R}$ is radial, we define the function $\tilde{f} : \mathbb{R}^+ \to \mathbb{R}$ by

$$f(x) = \tilde{f}(|x|).$$

In addition, for any $p \geq 1$ and $\alpha \in \mathbb{R}$ we have

$$\int_{\mathbb{R}^n} f(x)^p \, |x|^\alpha \, \mathrm{d}x = \left|S^{n-1}\right| \int_0^\infty \tilde{f}(t)^p \, t^{n-1+\alpha} \, \mathrm{d}t. \tag{2.5}$$

Hence, if we define the measure ν_α on $\mathbb{R}^n$ by

$$\mathrm{d}\nu_\alpha(x) = |x|^\alpha \mathrm{d}x \, ,$$

and the measure σ_n^α on $\mathbb{R}^+$ by

$$\mathrm{d}\sigma_n^\alpha(t) = t^{n-1+\alpha} \mathrm{d}t \, ,$$

equation (2.5) translates to

$$||f||_{L^p(\mathbb{R}^n, \, \mathrm{d}\nu_\alpha)} = \left|S^{n-1}\right|^{\frac{1}{p}} \, ||\tilde{f}||_{L^p(\mathbb{R}^+, \, \mathrm{d}\sigma_n^\alpha)}. \tag{2.6}$$

In the following computation we show how the operator $\mathcal{P}$ simplifies to a 1-dimensional operator when applied to radial functions. If f and g are radial, then

$$\begin{aligned} \mathcal{P}(f,g)(u) &= \int_{S^{n-1}} \tilde{f}\left(|u^-|\right) \tilde{g}\left(|u^+|\right) \, b(\hat{u}\cdot\omega) \, \mathrm{d}\omega \\ &= \int_{S^{n-1}} \tilde{f}\big(a_1(|u|, \hat{u}\cdot\omega)\big) \tilde{g}\big(a_2(|u|, \hat{u}\cdot\omega)\big) \, b(\hat{u}\cdot\omega) \, \mathrm{d}\omega \\ &= \left|S^{n-2}\right| \int_{-1}^1 \tilde{f}\big(a_1(|u|, s)\big) \tilde{g}\big(a_2(|u|, s)\big) b(s) \, (1-s^2)^{\frac{n-3}{2}} \, \mathrm{d}s. \end{aligned} \tag{2.7}$$

The functions a_1 and a_2 are defined on $\mathbb{R}^+ \times [-1,1] \to \mathbb{R}^+$ by

$$a_1(x,s) = \beta \; x \left(\tfrac{1-s}{2}\right)^{1/2} \quad \text{and} \quad a_2(x,s) = x \left[\left(\tfrac{1+s}{2}\right) + (1-\beta)^2 \left(\tfrac{1-s}{2}\right)\right]^{1/2}. \tag{2.8}$$

We conclude from (2.7) that

$$\widetilde{\mathcal{P}(f,g)}(x) = \left|S^{n-2}\right| \int_{-1}^1 \tilde{f}\left(a_1(x,s)\right) \, \tilde{g}\left(a_2(x,s)\right) \, \mathrm{d}\xi_n^b(s) \, , \tag{2.9}$$

where the measure ξ_n^b on $[-1,1]$ is defined as

$$\mathrm{d}\xi_n^b(s) = b(s)(1-s^2)^{\frac{n-3}{2}} \, \mathrm{d}s \, .$$

In virtue of equation (2.9) we define the following bilinear operator for any two bounded and continuous functions $f, g : \mathbb{R}^+ \to \mathbb{R}$,

$$\mathcal{B}(f,g)(x) := \int_{-1}^1 f\left(a_1(x,s)\right) \, g\left(a_2(x,s)\right) \mathrm{d}\xi_n^b(s). \tag{2.10}$$

Remark. It is worth to notice that in the case of constant parameter β (which includes elastic interactions) the functions a_1 and a_2 of the variable interactions are actually functions of the form $a_1 = x\alpha_1(s)$ and $a_2 = x\alpha_2(s)$; that is a_1 and a_2 are first order homogeneity in their radial part and their angular part is a positive, bounded by unity function of the angular parametrization s. This property is a signature of compactness properties of the spectral structure associated to the bilinear form (2.10) (see [9]).

For the operator in (2.10) we have the following bound.

Lemma 4. *Let $1 \le p, q, r \le \infty$ with $1/p + 1/q = 1/r$. For $f \in L^p(\mathbb{R}^+, \mathrm{d}\sigma_n^\alpha)$ and $g \in L^q(\mathbb{R}^+, \mathrm{d}\sigma_n^\alpha)$ we have*

$$\|\mathcal{B}(f,g)\|_{L^r(\mathbb{R}^+,\,\mathrm{d}\sigma_n^\alpha)} \le C\, \|f\|_{L^p(\mathbb{R}^+,\,\mathrm{d}\sigma_n^\alpha)}\, \|g\|_{L^q(\mathbb{R}^+,\,\mathrm{d}\sigma_n^\alpha)}\,, \tag{2.11}$$

where the constant C is given in (2.15)*. In the case of constant restitution coefficient e, corresponding to a constant parameter $\beta = (1+e)/2$, one can show that*

$$C(n,\alpha,p,q,b,\beta) = \beta^{-\frac{n+\alpha}{p}} \int_{-1}^{1} \left(\tfrac{1-s}{2}\right)^{-\frac{n+\alpha}{2p}} \left[\left(\tfrac{1+s}{2}\right) + (1-\beta)^2 \left(\tfrac{1-s}{2}\right)\right]^{-\frac{n+\alpha}{2q}} \mathrm{d}\xi_n^b(s) \tag{2.12}$$

is sharp.

Proof. Using Minkowski's inequality and Hölder's inequality with exponents p/r and q/r we obtain

$$\begin{aligned}\big\|\mathcal{B}(f,g)\big\|_{L^r(\mathbb{R}^+,\,\mathrm{d}\sigma_n^\alpha)} &\le \int_{-1}^{1} \left(\int_0^\infty |f(a_1(x,s))|^r\, |g(a_2(x,s))|^r\, \mathrm{d}\sigma_n^\alpha(x)\right)^{\frac{1}{r}} \mathrm{d}\xi_n^b(s) \\ &\le \int_{-1}^{1} \left(\int_0^\infty |f(a_1(x,s))|^p\, \mathrm{d}\sigma_n^\alpha(x)\right)^{\frac{1}{p}} \left(\int_0^\infty |g(a_2(x,s))|^q\, \mathrm{d}\sigma_n^\alpha(x)\right)^{\frac{1}{q}} \mathrm{d}\xi_n^b(s).\end{aligned}$$

Since the function $z \to z e(z)$ is non-decreasing, the change of variables $y = a_1(x,s)$ is valid for any fixed $s \in [-1,1)$, and its inverse Jacobian satisfies

$$\left|\frac{da_1}{dx}\right| \ge \frac{1}{2}\left(\tfrac{1-s}{2}\right)^{\frac{1}{2}}. \tag{2.13}$$

Moreover, using the fact that $\beta \ge 1/2$, we arrive at

$$\left(\int_0^\infty |f(a_1(x,s))|^p\, \mathrm{d}\sigma_n^\alpha(x)\right)^{\frac{1}{p}} \le 2^{\frac{n+\alpha}{p}} \left(\tfrac{1-s}{2}\right)^{-\frac{n+\alpha}{2p}} \|f\|_{L^p(\mathbb{R}^+,\,\mathrm{d}\sigma_n^\alpha)}\,.$$

Using a similar analysis for the change of variables $y = a_2(x,s)$, exploiting the fact that β is non-increasing, we obtain

$$\left|\frac{da_2}{dx}\right| \ge \left[\left(\tfrac{1+s}{2}\right) + (1-\beta_0)^2\left(\tfrac{1-s}{2}\right)\right]^{\frac{1}{2}}\,, \tag{2.14}$$

where $\beta_0 = \beta(0)$. We then arrive at

$$\left(\int_0^\infty |g(a_2(x,s))|^q\, \mathrm{d}\sigma_n^\alpha(x)\right)^{\frac{1}{q}} \le \left[\left(\tfrac{1+s}{2}\right) + (1-\beta_0)^2\left(\tfrac{1-s}{2}\right)\right]^{-\frac{n+\alpha}{2q}} \|g\|_{L^q(\mathbb{R}^+,\,\mathrm{d}\sigma_n^\alpha)}\,,$$

This gives (2.11) with constant

$$C = 2^{\frac{n+\alpha}{p}} \int_{-1}^{1} \left(\tfrac{1-s}{2}\right)^{-\frac{n+\alpha}{2p}} \left[\left(\tfrac{1+s}{2}\right) + (1-\beta_0)^2\left(\tfrac{1-s}{2}\right)\right]^{-\frac{n+\alpha}{2q}} \mathrm{d}\xi_n^b(s)\,. \tag{2.15}$$

In the case of constant β, the Jacobians (2.13) and (2.14) can be explicitly computed and the proposed change of variables leads to the constant (2.12). To prove that the constant (2.12) is the best possible in this case, one can consider the sequences $\{f_\epsilon\}$ and $\{g_\epsilon\}$ with $\epsilon > 0$ defined by

$$f_\epsilon(x) = \begin{cases} \epsilon^{1/p}\, x^{-(n+\alpha-\epsilon)/p} & \text{for } \ 0 < x < 1\,, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$g_\epsilon(x) = \begin{cases} \epsilon^{1/q}\, x^{-(n+\alpha-\epsilon)/q} & \text{for } \ 0 < x < 1\,, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly,

$$\|f_\epsilon\|_{L^p(\mathbb{R}^+,\,\mathrm{d}\sigma_n^\alpha)} = \|g_\epsilon\|_{L^q(\mathbb{R}^+,\,\mathrm{d}\sigma_n^\alpha)} = 1\,,$$

and one can check that

$$\|\mathcal{B}(f_\epsilon, g_\epsilon)\|_{L^r(\mathbb{R}^+,\,\mathrm{d}\sigma_n^\alpha)} \to C\,,$$

as $\epsilon \to 0$, where C is the constant defined in (2.12). The detailed argument is outlined in [2], in the case $\beta = 1$. □

From Lemma 3 we have

$$\|\mathcal{P}(f,g)\|_{L^r(\mathbb{R}^n,\,\mathrm{d}\nu_\alpha)} \leq \|\mathcal{P}(f_p^\star, g_q^\star)\|_{L^r(\mathbb{R}^n,\,\mathrm{d}\nu_\alpha)}\,,$$

where $1/p + 1/q = 1/r$. Using equations (2.6), (2.9) and Lemma 4 we obtain

$$\begin{aligned} \|\mathcal{P}(f_p^\star, g_q^\star)\|_{L^r(\mathbb{R}^n,\,\mathrm{d}\nu_\alpha)} &= \left|S^{n-1}\right|^{\frac{1}{r}} \ \left\|\widetilde{\mathcal{P}(f_p^\star, g_q^\star)}\right\|_{L^r(\mathbb{R}^+,\,\mathrm{d}\sigma_n^\alpha)} \\ &= \left|S^{n-1}\right|^{\frac{1}{r}} \ \left|S^{n-2}\right| \ \|\mathcal{B}(\tilde{f}_p^\star, \tilde{g}_q^\star)\|_{L^r(\mathbb{R}^+,\,\mathrm{d}\sigma_n^\alpha)} \\ &\leq C \ \left|S^{n-1}\right|^{\frac{1}{r}} \ \left|S^{n-2}\right| \ \|\tilde{f}_p^\star\|_{L^p(\mathbb{R}^+,\,\mathrm{d}\sigma_n^\alpha)} \ \|\tilde{g}_q^\star\|_{L^q(\mathbb{R}^+,\,\mathrm{d}\sigma_n^\alpha)} \\ &= C \ \left|S^{n-2}\right| \ \|f\|_{L^p(\mathbb{R}^n,\,\mathrm{d}\nu_\alpha)} \ \|g\|_{L^q(\mathbb{R}^n,\,\mathrm{d}\nu_\alpha)}\,, \end{aligned} \tag{2.16}$$

and thus we have proved the following result.

Theorem 5. *Let $1 \leq p, q, r \leq \infty$ with $1/p + 1/q = 1/r$, and $\alpha \in \mathbb{R}$. The bilinear operator $\mathcal{P}$ extends to a bounded operator from $L^p(\mathbb{R}^n, \mathrm{d}\nu_\alpha) \times L^q(\mathbb{R}^n, \mathrm{d}\nu_\alpha)$ to $L^r(\mathbb{R}^n, \mathrm{d}\nu_\alpha)$ via the estimate*

$$\|\mathcal{P}(f,g)\|_{L^r(\mathbb{R}^n,\,\mathrm{d}\nu_\alpha)} \leq C\ \|f\|_{L^p(\mathbb{R}^n,\,\mathrm{d}\nu_\alpha)}\, \|g\|_{L^q(\mathbb{R}^n,\,\mathrm{d}\nu_\alpha)}\,.$$

Moreover, in the case of constant restitution coefficient e, the constant

$$C = \left|S^{n-2}\right| \beta^{-\frac{n+\alpha}{p}} \int_{-1}^{1} \left(\tfrac{1-s}{2}\right)^{-\frac{n+\alpha}{2p}} \left[\left(\tfrac{1+s}{2}\right) + (1-\beta)^2 \left(\tfrac{1-s}{2}\right)\right]^{-\frac{n+\alpha}{2q}} \mathrm{d}\xi_n^b(s)$$

is sharp.

Remark. A simple application of Theorem 5 provides a sharp estimate for the L^2-norm in the case of Maxwell molecules and constant parameter β.

Corollary 6. *Let $f \in L^1(\mathbb{R}^n)$ and $g \in L^2(\mathbb{R}^n)$. Then*

$$\begin{aligned} \left\|Q^+(f,g)\right\|_{L^2(\mathbb{R}^n)} &= \left\|\widehat{Q^+(f,g)}\right\|_{L^2(\mathbb{R}^n)} = \left\|\mathcal{P}(\hat{f},\hat{g})\right\|_{L^2(\mathbb{R}^n)} \\ &\leq C_0\, \|\hat{f}\|_{L^\infty(\mathbb{R}^n)}\, \|\hat{g}\|_{L^2(\mathbb{R}^n)} \leq C_0\, \|f\|_{L^1(\mathbb{R}^n)}\, \|g\|_{L^2(\mathbb{R}^n)}. \end{aligned} \tag{2.17}$$

The constant is given by

$$C_0 = \left|S^{n-2}\right| \int_{-1}^{1} \left[\left(\tfrac{1+s}{2}\right) + (1-\beta)^2 \left(\tfrac{1-s}{2}\right)\right]^{-\frac{n}{4}} \mathrm{d}\xi_n^b(s)\,.$$

Similarly, for $f \in L^2(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$ we have

$$\left\|Q^+(f,g)\right\|_{L^2(\mathbb{R}^n)} \leq C_1\, \|f\|_{L^2(\mathbb{R}^n)}\, \|g\|_{L^1(\mathbb{R}^n)}, \tag{2.18}$$

where

$$C_1 = \left|S^{n-2}\right| \beta^{-\frac{n}{2}} \int_{-1}^{1} \left(\tfrac{1-s}{2}\right)^{-\frac{n}{4}} \, \mathrm{d}\xi_n^b(s).$$

Proof. The calculation of the constants C_0 and C_1 follow directly from Theorem 5. To guarantee that C_0 is indeed the sharp constant in the inequality (2.17) we need approximating sequences $\widetilde{\hat{f}}_\epsilon$ and $\widetilde{\hat{g}}_\epsilon$ slightly different from those presented in the end of the proof of Lemma 4, since we would like to impose the additional constraint $f \geq 0$ to have $\|\hat{f}\|_{L^\infty(\mathbb{R}^n)} = \|f\|_{L^1(\mathbb{R}^n)}$. Heuristically, this can be done by considering $f = \delta(x)$ the Dirac delta and so $\hat{f} \equiv 1$. In practice we should choose f_ϵ a Gaussian approximation of the identity by putting

$$\widetilde{\hat{f}}_\epsilon(x) = e^{-\pi\epsilon^2 x^2},$$

and

$$\widetilde{\hat{g}}_\epsilon(x) = \begin{cases} \epsilon^{1/2}\, x^{-(n-\epsilon)/2} & \text{for } \ 0 < x < 1\,, \\ 0 & \text{otherwise.} \end{cases}$$

A similar consideration applies to the inequality (2.18). Inequalities (2.17) and (2.18) are particular cases of the Young's inequality for Q^+ that will be treated in the next section. These are the only cases where we are able to explicitly find the sharp constant. □

3. Young's inequality for hard potentials

The goal of this section is to prove Theorem 1. First we treat the case $\alpha = \lambda = 0$. The main idea is to use the relation (1.8) that establishes a connection between the operators Q^+ and $\mathcal{P}$, and then use the knowledge from the previous section. From (1.8) we have

$$I := \int_{\mathbb{R}^n} Q^+(f,g)(v)\psi(v)\,\mathrm{d}v = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(v)g(v-u)\mathcal{P}(\tau_v\mathcal{R}\psi, 1)(u)\,\mathrm{d}u\,\mathrm{d}v. \quad (3.1)$$

The exponents p, q, r in Theorem 1 satisfy $1/p' + 1/q' + 1/r = 1$, and thus we can regroup the terms conveniently and use Hölder's inequality

$$\begin{aligned} I = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} &\left(f(v)^{\frac{p}{r}} g(v-u)^{\frac{q}{r}}\right)\left(f(v)^{\frac{p}{q'}}\mathcal{P}(\tau_v\mathcal{R}\psi,1)(u)^{\frac{r'}{q'}}\right) \\ &\left(g(v-u)^{\frac{q}{p'}}\mathcal{P}(\tau_v\mathcal{R}\psi,1)(u)^{\frac{r'}{p'}}\right)\mathrm{d}u\,\mathrm{d}v \ \leq \ I_1\, I_2\, I_3, \end{aligned} \quad (3.2)$$

where

$$I_1 := \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(v)^p g(v-u)^q\,\mathrm{d}u\,\mathrm{d}v\right)^{\frac{1}{r}}$$

$$I_2 := \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(v)^p \mathcal{P}(\tau_v\mathcal{R}\psi,1)(u)^{r'}\mathrm{d}u\,\mathrm{d}v\right)^{\frac{1}{q'}}$$

$$\begin{aligned} I_3 &:= \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} g(v-u)^q \mathcal{P}(\tau_v\mathcal{R}\psi,1)(u)^{r'}\mathrm{d}u\,\mathrm{d}v\right)^{\frac{1}{p'}} \\ &= \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} g(v)^q \mathcal{P}(1,\tau_{-v}\psi)(u)^{r'}\mathrm{d}u\,\mathrm{d}v\right)^{\frac{1}{p'}}. \end{aligned}$$

Recall that τ and $\mathcal{R}$ are unitary operators in the L^p spaces, thus, from (3.2) and Theorem 5 we obtain

$$I \leq C\, \|f\|_{L^p(\mathbb{R}^n)}\, \|g\|_{L^q(\mathbb{R}^n)}\, \|\psi\|_{L^{r'}(\mathbb{R}^n)},$$

with constant given by

$$\begin{aligned} C = \left|S^{n-2}\right| &\left(2^{\frac{n}{r'}} \int_{-1}^{1} \left(\tfrac{1-s}{2}\right)^{-\frac{n}{2r'}} \mathrm{d}\xi_n^b(s)\right)^{\frac{r'}{q'}} \\ &\left(\int_{-1}^{1} \left[\left(\tfrac{1+s}{2}\right) + (1-\beta_0)^2 \left(\tfrac{1-s}{2}\right)\right]^{-\frac{n}{2r'}} \mathrm{d}\xi_n^b(s)\right)^{\frac{r'}{p'}}, \end{aligned} \tag{3.3}$$

which concludes the proof in this case. In the case where $\alpha + \lambda > 0$, we shall use two additional inequalities. From the energy dissipation we have $|v'|^2 + |v'_*|^2 \leq |v|^2 + |v_*|^2$ and thus

$$|v'|^\alpha = |v - u^-|^\alpha \leq \left(|v|^2 + |v_*|^2\right)^{\alpha/2} \leq 2^{\alpha/2}\left(|v|^\alpha + |v-u|^\alpha\right). \tag{3.4}$$

Also, we shall use

$$|u|^\lambda \leq (|v-u| + |v|)^\lambda \leq 2^\lambda \left(|v-u|^\lambda + |v|^\lambda\right). \tag{3.5}$$

Let $\psi_\alpha(v) = \psi(v)|v|^\alpha$ and repeat the procedure above for the case $\alpha = \lambda = 0$ using (3.4) and (3.5) to obtain

$$\begin{aligned} \int_{\mathbb{R}^n} Q^+(f,g)(v)\psi_\alpha(v)\,\mathrm{d}v &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(v)g(v-u)\mathcal{P}(\tau_v\mathcal{R}\psi_\alpha,1)(u)\ |u|^\lambda\,\mathrm{d}u\,\mathrm{d}v \\ &\leq 4\ 2^{\alpha/2}\ 2^\lambda\, C\,\|f\|_{L^p_{\alpha+\lambda}(\mathbb{R}^n)}\,\|g\|_{L^q_{\alpha+\lambda}(\mathbb{R}^n)}\,\|\psi\|_{L^{r'}(\mathbb{R}^n)}. \end{aligned}$$

This proves that

$$\left\|Q^+(f,g)(v)|v|^\alpha\right\|_{L^r(\mathbb{R}^n)} \leq\ 2^{\alpha/2}\ 2^{\lambda+2}\, C\,\|f\|_{L^p_{\alpha+\lambda}(\mathbb{R}^n)}\,\|g\|_{L^q_{\alpha+\lambda}(\mathbb{R}^n)}.$$

A similar reasoning provides

$$\left\|Q^+(f,g)(v)\right\|_{L^r(\mathbb{R}^n)} \leq\ 2^{\lambda+1}\, C\,\|f\|_{L^p_{\alpha+\lambda}(\mathbb{R}^n)}\,\|g\|_{L^q_{\alpha+\lambda}(\mathbb{R}^n)},$$

and finally

$$\left\|Q^+(f,g)(v)\right\|_{L^r_\alpha(\mathbb{R}^n)} \leq 2^{1/r}\ \ 2^{\alpha/2}\ 2^{\lambda+2}\, C\,\|f\|_{L^p_{\alpha+\lambda}(\mathbb{R}^n)}\,\|g\|_{L^q_{\alpha+\lambda}(\mathbb{R}^n)}, \tag{3.6}$$

with C given in (3.3). This concludes the proof.

4. Hardy-Littlewood-Sobolev inequality for soft potentials

In this section we study the collision operator for soft potentials and prove Theorem 2. From (1.8) we have

$$\begin{aligned} I := \int_{\mathbb{R}^n} Q^+(f,g)(v)\,\psi(v)\,\mathrm{d}v &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(v)g(v-u)\mathcal{P}(\tau_v\mathcal{R}\psi,1)(u)\,|u|^\lambda\,\mathrm{d}u\,\mathrm{d}v \\ &= \int_{\mathbb{R}^n} f(v)\left(\int_{\mathbb{R}^n}\tau_v\mathcal{R}g(u)\,\mathcal{P}(\tau_v\mathcal{R}\psi,1)(u)\ |u|^\lambda\mathrm{d}u\right)\mathrm{d}v. \end{aligned} \tag{4.1}$$

Applying Hölder's inequality and then Theorem 5 to the inner integral of (4.1), with $(p,q,r) = (a,\infty,a)$, we obtain

$$\int_{\mathbb{R}^n}\tau_v\mathcal{R}g(u)\,\mathcal{P}(\tau_v\mathcal{R}\psi,1)(u)\ |u|^\lambda\mathrm{d}u \leq\ \|\mathcal{P}(\tau_v\mathcal{R}\psi,1)\|_{L^a(\mathbb{R}^n,\,\mathrm{d}\nu_\lambda)}\,\|\tau_v\mathcal{R}g\|_{L^{a'}(\mathbb{R}^n,\,\mathrm{d}\nu_\lambda)}$$

$$\leq C_1 \|\tau_v \mathcal{R}\psi\|_{L^a(\mathbb{R}^n,\,\mathrm{d}\nu_\lambda)} \|\tau_v \mathcal{R} g\|_{L^{a'}(\mathbb{R}^n,\,\mathrm{d}\nu_\lambda)}$$
$$= C_1 \Big[\big(|\psi|^a * |u|^\lambda\big)(v)\Big]^{1/a} \Big[\big(|g|^{a'} * |u|^\lambda\big)(v)\Big]^{1/a'},$$

where $1/a + 1/a' = 1$ (a to be chosen later), and the constant C_1 given by

$$C_1 = \big|S^{n-2}\big|\, 2^{\frac{n+\lambda}{a}} \int_{-1}^{1} \big(\tfrac{1-s}{2}\big)^{-\frac{n+\lambda}{2a}}\, \mathrm{d}\xi_n^b(s)\,.$$

We note that this choice of integrability exponents allowed to get rid of the integrand singularity at $s = -1$, thus, producing a uniform control with respect to the inelasticity parameter β.

Therefore we obtain

$$I \leq C_1 \int_{\mathbb{R}^n} f(v) \Big[\big(|\psi|^a * |u|^\lambda\big)(v)\Big]^{1/a} \Big[\big(|g|^{a'} * |u|^\lambda\big)(v)\Big]^{1/a'} \mathrm{d}v. \tag{4.2}$$

Applying Hölder's inequality in (4.2) with exponents $1/p + 1/b + 1/c = 1$ (b and c to be chosen later) we arrive at

$$I \leq C_1\, \|f\|_{L^p(\mathbb{R}^n)}\, \big\||\psi|^a * |u|^\lambda\big\|^{1/a}_{L^{b/a}(\mathbb{R}^n)}\, \Big\||g|^{a'} * |u|^\lambda\Big\|^{1/a'}_{L^{c/a'}(\mathbb{R}^n)} \tag{4.3}$$

We now use the classical Hardy-Littlewood-Sobolev inequality to obtain

$$\big\||\psi|^a * |u|^\lambda\big\|_{L^{b/a}(\mathbb{R}^n)} \leq C_2\, \|\psi\|^a_{L^{ad}(\mathbb{R}^n)} \tag{4.4}$$

and

$$\Big\||g|^{a'} * |u|^\lambda\Big\|_{L^{c/a'}(\mathbb{R}^n)} \leq C_3\, \|g\|^{a'}_{L^{a'e}(\mathbb{R}^n)}\,, \tag{4.5}$$

where

$$1 + \frac{a}{b} = \frac{1}{d} - \frac{\lambda}{n} \quad \text{and} \quad 1 + \frac{a'}{c} = \frac{1}{e} - \frac{\lambda}{n}\,.$$

The constants C_2 and C_3 (generally not sharp) are explicit in [19, p. 106]. Finally putting together (4.4) and (4.5) with (4.3) we arrive at

$$I \leq C_1\, C_2^{1/a}\, C_3^{1/a'} \|f\|_{L^p(\mathbb{R}^n)}\, \|g\|_{L^{a'e}(\mathbb{R}^n)}\, \|\psi\|_{L^{ad}(\mathbb{R}^n)}. \tag{4.6}$$

To conclude the proof of the theorem it would suffice to have in (4.6) the relations $a'e = q$ and $ad = r'$. Now it comes the moment to choose our variables. All the inequalities we used above will be well-posed if the following relations are satisfied

$$(*)\begin{cases} \dfrac{1}{a} + \dfrac{1}{a'} = 1, & 1 \leq a \leq \infty \\[2ex] \dfrac{1}{p} + \dfrac{1}{b} + \dfrac{1}{c} = 1, & 1 < b, c < \infty \\[2ex] 1 + \dfrac{a}{b} = \dfrac{1}{d} - \dfrac{\lambda}{n}, \quad b > a, & 1 < d < \infty \\[2ex] 1 + \dfrac{a'}{c} = \dfrac{1}{e} - \dfrac{\lambda}{n}, \quad c > a', & 1 < e < \infty \\[2ex] a'e = q \\[2ex] ad = r' \end{cases}$$

The last two equations determine d and e in terms of a. The remaining linear system (in the variables $1/a$, $1/a'$, $1/b$ and $1/c$) in undetermined because of the original relation

$$\frac{1}{p}+\frac{1}{q}=1+\frac{\lambda}{n}+\frac{1}{r}.$$

One can check that the choice

$$\frac{1}{b}=\frac{1}{r'}-\frac{1}{a}\left(1+\frac{\lambda}{n}\right)$$

and

$$\frac{1}{c}=\frac{1}{q}-\frac{1}{a'}\left(1+\frac{\lambda}{n}\right)$$

with any $1/a$ in the non-empty interval

$$\max\left\{\frac{1}{r'(2+\frac{\lambda}{n})}\,,\,1-\frac{1}{q(1+\frac{\lambda}{n})}\right\}<\frac{1}{a}<\min\left\{\frac{1}{r'(1+\frac{\lambda}{n})}\,,\,1-\frac{1}{q(2+\frac{\lambda}{n})}\right\}$$

provides a solution for $(*)$.

Remark. For a quadratic operator $Q^{+}(f,f)$, the angular cross section function b can be defined in the lower half sphere, so just integrability of b as an angular function on the sphere is enough to control the estimates.

5. Inequalities with Maxwellian weights

As an application of the ideas of Section 3, we now prove a Young type estimate for the non-symmetric Boltzmann collision operator with Maxwellian weights. These inequalities have been important in the study of propagation of moments [10] and L^1-L^∞ comparison principles [14], still for the case of inelastic collisions depending on the function β. The main contribution of this section relies in the generality of the statements and the extension to inelastic interactions.

Throughout this section we will assume that the angular kernel $b(s)$ vanishes for $s<0$ (non-symmetric assumption). For any $a>0$ define the global Maxwellian as

$$\mathcal{M}_a(v):=\exp\left(-a|v|^2\right).$$

Theorem 7. *Let $1\le p,\ q,\ r\le\infty$ with $1/p+1/q=1+1/r$. Assume that*

$$B(|u|,\hat{u}\cdot\omega)=|u|^{\lambda}\,b(\hat{u}\cdot\omega)\,,$$

with $\lambda\ge 0$. Then, for $a>0$,

$$\left\|Q^{+}(f,g)\ \mathcal{M}_a^{-1}\right\|_{L^r(\mathbb{R}^n)}\le C\,\|f\ \mathcal{M}_a^{-1}\|_{L^p_\lambda(\mathbb{R}^n)}\,\|g\ \mathcal{M}_a^{-1}\|_{L^q(\mathbb{R}^n)}. \tag{5.1}$$

Proof. Using (1.2) and (1.3) we obtain

$$\begin{aligned} I&:=\int_{\mathbb{R}^n}Q^{+}(f,g)(v)\left(\mathcal{M}_a^{-1}\psi\right)(v)\,\mathrm{d}v\\ &=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}f(v)g(v-u)\int_{S^{n-1}}\left(\mathcal{M}_a^{-1}\psi\right)(v')\,|u|^{\lambda}\,b(\hat{u}\cdot\omega)\,\mathrm{d}\omega\,\mathrm{d}u\,\mathrm{d}v. \end{aligned} \tag{5.2}$$

From the energy dissipation we have $|v'|^2+|v'_*|^2\le|v|^2+|v_*|^2$, and thus

$$\mathcal{M}_a^{-1}(v')\le\mathcal{M}_a^{-1}(v)\,\mathcal{M}_a^{-1}(v_*)\,\mathcal{M}_a(v'_*). \tag{5.3}$$

Using (5.3) in (5.2) we obtain

$$I \leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left(\mathcal{M}_a^{-1} f\right)(v) \left(\mathcal{M}_a^{-1} g\right)(v-u) \int_{S^{n-1}} \psi(v')\, \mathcal{M}_a(v'_*)\, |u|^\lambda\, b(\hat{u}\cdot\omega)\, \mathrm{d}\omega\, \mathrm{d}u\, \mathrm{d}v. \tag{5.4}$$

Recall from (2.7) and (2.8) that

$$|u^+| = a_2(|u|, \hat{u}\cdot\omega) = |u| \left[\left(\tfrac{1+\hat{u}\cdot\omega}{2}\right) + (1-\beta)^2 \left(\tfrac{1-\hat{u}\cdot\omega}{2}\right)\right]^{1/2} .$$

Since $b(\hat{u}\cdot\omega)$ vanishes for $\hat{u}\cdot\omega \leq 0$ one has that, in the support of b,

$$\left[\left(\tfrac{1+\hat{u}\cdot\omega}{2}\right) + (1-\beta)^2 \left(\tfrac{1-\hat{u}\cdot\omega}{2}\right)\right]^{1/2} \geq \tfrac{1}{\sqrt{2}}\,,$$

thus yielding

$$|u| \leq \sqrt{2}\; |u^+|.$$

Therefore,

$$\begin{aligned}
&\int_{S^{n-1}} \psi(v')\, \mathcal{M}_a(v'_*)\, |u|^\lambda\, b(\hat{u}\cdot\omega)\, \mathrm{d}\omega \\
&\qquad = \int_{S^{n-2}} \psi(v-u^-)\mathcal{M}_a(v-u^+)\, |u|^\lambda\, b(\hat{u}\cdot\omega)\, \mathrm{d}\omega \qquad (5.5)\\
&\qquad \leq 2^{\lambda/2} \int_{S^{n-2}} \psi(v-u^-)\mathcal{M}_a(v-u^+)\, \left|u^+\right|^\lambda b(\hat{u}\cdot\omega)\, \mathrm{d}\omega.
\end{aligned}$$

In addition, note that

$$\mathcal{M}_a(v-u^+)|u^+|^\lambda \leq 2^\lambda\, \mathcal{M}_a(v-u^+) \left(|v-u^+|^\lambda + |v|^\lambda\right) \leq C_{\lambda,a}\,(1+|v|^\lambda), \tag{5.6}$$

where the constant $C_{\lambda,a}$ depends on λ and a. Using (5.6) and (5.5) in expression (5.4) we arrive at

$$I \leq 2^{\lambda/2} C_{\lambda,a} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left(\mathcal{M}_a^{-1} f\right)(v)(1+|v|^\lambda) \left(\mathcal{M}_a^{-1} g\right)(v-u)\mathcal{P}(\tau_v \mathcal{R}\psi, 1)(u)\, \mathrm{d}u\, \mathrm{d}v.$$

We now have arrived at the same expression given in (3.1), with $f(v)$ changed by $\left(\mathcal{M}_a^{-1} f\right)(v)(1+|v|^\lambda)$ and $g(v)$ changed by $\left(\mathcal{M}_a^{-1} g\right)(v)$. Repeating the argument for the Young's inequality in Section 3 we will conclude that

$$\left\|Q^+(f,g)\; \mathcal{M}_a^{-1}\right\|_{L^r(\mathbb{R}^n)} \leq 2^{\lambda/2} C_{\lambda,a} 2C \left\|f\; \mathcal{M}_a^{-1}\right\|_{L^p_\lambda(\mathbb{R}^n)} \left\|g\; \mathcal{M}_a^{-1}\right\|_{L^q(\mathbb{R}^n)}, \tag{5.7}$$

with C given by (3.3). This concludes the proof. □

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