The Vlasov-Poisson-Landau System in a Periodic Box

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Abstract

The classical Vlasov-Poisson-Landau system describes dynamics of a collisional plasma interacting with its own electrostatic field as well as its grazing collisions. Such grazing collisions are modeled by the famous Landau (Fokker-Planck) collision kernel, proposed by Landau in 1936. We construct global unique solutions to such a system for initial data which have small weighted H^2 norms, but can have large $H^k (k \geq 3)$ norms with high velocity moments. Our construction is based on accumulative study on the Landau kernel in the past decade [G1] [SG1-3], with four extra ingredients to overcome the specific mathematical difficulties present in the Vlasov-Poisson-Landau system: a new exponential weight of electric potential to cancel the growth of the velocity, a new velocity weight to capture the weak velocity diffusion in the Landau kernel, a decay of the electric field to close the energy estimate, and a new bootstrap argument to control the propagation of the high moments and regularity with large amplitude.

1 Introduction

In the absence of magnetic effects, the dynamics of charged dilute particles (e.g., electrons and ions) is described by the Vlasov–Poisson-Landau system:

$$
\begin{aligned}
&\partial_t F_+ + v \cdot \nabla_x F_+ + \frac{e_+}{m_+} E \cdot \nabla_v F_+ = Q(F_+, F_+) + Q(F_-, F_+), \\
&\partial_t F_- + v \cdot \nabla_x F_- - \frac{e_-}{m_-} E \cdot \nabla_v F_- = Q(F_+, F_-) + Q(F_-, F_-), \qquad (1)\\
&F_\pm(0, x, v) = F_{0,\pm}(x, v).
\end{aligned}
$$

Here $F_\pm(t, x, v) \geq 0$ are the spatially periodic number density functions for the ions (+) and electrons (-) respectively, at time $t \geq 0$, position $x = (x_1, x_2, x_3) \in [-\pi,\pi]^3 = \mathbf{T}^3$, velocity $v = (v_1, v_2, v_3) \in \mathbf{R}^3$, and $e_\pm$, $m_\pm$ the magnitude of their charges and masses, c the speed of light. The collision between charged particles is given by

$$Q(G_1,G_2)(v)=\frac{c_{12}}{m_1}\nabla_v\cdot\int_{\mathbf{R}^3}\Phi(v-v')\left\{\frac{G_1(v')\nabla_v G_2(v)}{m_1}-\frac{G_2(v)\nabla_{v'}G_1(v')}{m_2}\right\}dv' \tag{2}$$

where Φ is the famous Landau (Fokker-Planck) kernel [G1]:

$$\Phi(u)=\frac{1}{|u|}\left(I-\frac{u\otimes u}{|u|^2}\right) \tag{3}$$

and $c_{12}=2\pi e_1^2e_2^2\ln\Lambda$, $\ln\Lambda=\ln(\frac{\lambda_D}{b_0})$, $\lambda_D=(\frac{T}{4\pi n_e e^2})^{1/2}$ being the Debye shielding distance and $b_0=\frac{e^2}{3T}$ being a typical 'distance of closest approach' for a thermal particle [H]. The self-consistent electrostatic field $E(t,x)=-\nabla\phi$, and the electric potential ϕ satisfies:

$$-\Delta\phi=4\pi\rho=4\pi\int_{\mathbf{R}^3}\{e_+F_+-e_-F_-\}dv,\quad \int_{\mathbf{T}^3}\phi(t,x)dx=0. \tag{4}$$

It is well-known that for classical solutions to the Vlasov-Poisson-Landau system, the following conservation laws of mass, total momentum, and total energy hold:

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbf{T}^3\times\mathbf{R}^3}m_\pm F_\pm(t)=0,\\
&\frac{d}{dt}\left\{\int_{\mathbf{T}^3\times\mathbf{R}^3}v(m_+F_+(t)+m_-F_-(t))\right\}=0,\\
&\frac{d}{dt}\left\{\frac{1}{2}\int_{\mathbf{T}^3\times\mathbf{R}^3}|v|^2(m_+F_+(t)+m_-F_-(t))+\frac{1}{8\pi}\int_{\mathbf{T}^3}|E(t)|^2\right\}=0.
\end{aligned}$$

Moreover, we also have the following celebrated H-Theorem of Boltzmann

$$\frac{d}{dt}\left\{\int_{\mathbf{T}^3\times\mathbf{R}^3}(F_+(t)\ln F_+(t)+F_-(t)\ln F_-(t))\right\}\le 0. \tag{5}$$

It is our purpose in this article to construct unique global solutions for the Vlasov-Poisson-Landau system (1) and (4) near global Maxwellians:

$$\mu_+(v)=\frac{n_0}{e_+}(\frac{m_+}{2\pi\kappa T_0})^{3/2}e^{-m_+|v|^2/2\kappa T_0},\quad \mu_-(v)=\frac{n_0}{e_-}(\frac{m_-}{2\pi\kappa T_0})^{3/2}e^{-m_-|v|^2/2\kappa T_0}.$$

For notational simplicity, we normalize all constants in the Vlasov-(Poisson)-Landau system to be one. Accordingly, we normalized the Maxwellian as

$$\mu(v)\equiv\mu_+(v)=\mu_-(v)=e^{-|v|^2}. \tag{6}$$

We define the standard perturbation $f_\pm(t,x,v)$ to μ as

$$F_\pm=\mu+\sqrt{\mu}f_\pm. \tag{7}$$

Let $f(t,x,v)=\binom{f_+(t,x,v)}{f_-(t,x,v)}$, the Vlasov-Poisson-Landau system for the perturbation now takes the form

$$\{\partial_t+v\cdot\nabla_x\pm E\cdot\nabla_v\}f_\pm\mp 2\{E\cdot v\}\sqrt{\mu}+L_\pm f \quad = \quad \pm\{E\cdot v\}f_\pm+\Gamma_\pm(f,f) \tag{8}$$

$$-\Delta\phi \quad = \quad \int\sqrt{u}[f_+-f_-]dv \tag{9}$$

with $\int_{\mathbf{T}^3}\phi dx=0$. For any $g=\binom{g_1}{g_2}$, the linearized collision operator Lg in (8) is given by the vector

$$Lg\equiv\begin{pmatrix}L_+g\\ L_-g\end{pmatrix}\equiv-\frac{1}{\sqrt{\mu}}\begin{pmatrix}2Q(\mu,\sqrt{\mu}g_1)+Q(\sqrt{\mu}\{g_1+g_2\},\mu)\\ 2Q(\mu,\sqrt{\mu}g_2)+Q(\sqrt{\mu}\{g_1+g_2\},\mu)\end{pmatrix}. \tag{10}$$

For $g=[g_1,g_2]$ and $h=[h_1,h_2]$, the nonlinear collision operator $\Gamma(g,h)$ is given by the vector

$$\Gamma(g,h)\equiv\begin{pmatrix}\Gamma_+(g,h)\\ \Gamma_-(g,h)\end{pmatrix}\equiv\frac{1}{\sqrt{\mu}}\begin{pmatrix}Q(\sqrt{\mu}g_1,\sqrt{\mu}h_1)+Q(\sqrt{\mu}g_2,\sqrt{\mu}h_1)\\ Q(\sqrt{\mu}g_1,\sqrt{\mu}h_2)+Q(\sqrt{\mu}g_2,\sqrt{\mu}h_2)\end{pmatrix}. \tag{11}$$

By assuming that initially $F_{0\pm}$ has the same mass, total momentum and total energy as the steady state μ, we can then rewrite the conservation laws in terms of the perturbation f as

$$\int_{\mathbf{T}^3\times\mathbf{R}^3}f_+(t)\sqrt{\mu}\equiv\int_{\mathbf{T}^3\times\mathbf{R}^3}f_-(t)\sqrt{\mu}\equiv 0, \tag{12}$$

$$\int_{\mathbf{T}^3\times\mathbf{R}^3}v\{f_+(t)+f_-(t)\}\sqrt{\mu}\equiv 0, \tag{13}$$

$$\int_{\mathbf{T}^3\times\mathbf{R}^3}|v|^2\{f_+(t)+f_-(t)\}\sqrt{\mu}\equiv-\int_{\mathbf{T}^3}|E(t)|^2, \tag{14}$$

In an attempt [G1-2] to construct global smooth solution near Maxwellian for the Vlasov-Poisson-Landau system, the author initiated a nonlinear energy method for a general dissipative problem:

$$\partial_t g+\mathcal{L}g=\mathcal{N}(g). \tag{15}$$

We denote $||\cdot||_2$ to be the L^2 norm. Upon taking L^2 inner product with g, we obtain

$$\frac{1}{2}\frac{d}{dt}||g||_2^2+(\mathcal{L}g,g)=(\mathcal{N}(g),g). \tag{16}$$

Global solutions with small L^2 norm can be constructed if one can identify a dissipation rate $|||\cdot|||$ such that the following estimates can be established

$$(\mathcal{L}g,h) \quad \gtrsim \quad |||g|||^2, \tag{17}$$

$$(\mathcal{N}(g),g) \quad \lesssim \quad ||g||_2\cdot|||g|||^2. \tag{18}$$

For $||h||_2 << 1$ we obtain

$$\frac{1}{2}\frac{d}{dt}||h||_2^2 + \delta|||h|||^2 \leq 0,$$

with some $\delta > 0$. This implies global uniform bound for $||g||_2$ and hence stability. Several remarks are needed: (1) Usually higher order Sobolev norms are needed to close the argument. (2) Such a dissipation rate $|||\cdot|||$ satisfying both (17) and (18), if exists, usually is unique up to a constant. The proof of (17) and (18) can be very challenging [G1] [GrS1-2]. (3) Such an approach is designed to treat cases when $|||\cdot|||$ is weaker, or it can not be compared with $||\cdot||_2$ so that standard spectra analysis and semigroup approach is difficult to apply. (4) In many interesting applications, such $(\mathcal{L}g, g)$ can not control full $|||g|||^2$, and one has to control the missing part via further study of the nonlinear equation (15). In particular, in the context of Landau or Boltzmann equations, the linear collision operator L as in (10) can be shown to be positive definite ([G1], [SG1-3]) along the nonlinear dynamics even it has a kernel.

Such a flexible approach turns out to be robust, leading to constructions of global solutions to several different applied PDE [G1-5], [GS], [GrS1-2], [SG1-3], [GT], [Ha]. Unfortunately, despite these advances and many attempts, the original motivation in this program, the stability of Maxwellian for the Vlasov-Poisson-Landau system has remained out of reach, as pointed out in [SG3]. For other work related to the Landau equation from different approaches, see [AB], [AV], [CDL], [HY], [L], [V], and [Z1-2] among others. Let $w(v) \geq 1$ be a weight function and $||\cdot||_{2,w}$ to denote the weighted L^2 norm. We define

$$||f||_{\sigma,w} \equiv ||f||_{H^\sigma_w} \equiv ||\langle v\rangle^{-\frac{1}{2}} f||_{2,w} + ||\langle v\rangle^{-\frac{3}{2}} \nabla_v f \cdot \frac{v}{|v|}||_{2,w} + ||\langle v\rangle^{-\frac{1}{2}} \nabla_v f \times \frac{v}{|v|}||_{2,w} \tag{19}$$

with $\langle v\rangle = \sqrt{1+|v|^2}$. It is well-known that $||f||_\sigma \equiv ||f||_{\sigma,1}$ captures the dissipation for the Landau kernel [DL], [G1]. There are two intrinsic difficulties [G1] associated with the Landau kernel: the vanishing factors $\langle v\rangle^{-\frac{1}{2}}$ and $\langle v\rangle^{-\frac{3}{2}}$ make $||f||_\sigma$ 'soft', and their different vanishing rates along different directions makes $||f||_\sigma$ non-isotropic for $\nabla_v f$.

There are two major mathematical difficulties in the study of stability in Vlasov-Poisson-Landau system. The first difficulty is created by the (innocent looking!) nonlinear term $E \cdot v f_\pm$ in (8). Such a term comes from the factor $\sqrt{\mu}$ in our linearization $F_\pm = \mu + \sqrt{\mu} f_\pm$, which is the only known choice so far capturing the linear dissipation rate of L, a linear analog to the entropy production in the fundamental Boltzmann's H-theorem. Hence, the presence of the term $E \cdot v f_\pm$ is a basic feature of interaction between the electric field and the particles in the near Maxwellian context. Upon multiplying $f_\pm$ one hopes to bound

$$\int |vEf^2| \tag{20}$$

in terms of $||f||_2 \cdot ||f||_\sigma^2$ (19). The electric field $E = E_f$ behaves nicely, but the extra velocity factor v makes the control by $||f||_\sigma^2$ (which only controls

$||\langle v\rangle^{-1/2}f||_2^2$ in (19)) impossible. In a simpler model problem [G2], the Landau collision is replaced by the hard-sphere interaction so that $\int|v|f^2$ can be exactly bounded by the dissipation rate of the hard-sphere kernel. In a relativistic Landau collision kernel [SG3] [DL], the relativistic counterpart is $\int|E\cdot\hat{p}f^2|$, where the velocity $\hat{p}$ is bounded, and $||f||_2^2$ is controlled by the relativistic Landau dissipation. These two facts led to the resolution for the relativistic Landau problem [SG3]. As a matter of fact, this extra v factor is the key reason that only hard-sphere like interaction can be treated for the Boltzmann type of equations in the presence if a force term. With even a bit weaker (softer) than hard-sphere interaction, (20) is beyond the control of either the energy or dissipation rate so that even the local in-time solutions can not be constructed within this framework.

The second main difficulty stems from controlling the velocity v derivative of f. Taking v derivative of (8), we estimate $||\nabla_v f||_2^2$ via the standard energy method. In this process, the free streaming term $v\cdot\nabla_x f$ produces a quadratic term as:

$$\int\nabla_x f\cdot\nabla_v f. \tag{21}$$

It is well-known that $\nabla_v f$ can produce growth in time in the kinetic theory. To estimate (21) by the norm (19) is tricky, again due to the negative weight. In the absence of the electric field E [G1], the author designed a weighted norm to overcome this difficulty with more v derivatives associated with more negative velocity weights. The first step was to take pure x derivative (with no weight and no electric field E) and to bound

$$\int_0^t||\nabla_x f||_\sigma^2.$$

In the next step, $||\langle v\rangle^{-1}\nabla_v f||_2^2$ (instead of $||\nabla_v f||_2^2$) was estimated in the $v-$derivative of (8), which contains a dissipation rate of $||\nabla_v f||_{\sigma,\langle v\rangle^{-1}}^2$. The weighted mixed terms (21) could be bounded by (see (19)):

$$\int\langle v\rangle^{-2}|\nabla_x f\cdot\nabla_v f|\leq C_\varepsilon\int_0^t||\nabla_x f||_\sigma^2+\varepsilon\int_0^t||\nabla_v f||_{\sigma,\langle v\rangle^{-1}}^2,$$

which could be closed for ε small. Such a weight disparity between x and v derivatives of f has played an important role in treating soft potentials [G3][GrS1-2]. Unfortunately, in the presence of E, this strategy fails completely because $\int_0^t||\nabla_x f||_\sigma^2$ can not be estimated independently of $\nabla_v f$ at the first step. In fact, taking $x-$derivatives of (8) produces new contribution

$$\int E\nabla_x f\cdot\nabla_v f \tag{22}$$

Now this can not be controlled with a norm for $\nabla_v f$ with a negative weight $\langle v\rangle^{-1}$.

The key to overcome the first difficulty (20) is to realize that, instead of treating $E\cdot vf_\pm$ as a second order perturbation in (8), we need to combine or cancel it with the linear term streaming term $v\cdot\nabla_x f$ in (8), which also contains an extra v factor! Upon using the fact that $E=-\nabla_x\phi$ in this problem, upon multiplying with $e^{\pm\phi}$, we can rewrite

$$e^{\pm\phi}[v\cdot\nabla_x f_\pm \pm \nabla_x\phi\cdot v f_\pm] = v\cdot\nabla_x\{e^{\pm\phi}f_\pm\}.$$

Indeed, such a perfect derivative leads no contribution in the integration (see (74)). Even though such a spatial weight destroys the exact energy structure from the Poisson equation, fortunately, new error contributions are of the type $\int \nabla_x\phi\cdot v\sqrt{\mu}f_\pm(e^{\pm 2\phi}-1)$ which can be controlled if ϕ is small. Our observation works for all forces given by a potential.

To overcome the second main difficulty and to control the $v-$derivative of f, we need to design new weight function in v. In light of (22), we need to assign same weight functions for both $\nabla_x f$ and $\nabla_v f$, which seems to contradict to the weight disparity for controlling (21). We observe crucially that $||f||_\sigma$ contains (weak) $v-$derivative of f. Hence $\nabla_v f$ can be also viewed, not as a part of $||\nabla_v f||_\sigma$, but as a part of $||f\ ||_{\sigma,\langle v\rangle^2}$ with no $v-$derivative but with an extra stronger weight $\langle v\rangle^2$. In fact, thanks to (19), we can estimated (21) by

$$\int \nabla_x f\cdot\nabla_v f \leq ||\nabla_x f||_\sigma ||f||_{\sigma,\langle v\rangle^2}$$

provided $||f||_{\sigma,\langle v\rangle^2}$ is controlled at an earlier step. The weight disparity is not between x and v derivatives, but between $||f||_{\sigma,\langle v\rangle^2}$ and $||\nabla_x f||_\sigma$. That is, less derivatives of f should requires stronger weight. Since higher spatial derivatives are associated with weaker velocity weight in our norm (26), more careful analysis is needed for spatial Sobolev imbedding to close the energy estimate, especially when we take one derivative (Step 2 in the proof of Proposition 6). Such a cascade of weight takes advantage of the crucial feature of the Landau operator: a weak gain of ∇_v. Because of this reason, our new strategies do not work for a soft potential with Grad's angular cutoff, even if it just a bit weaker than the hard-sphere interaction. On the other hand, it is interesting to use such a weight for the full inverse power law without angular cutoff [GrS1-2].

For notational simplicity, we use $||\cdot||_p$ to denote L^p norms with weight $w(v)$ in $\mathbf{T}^3\times\mathbf{R}^3$ or $\mathbf{T}^3$, and $||\cdot||_{p,w}$ for L^p norms with weight $w(v)$. Let the multi-indices α and β be $\alpha=[\alpha_1,\alpha_2,\alpha_3]$, $\quad\beta=[\beta_1,\beta_2,\beta_3]$, and we define $\partial^\alpha_\beta \equiv \partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\alpha_3}_{x_3}\partial^{\beta_1}_{v_1}\partial^{\beta_2}_{v_2}\partial^{\beta_3}_{v_3}$. If each component of θ is not greater than that of $\bar\theta$'s, we denote by $\theta\leq\bar\theta$; $\theta<\bar\theta$ means $\theta\leq\bar\theta$, and $|\theta|<|\bar\theta|$. We define the velocity weight

$$w(\alpha,\beta)(v)\equiv e^{\frac{q|v|^2}{2}}\langle v\rangle^{2(l-|\alpha|-|\beta|)},\qquad l\geq|\alpha|+|\beta|,\quad 0\leq q<1. \tag{23}$$

The presence of $e^{\frac{q|v|^2}{2}}$ would lead to stretched exponential decay. Recall (19).

Let the instant energy and dissipation rate are:

$$\mathcal{E}_{m;l,q}(f)(t) \equiv \sum_{|\alpha|+|\beta|\leq m} \sum_{\pm} ||\partial^{\alpha}_{\beta} f_{\pm}(t)||_{2,w(\alpha,\beta)}, \tag{24}$$

$$\mathcal{D}_{m;l,q}(f)(t) \equiv \sum_{|\alpha|+|\beta|\leq m} \sum_{\pm} ||\partial^{\alpha}_{\beta} f_{\pm}(t)||_{\sigma,w(\alpha,\beta)}, \tag{25}$$

We remark that from the definition,

$$\begin{aligned} \mathcal{E}_{0;l+m,q}(f) &\subset \mathcal{E}_{1;l+m-1,q}(f) \subset ... \subset \mathcal{E}_{m;l,q}(f), \\ \mathcal{D}_{0;l+m,q}(f) &\subset \mathcal{D}_{1;l+m-1,q}(f) \subset ... \subset \mathcal{D}_{m;l,q}(f), \end{aligned} \tag{26}$$

so that less derivatives of f demands stronger velocity weight. It is important to note that due to the presence of (20), all our estimates can only be obtained with (nonlinear) exponential weight of $e^{\pm(1+q)\phi_f}$ (see Eqs (76) to (86) and Lemma 14). But if the electric potential $||\phi_f||_\infty$ is bounded (as we shall prove), such weighted norms are equivalent to $\mathcal{E}_{m;l,q}(f)$ and $\mathcal{D}_{m;l,q}(f)$, and we need to use $\mathcal{E}_{m;l,q}(f)$ and $\mathcal{D}_{m;l,q}(f)$ without the nonlinear weight to close our continuity argument. Our main result is as follows.

Theorem 1 *Assume that f_0 satisfies the conservation laws (12), (13), (14) with $F_{0,\pm}(x,v) = \mu + \sqrt{\mu} f_{0,\pm}(x,v) \geq 0$. There exists a sufficiently small $M > 0$ such that if*

$$\mathcal{E}_{2;2,0}(f_0) \leq M,$$

then there exists a unique global solution $f(t,x,v)$ to the Vlasov-Poisson-Laudau system (8) and (9) with $F_{\pm}(t,x,v) = \mu + \sqrt{\mu} f_{\pm}(t,x,v) \geq 0$.

(1) If $\mathcal{E}_{2;l,q}(f_0) < +\infty$ for $l \geq 2$ and $q \geq 0$, then there exists $C_l > 0$,

$$\sup_{0\leq s\leq\infty} \mathcal{E}_{2;l,q}(f(s)) + \int_0^\infty \mathcal{D}_{2;l,q}(f(s))ds \leq C_l \mathcal{E}_{2;l,q}(f_0). \tag{27}$$

Furthermore,

$$||\partial_t\phi(t)||_\infty + ||\nabla_x\phi(t)||_\infty + ||f(t)||_2 \leq C_l(1+t)^{-2l+2}\mathcal{E}_{2;l,0}(f_0), \tag{28}$$

$$||\partial_t\phi(t)||_\infty + ||\nabla_x\phi(t)||_\infty + ||f(t)||_2 \leq C_l e^{-C_l t^{2/3}} \mathcal{E}_{2;l,q}(f_0) \quad \text{for } q > 0 \tag{29}$$

(2) In addition, if $\mathcal{E}_{m;l,q}(f_0) < \infty$ for any $l > 2$, $l \geq m \geq 2$, $q \geq 0$, there exists an increasing continuous function $P_{m,l}(\cdot)$ with $P_{m,l}(0) = 0$ such that the unique solution satisfies

$$\sup_{0\leq t\leq\infty} \mathcal{E}_{m;l,q}(f(t)) + \int_0^\infty \mathcal{D}_{m;l,q}(f(s))ds \leq P_{m,l}(\mathcal{E}_{m;l,q}(f_0)). \tag{30}$$

The continuous function $P_{m,l}(\cdot)$ can be determined inductively on m. We remark that we only require $\mathcal{E}_{2;2,0}$ (f_0) to be small, but for $l > 2$ and $m \geq 2$, the high momentum or high Sobolev norm $\mathcal{E}_{m;l,q}(f_0)$ can be arbitrarily large. A H^2

type of construction was first carried out in [GrS1-2] for the Boltzmann equation with an non-cutoff inverse power collision kernel. Note from the Sobolev's imbedding, L^∞ is not necessarily bounded by $\mathcal{E}_{2;2,0}\ (f_0)$. We also note that the estimate (30) is uniform in time, so that we verify the bounds in [DV] and more decay can be obtained (see also [SG1-2]). Such an estimate (30) also gives a natural approximation mechanism to establish the gain of smoothness for $t > 0$ [CDH].

Throughout the paper, we introduce the notation $A \lesssim B$ ($A \gtrsim B$ and $A \backsim B$) if A is bounded by B up to a universal constant C which *does not* depend on either l or m.

The introduction of the weight spatial weight function $e^{\pm(1+q)\phi}$ creates a new analytical difficulty: we need to control (see Eqs. (115) and (116))

$$\int_0^\infty [||\partial_t \phi(s)||_\infty + ||\nabla_x \phi(s)||_\infty] ds \tag{31}$$

to close the global energy estimate for $\mathcal{E}_{2;2,0}\ (f)$. Note that (31) is different from the dissipation estimate $\int_0^\infty \mathcal{D}_{2;2,0}(f(s))ds < \infty$ since

$$||\partial_t \phi(s)||_\infty + ||\nabla_x \phi(s)||_\infty \lesssim \sqrt{\mathcal{D}_{2;2,0}(f(s))},$$

and $\mathcal{D}_{2;2,0}(f(s))$ is expected to be small most of the time. It is a typical difficulty that $\int_0^\infty \sqrt{\mathcal{D}_{2;2,0}(f(s))}ds < \infty$ can not be derived directly from the energy-dissipation estimate (27). We need to make an interplay (see (143)) between the decay estimate of (28) (with $l = 2, q = 0$) and the energy estimate (27). This is different from the program in [SG1-2], in which the energy estimate can be close alone first and the decay is obtained after. In fact, the proof of (28) is intertwined with (27) with $l = 2$, and we are able to close a differential inequality (142) with up to only one spatial derivative. This leads to a decay rate of

$$||\partial_t \phi(s)||_\infty + ||\nabla_x \phi(s)|| \backsim \frac{1}{s^2}$$

in terms of (27) with $l = 2$, which is sufficient to close the estimate. The strong decay rate of $\frac{1}{s^2}$ is a consequence of the periodic box $\mathbf{T}^3$ and it remains an open question if sufficient decay rate can be obtained for the Vlasov-Poisson-Landau system in the whole space $\mathbf{R}^3$.

The last novelty of the paper is the proof of high moments and high regularity (30) with only small $\mathcal{E}_{2;2,0}(f_0)$. It is important to note, that we require $l \geq m$, the total number of derivatives in $w(\alpha, \beta)$ (23). This is because that the starting point of our method is to control pure $x-$derivatives of f without any weight, which demands that $l \geq m$ at the highest level of derivatives. It is also important for our analysis to require $w(\alpha, \beta) \geq 1$. This implies that $||f||_{2,l} \leq \mathcal{E}_{m;l,q}(f_0)$, which can not be small for all l! Therefore it is natural to assume $\mathcal{E}_{m;l,q}(f_0)$ is only finite but not small for $l > 2$. The hope is to obtain, at the highest order derivatives,

$$\int |w^2(\alpha,\beta)\Gamma(\partial_\beta^\alpha f, f)\partial_\beta^\alpha f| + \int |w^2(\alpha,\beta)\Gamma(f, \partial_\beta^\alpha f)\partial_\beta^\alpha f| \tag{32}$$
$$\lesssim \sqrt{\mathcal{E}_{2;2,0}(f)}||\partial_\beta^\alpha f(t)||^2_{\sigma,w(\alpha,\beta)},$$

so that the smallness of $\mathcal{E}_{2;2,0}(f_0)$ would be sufficient. Unfortunately, due to the combination of the non-local feature of the Landau operator as well as the velocity weight $w(\alpha,\beta)$, as in Lemma 10 [SG2], a term like

$$\sqrt{\mathcal{E}_{2;l,q}(f)}||\partial_\beta^\alpha f(t)||_\sigma ||\partial_\beta^\alpha f(t)||_{\sigma,w(\alpha,\beta)}$$

will occur in the upper bound for (32). Since $\sqrt{\mathcal{E}_{2;l,q}(f)}$ is large for $l > 2$, we can not absorb the whole product by the dissipation rate of $||\partial_\beta^\alpha f(t)||^2_{\sigma,w(\alpha,\beta)}$. In Proposition 6, we are able to move the weight function from $\sqrt{\mathcal{E}_{2;l,q}(f)}$ to $||\partial_\beta^\alpha f(t)||_\sigma$ and the price to pay is an additional contribution of

$$C_l\mathcal{D}_{2;l,q}(f)||\partial_\beta^\alpha f(t)||^2_{2,w(\alpha,\beta)}.$$

Even though $C_l\mathcal{D}_{2;l,q}(f)$ can be very large, but $||\partial_\beta^\alpha f(t)||^2_{2,w(\alpha,\beta)}$ belongs to the instant energy not the dissipation rate, therefore we can still control this via the Gronwall lemma and the fact $\int_0^t \mathcal{D}_{2;l,q}(f)(s)ds < \infty$. A new splitting of the domain needs to designed to achieve this goal. We believe that our method will lead to new estimates for solutions recently constructed in [GrS1-2] for the inverse power law. Moreover, the presence of E also makes the construction of the local solutions more delicate and we need to employ fractional Sobolev spaces to gain compactness of the approximate solutions.

Our paper is organized as follows. In section 2, new refined estimates are developed to cope with nonlinear terms in (8). In section 3, local in time solutions are constructed via estimates of the pseudo energy and dissipation rate (24) and (25). In section 4, decay estimates (28) and (29) are obtained to bootstrap into global in time solutions.

2 Basic Estimates

We use $\langle \cdot,\cdot \rangle$ to denote the standard L^2 inner product in $\mathbf{R}_v^3$ for a pair of functions $g = \binom{g_+}{g_-}$ and $h = \binom{h_+}{h_-}$ and define:

$$\langle f, h \rangle = \langle f_+, h_+ \rangle + \langle f_-, h_- \rangle. \tag{33}$$

We also denote $|\cdot|_{2,w}$ and $|\cdot|_{\sigma,w}$ to be corresponding $w-$weighted L^2 and H^σ norms (19) in $\mathbf{R}_v^3$.

Recall the linear Landau operator L in (10). We first recall a basic property of L (see [H1] [G1] [SG3] for a proof).

Lemma 2 *We have $\langle Lg,h\rangle=\langle Lh,g\rangle$, $\langle Lg,g\rangle\geq 0$, and $Lg=0$ if and only if $g=Pg$ where P being the $L^2_v(\mathbf{R}^3)$ projection with respect to the vector L^2 inner product (33) onto the null space of L :*

$$\mathrm{span}\left\{\sqrt{\mu}\begin{pmatrix}1\\0\end{pmatrix},\ \sqrt{\mu}\begin{pmatrix}0\\1\end{pmatrix},\ v_i\sqrt{\mu}\begin{pmatrix}1\\1\end{pmatrix},\ |v|^2\sqrt{\mu}\begin{pmatrix}1\\1\end{pmatrix}\right\}, \tag{34}$$

with $1\leq i\leq 3$. Moreover

$$\langle Lg,g\rangle\gtrsim|\{I-P\}g|^2_\sigma\equiv\sum_{\pm}|[\{I-P\}g]_\pm|^2_\sigma \tag{35}$$

where $[\{I-P\}g]_\pm$ are two components of $\{I-P\}g$.

Lemma 3 *Let $w(v)\geq 0$. Then for any $\eta>0$*

$$\begin{aligned}\sup_x|g(x,\cdot)|_w &\lesssim \||g|_{2,w}\|_{H^{\frac{7}{4}}}\lesssim\eta\sum_{|\gamma|=2}||\partial^\gamma g||_{2,w}+C_\eta||g||_{2,w},\\ \sup_x|g(x,\cdot)|_{\sigma,w} &\lesssim \||g|_{\sigma,w}\|_{H^{\frac{7}{4}}}\lesssim\eta\sum_{|\gamma|=2}||\partial^\gamma g||_{\sigma,w}+C_\eta||g||_{\sigma,w},\\ ||g||_{L^4_x(L^2_w)} &\lesssim \||g|_{2,w}\|_{H^{\frac{3}{4}}}\lesssim\eta\sum_{|\gamma|=1}||\partial^\gamma g||_{2,w}+C_\eta||g||_{2,w},\\ ||g||_{L^4(H^\sigma_w)} &\lesssim \||g|_{\sigma,w}\|_{H^{\frac{3}{4}}}\lesssim\eta\sum_{|\gamma|=1}||\partial^\gamma g||_{\sigma,w}+C_\eta||g||_{\sigma,w}.\end{aligned}$$

We note that $L^\infty(\mathbf{T}^3)\subset H^{\frac{7}{4}}(\mathbf{T}^3)$, $L^4(\mathbf{T}^3)\subset H^{\frac{3}{4}}(\mathbf{T}^3)$. Moreover, $H^2\subset H^{\frac{7}{4}}$, and $H^1\subset H^{\frac{3}{4}}$ compactly, which gives rise to the arbitrary small constant η. The Lemma then follows from general functional Sobolev inequality of H^k, see [GrS1-2].

We also need the following version of the Gronwall Lemma.

Lemma 4 *Let $A(t),B(t),y(t)\geq 0$ satisfy $y(t)\leq\int_0^t A(s)y(s)ds+B(t)$, then*

$$y(t)\leq e^{\int_0^t A(s)ds}\int_0^t A(s)B(s)ds+B(t).$$

The following lemma is the key to treat the streaming term $\nabla_v f\cdot\nabla_x f$ via our dissipation rate (25). Denote

$$\delta^{\mathbf{e}_i}_\beta=1\ \text{ if }\quad \mathbf{e}_i\leq\beta;\quad\text{or}\quad\delta^{\mathbf{e}_i}_\beta=0,\ \text{ otherwise.} \tag{36}$$

Lemma 5 *We have*

$$\int w^2(\alpha,\beta)\delta^{\mathbf{e}_i}_\beta\partial^{\alpha+\mathbf{e}_i}_{\beta-\mathbf{e}_i}f\partial^\alpha_\beta f\lesssim||\delta^{\mathbf{e}_i}_\beta\partial^\alpha_{\beta-\mathbf{e}_i}f||_{\sigma,w(\alpha,\beta-\mathbf{e}_i)}||\partial^{\alpha+\mathbf{e}_i}_{\beta-\mathbf{e}_i}f_\pm||_{\sigma,w(\alpha+\mathbf{e}_i,\beta-\mathbf{e}_i)}.$$

Proof. Recalling (23), we note $w(\alpha+\mathbf{e}_i,\beta-\mathbf{e}_i)(v)=w(\alpha,\beta)(v)$ and $w(\alpha,\beta-\mathbf{e}_i)(v)=\langle v\rangle^{-2}w(\alpha,\beta)(v)$. We rewrite from (36) $\partial^\alpha_\beta=\partial_{\mathbf{e}_i}\partial^\alpha_{\beta-\mathbf{e}_i}\delta^{\mathbf{e}_i}_\beta$ and we use (19) for $\partial^{\alpha+\mathbf{e}_i}_{\beta-\mathbf{e}_i}f_\pm$ to get

$$\begin{aligned}
&\int w^2(\alpha,\beta)\delta^{\mathbf{e}_i}_\beta\partial^{\alpha+\mathbf{e}_i}_{\beta-\mathbf{e}_i}f\partial^\alpha_\beta f\\
\le&\int|w(\alpha,\beta)\langle v\rangle^{1/2}\delta^{\mathbf{e}_i}_\beta\partial_{\mathbf{e}_i}\partial^\alpha_{\beta-\mathbf{e}_i}f_\pm||w(\alpha,\beta)\langle v\rangle^{-1/2}\partial^{\alpha+\mathbf{e}_i}_{\beta-\mathbf{e}_i}f_\pm|\\
\le&\ ||w(\alpha,\beta-\mathbf{e}_i)\langle v\rangle^{-3/2}\delta^{\mathbf{e}_i}_\beta\partial_{\mathbf{e}_i}\partial^\alpha_{\beta-\mathbf{e}_i}f_\pm||_2||\partial^{\alpha+\mathbf{e}_i}_{\beta-\mathbf{e}_i}f_\pm||_{\sigma,w(\alpha,\beta)},
\end{aligned}$$

where $\langle v\rangle^{1/2}w(\alpha,\beta)=\langle v\rangle^{-3/2}w(\alpha,\beta-\mathbf{e}_i)(v)$. We now use (19) again for $\partial^\alpha_{\beta-\mathbf{e}_i}f_\pm$ to conclude the proof

$$||w(\alpha,\beta-\mathbf{e}_i)\langle v\rangle^{-3/2}\delta^{\mathbf{e}_i}_\beta\partial_{\mathbf{e}_i}\partial^\alpha_{\beta-\mathbf{e}_i}f_\pm||_2\lesssim||\delta^{\mathbf{e}_i}_\beta\partial^\alpha_{\beta-\mathbf{e}_i}f_\pm||_{\sigma,w(\alpha,\beta-\mathbf{e}_i)}.$$

■

The following proposition is a refined estimate for the non-linear collision term $\Gamma(g,g)$ in [G1] [SG1-2]. The key improvement is that the factors in front of the highest order dissipation rate is bounded by only $\sqrt{\mathcal{E}_{2;2,0}(g_1)}+\sqrt{\mathcal{E}_{2;2,0}(g_2)}$.

Proposition 6 *(1) For $|\alpha|+|\beta|=m\le 2$, recall $w=w(\alpha,\beta)$ in (23). We have*

$$\begin{aligned}
&\langle w^2\partial^\alpha_\beta\Gamma[g_1,g_2],\partial^\alpha_\beta g_2\rangle\\
\lesssim&\ C_l\sum_{\bar\beta\le\beta_1\le\beta}|\partial^{\alpha_1}_{\bar\beta}g_1|_\sigma|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|_{\sigma,w}|\partial^\alpha_\beta g_2|_{2,w}\\
&+\sum_{\substack{\alpha_1\le\alpha,\\ \bar\beta\le\beta_1\le\beta}}[|\partial^{\alpha_1}_{\bar\beta}g_1|_2|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|_{\sigma,w}+|\partial^{\alpha_1}_{\bar\beta}g_1|_\sigma|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|_{2,w}]|\partial^\alpha_\beta g_2|_{\sigma,w}.
\end{aligned}\tag{37}$$

Moreover, for any $\eta>0$, there exists $C_{l,\eta}>0$ such that

$$\begin{aligned}
&\int_{\mathbf{T}^3}|\langle w^2\partial^\alpha_\beta\Gamma[g_1,g_2],\partial^\alpha_\beta g_2\rangle|dx\\
\lesssim&\ (\mathcal{E}_{2;2,0}(g_1)+\eta)\mathcal{D}_{2;l,q}(g_2)+C_{l,\eta}\sum_{|\alpha'|+|\beta'|\le1}\left\||\partial^{\alpha'}_{\beta'}g_1|_\sigma\right\|^2_{H^{\frac{3}{4}}}\mathcal{E}_{2;l,q}(g_2).
\end{aligned}\tag{38}$$

(2) For $|\alpha|+|\beta|=m\ge3$, we have

$$\begin{aligned}
&\langle w^2\partial^\alpha_\beta\Gamma[g_1,g_2],\partial^\alpha_\beta g_2\rangle\\
\lesssim&\sum_{\alpha_1\le\alpha,\beta_1\le\beta}C^{\alpha_1}_\alpha C^{\beta_1}_\beta C_l\sum_{\bar\beta\le\beta_1}|\partial^{\alpha_1}_{\bar\beta}g_1|_\sigma|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|_{\sigma,w}|\partial^\alpha_\beta g_2|_{2,w}\\
&+\sum_{|\alpha_1|+|\bar\beta|\le\frac{|\alpha|+|\beta|}{2}}C^{\alpha_1}_\alpha C^{\beta_1}_\beta[|\partial^{\alpha_1}_{\bar\beta}g_1|_2|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|_{\sigma,w}+|\partial^{\alpha_1}_{\bar\beta}g_1|_\sigma|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|_{2,w}]|\partial^\alpha_\beta g_2|_{\sigma,w}\\
&+\sum_{|\alpha_1|+|\bar\beta|\ge\frac{|\alpha|+|\beta|}{2}}C^{\alpha_1}_\alpha C^{\beta_1}_\beta[|\partial^{\alpha_1}_{\bar\beta}g_1|_{2,w}|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|_\sigma+|\partial^{\alpha_1}_{\bar\beta}g_1|_{\sigma,w}|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|_2]|\partial^\alpha_\beta g_2|_{\sigma,w}\\
&+\sum_{|\alpha_1|+|\bar\beta|\ge\frac{|\alpha|+|\beta|}{2}}C^{\alpha_1}_\alpha C^{\beta_1}_\beta C_{l,\alpha,\beta}|\partial^{\alpha_1}_{\bar\beta}g_1|_2|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|_{\sigma,w}|\partial^\alpha_\beta g_2|_{\sigma,w}.
\end{aligned}\tag{39}$$

Furthermore, for any $\eta > 0$, there exists $C_{l,m,\eta} > 0$

$$
\begin{aligned}
&\int_{\mathbf{T}^3} |\langle w^2(\alpha,\beta)\partial^\alpha_\beta \Gamma[g_1,g_2], \partial^\alpha_\beta g_2\rangle| dx \\
\lesssim\ & \{\sqrt{\mathcal{E}_{2;2,0}(g_1)} + \sqrt{\mathcal{E}_{2;2,0}(g_2)}\}\{||\partial^\alpha_\beta g_1||^2_{\sigma,w} + ||\partial^\alpha_\beta g_2||^2_{\sigma,w}\} \\
&+\eta\{\sqrt{\mathcal{E}_{2;2,0}(g_1)} + \sqrt{\mathcal{E}_{2;2,0}(g_2)} + 1\} \sum_{\substack{|\alpha'|+|\beta'|=m \\ \beta'\leq\beta}} \{||\partial^{\alpha'}_{\beta'} g_1||^2_{\sigma,w} + ||\partial^{\alpha'}_{\beta'} g_2||^2_{\sigma,w}\} \\
&+C_{l,m,\eta} \sum_{|\alpha'|+|\beta'|\leq[\frac{m}{2}]} \left\{ \left\| |\partial^{\alpha'}_{\beta'} g_1|_{\sigma,\frac{w(\alpha',\beta')}{\langle v\rangle^2}} \right\|^2_{H^{\frac{3}{4}}} + \left\| |\partial^{\alpha'}_{\beta'} g_2|_{\sigma,\frac{w(\alpha',\beta')}{\langle v\rangle^2}} \right\|^2_{H^{\frac{3}{4}}} \right\} \{\mathcal{E}_{m;l,q}(g_1) + \mathcal{E}_{m;l,q}(g_2)\} \\
&+C_{l,m,\eta}\{\mathcal{E}_{m-1;l,q}(g_1) + \mathcal{E}_{m-1;l,q}(g_2) + 1\}\{\mathcal{D}_{m-1;l,q}(g_1) + \mathcal{D}_{m-1;l,q}(g_2)\}. \qquad (40)
\end{aligned}
$$

We remark that from (23), (25) and Lemma 3,

$$
\sum_{|\alpha'|+|\beta'|\leq 1} \left\| |\partial^{\alpha'}_{\beta'} g|_\sigma \right\|^2_{H^{\frac{3}{4}}} \lesssim \sum_{|\alpha'|+|\beta'|\leq 1, |\gamma|\leq 1} ||\partial^\gamma \partial^{\alpha'}_{\beta'} g||^2_\sigma \lesssim \mathcal{D}_{2;2,0}(g), \qquad (41)
$$

$$
\sum_{|\alpha'|+|\beta'|\leq[\frac{m}{2}]} \left\{ \left\| |\partial^{\alpha'}_{\beta'} g|_{\sigma,\frac{w(\alpha',\beta')}{\langle v\rangle^2}} \right\|^2_{H^{\frac{3}{4}}} \right\} \lesssim \sum_{\substack{|\alpha'|+|\beta'|\leq[\frac{m}{2}] \\ |\gamma|\leq 1}} ||\partial^\gamma \partial^{\alpha'}_{\beta'} g||^2_{\sigma,w(\alpha'+\gamma,\beta')} \lesssim \mathcal{D}_{[\frac{m}{2}]+1;l,q}(g),
$$

(replacing $H^{\frac{3}{4}}$ by H^1) which are sufficient for our estimates for global solutions. However, for the construction of the local solutions, such sharper estimates are important for compactness of the approximate solutions.

Proof. Recall Lemma 10 in [SG2] and Theorem 3 in [G1]. We first separate:

$$
\partial_i[w^2] = \{4qv_i w^2\} + \left\{ \frac{2(l-|\alpha|-|\beta|)}{1+|v|^2} v_i w^2 \right\}. \qquad (42)
$$

We can express $\langle w^2\partial^\alpha_\beta \Gamma[g_1,g_2], \partial^\alpha_\beta g_2\rangle = \sum C^{\alpha_1}_\alpha C^{\beta_1}_\beta \times G_{\alpha_1,\beta_1}$ with G_{α_1,β_1} as

$$
-\langle w^2\{\Phi^{ij} * \partial_{\beta_1}[\mu^{1/2}\partial^{\alpha_1} g_1]\}\partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2, \partial_i \partial^\alpha_\beta g_2\rangle \qquad (43)
$$

$$
-(1+4q)\langle w^2\{\Phi^{ij} * \partial_{\beta_1}[v_i\mu^{1/2}\partial^{\alpha_1} g_1]\}\partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2, \partial^\alpha_\beta g_2\rangle \qquad (44)
$$

$$
+\langle w^2\{\Phi^{ij} * \partial_{\beta_1}[\mu^{1/2}\partial_j\partial^{\alpha_1} g_1]\}\partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2, \partial_i \partial^\alpha_\beta g_2\rangle \qquad (45)
$$

$$
+(1+4q)\langle w^2\{\Phi^{ij} * \partial_{\beta_1}[v_i\mu^{1/2}\partial_j\partial^{\alpha_1} g_1]\}\partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2, \partial^\alpha_\beta g_2\rangle \qquad (46)
$$

$$
-\langle \frac{2(l-|\alpha|-|\beta|)}{1+|v|^2} w^2\{\Phi^{ij} * \partial_{\beta_1}[v_i\mu^{1/2}\partial^{\alpha_1} g_1]\}\partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2, \partial^\alpha_\beta g_2\rangle \qquad (47)
$$

$$
+\langle \frac{2(l-|\alpha|-|\beta|)}{1+|v|^2} w^2\{\Phi^{ij} * \partial_{\beta_1}[\mu^{1/2}v_i\partial_j\partial^{\alpha_1} g_1]\}\partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2, \partial^\alpha_\beta g_2\rangle. \qquad (48)
$$

with double summations over $1 \leq i,j \leq 3$ and $\partial_{\mathbf{e}_i} = \partial_{v_i} = \partial_i$.

Step 1. Estimate of (47) *and* (48).

We note that there is a (large) factor $2(l-|\alpha|-|\beta|)$ in both (47) and (48). We follow exactly as in the proof of Lemma 10 in [SG2]. From $\Phi^{ij}*\mu^{1/8}\leq\frac{1}{1+|v|}$ and the Cauchy-Schwarz inequality,

$$|\Phi^{ij}*\partial_{\beta_1}[v_i\mu^{1/2}\partial_j\partial^{\alpha_1}g_1]|+|\Phi^{ij}*\partial_{\beta_1}[v_i\mu^{1/2}\partial^{\alpha_1}g_1]|\lesssim\langle v\rangle^{-1}\sum_{\bar{\beta}\leq\beta_1}|\partial^{\alpha}_{\bar{\beta}}g_1|_\sigma,$$

and we use the exponential weight $v_i\mu^{1/2}$ to bound $|\partial^{\alpha}_{\bar{\beta}}g_1|_\sigma$ in (19). Therefore

$$\begin{aligned}
&2(l-|\alpha|-|\beta|)|\langle\frac{w^2}{\langle v\rangle^2}\Phi^{ij}*\{\partial_{\beta_1}[v_i\mu^{1/2}\partial^{\alpha_1}g_1]\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2,\partial^\alpha_\beta g_2\rangle|\\
&+2(l-|\alpha|-|\beta|)|\langle\frac{w^2}{\langle v\rangle^2}\Phi^{ij}*\{\partial_{\beta_1}[v_i\mu^{1/2}\partial_j\partial^{\alpha_1}g_1]\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2,\partial^\alpha_\beta g_2\rangle|\\
\leq\;&C_{l,m}|\partial^{\alpha_1}_{\bar{\beta}}g_1|_\sigma\int\frac{w^2}{\langle v\rangle^3}(|\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|+|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|)|\partial^\alpha_\beta g_2|dv\\
\leq\;&C_{l,m}\sum_{\bar{\beta}}|\partial^{\alpha_1}_{\bar{\beta}}g_1|_\sigma|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|_{\sigma,w}|\partial^\alpha_\beta g_2|_{2,w},
\end{aligned}$$

where we have used $|w\langle v\rangle^{-3}\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|_2\lesssim|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|_{\sigma,w}$ by (19). This concludes the control of (47) and (48) via the first terms on the right hand side of both (37) and (39).

Step 2. Proof of (37) and (38) for $|\alpha|+|\beta|=m\leq 2$.

In light of step 1, we only need to bound (43) to (46), which are precisely bounded as in Lemma 10 of [SG2] leading to (37). Note that one can eliminate the weight function in the g_1 factor due to exponential decay factor $\mu^{1/2}$ in the Landau integral.

To conclude (38), we need to take x integration of (37) and we need to separate three cases.

If $|\alpha|+|\beta|=0$, $w=w(0,0)$, we have

$$\begin{aligned}
&\int_{\mathbf{T}^3}\langle w^2\Gamma[g_1,g_2],g_2\rangle\\
\lesssim\;&C_l\sup_x|g_1|_\sigma\int_{\mathbf{T}^3}|g_2|_{\sigma,w}|g_2|_{2,w}+\sup_x|g_1|_2\int_{\mathbf{T}^3}|g_2|^2_{\sigma,w}+\sup_x|g_1|_\sigma\int_{\mathbf{T}^3}|g_2|_{2,w}|g_2|_{\sigma,w}\\
\lesssim\;&C_l\,\||g_1|_\sigma\|_{H^{\frac{7}{4}}}\sqrt{\mathcal{D}_{2,l,q}(g_2)}\sqrt{\mathcal{E}_{2,l,q}(g_2)}+\sqrt{\mathcal{E}_{2,2,0}(g_1)}\mathcal{D}_{2,l,q}(g_2)\\
\lesssim\;&(\sqrt{\mathcal{E}_{2,2,0}(g_1)}+\eta)\mathcal{D}_{2,l,q}(g_2)+C_{l,\eta}\,\||g_1|_\sigma\|^2_{H^{\frac{7}{4}}}\mathcal{E}_{2,l,q}(g_2).
\end{aligned}$$

We have used from (24) and (25) and Lemma 3:

$$\sup_x|g_1|_\sigma\lesssim\||g_1|_\sigma\|_{H^{\frac{7}{4}}}\text{ and }\sup_x|g_1|_2\lesssim\sqrt{\mathcal{E}_{2,2,0}(g_1)}.\tag{49}$$

If $|\alpha|+|\beta|=1$, either $(\alpha_1,\beta_1)=0$ or $(\alpha-\alpha_1,\beta-\beta_1)=0$. In the case

$(\alpha_1, \beta_1) = 0$, we use (49) to bound similarly

$$\begin{aligned}
& C_l \sup_x |g_1|_\sigma \int_{\mathbf{T}^3} |\partial_\beta^\alpha g_2|_{\sigma,w} |\partial_\beta^\alpha g_2|_{2,w} \\
& + \sup_x |g_1|_2 \int_{\mathbf{T}^3} |\partial_\beta^\alpha g_2|_{\sigma,w}^2 + \sup_x |g_1|_\sigma \int_{\mathbf{T}^3} |\partial_\beta^\alpha g_2|_{2,w} |\partial_\beta^\alpha g_2|_{\sigma,w} \\
\lesssim \; & (\sqrt{\mathcal{E}_{2,2,0}(g_1)} + \eta)\mathcal{D}_{2,l,q}(g_2) + C_{l,\eta} \, \| |g_1|_\sigma \|_{H^{\frac{7}{4}}}^2 \, \mathcal{E}_{2,l,q}(g_2).
\end{aligned}$$

On the other hand, if $(\alpha - \alpha_1, \beta - \beta_1) = 0$, we take $L^4 - L^4 - L^2$ and use Lemma 3 to get

$$\begin{aligned}
& C_l \int_{\mathbf{T}^3} |\partial_{\bar\beta}^\alpha g_1|_\sigma |g_2|_{\sigma,w} |\partial_\beta^\alpha g_2|_{2,w} + \int_{\mathbf{T}^3} [|\partial_{\bar\beta}^{\alpha_1} g_1|_2 |g_2|_{\sigma,w} + |\partial_{\bar\beta}^{\alpha_1} g_1|_\sigma |g_2|_{2,w}] |\partial_\beta^\alpha g_2|_{\sigma,w} \\
\lesssim \; & C_l \left\| |\partial_{\bar\beta}^\alpha g_1|_\sigma \right\|_{H^{\frac{3}{4}}} ||\partial^\gamma g_2||_{\sigma,w} ||\partial_\beta^\alpha g_2||_{2,w} + ||\partial^\gamma \partial_{\bar\beta}^{\alpha_1} g_1||_2 ||\partial^\gamma g_2||_{\sigma,w} ||\partial_\beta^\alpha g_2||_{\sigma,w} \qquad (50)
\end{aligned}$$

where $|\gamma| \leq 1$. Note $||\partial^\gamma \partial_{\bar\beta}^{\alpha_1} g_1||_2 \lesssim \sqrt{\mathcal{E}_{2,2,0}(g_1)}$. For $w = w(\alpha, \beta)$,

$$||\partial^\gamma g_2||_{\sigma,w} \lesssim \sqrt{\mathcal{D}_{1,l,q}(g_2)} \lesssim \sqrt{\mathcal{D}_{2,l,q}(g_2)}$$

from (24) and (25). Therefore, (50) is further bounded by

$$\begin{aligned}
& C_l \left\| |\partial_{\bar\beta}^\alpha g_1|_\sigma \right\|_{H^{\frac{3}{4}}} \sqrt{\mathcal{D}_{2,l,q}(g_2)} \sqrt{\mathcal{E}_{2,l,q}(g_1)} + \sqrt{\mathcal{E}_{2,2,0}(g_1)} \mathcal{D}_{2,l,q}(g_2) \\
\lesssim \; & (\sqrt{\mathcal{E}_{2,2,0}(g_1)} + \eta)\mathcal{D}_{2,l,q}(g_2) + C_{l,\eta} \left\| |\partial_{\bar\beta}^\alpha g_1|_\sigma \right\|_{H^{\frac{3}{4}}}^2 \mathcal{E}_{2,l,q}(g_2).
\end{aligned}$$

We remark that we can not take L_x^∞ of $\sup_x |g_2|_{\sigma,w(\alpha,\beta)}$ and $\sup_x |g_2|_{2,w(\alpha,\beta)}$ in this case, because $w(\alpha, \beta)$ now is associated with $|\alpha| + |\beta| = 1$, and stronger than w allowed for second order derivatives from (23): for $|\gamma| \geq 2$,

$$w(\gamma, 0) < w(\alpha, \beta).$$

As a consequence, $\sum_{|\gamma|\leq 2} ||\partial^\gamma g_2||_{\sigma,w(\alpha,\beta)}$ and $\sum_{|\gamma|\leq 2} ||\partial^\gamma g_2||_{2,w(\alpha,\beta)}$ can not be bounded by $\sqrt{\mathcal{D}_{2,l,q}(g_2)}$ or $\sqrt{\mathcal{E}_{2,l,q}(g_2)}$.

We now consider the third case $|\alpha| + |\beta| = 2$. We first consider

$$\text{either } \alpha_1 = \bar\beta = 0 \;\text{ or } \alpha - \alpha_1 = \beta - \beta_1 = 0. \qquad (51)$$

We take L_x^∞ of the term without derivatives and L_x^2 of the other two terms. We note that by (23), accordingly,

$$w(\alpha, \beta) \leq w(\gamma + \alpha_1, \bar\beta) \text{ or } w(\alpha, \beta) \leq w(\gamma + \alpha - \alpha_1, \beta - \beta_1) \qquad (52)$$

for $|\gamma| \leq 2$. Therefore for any g

$$\begin{aligned}
\sup_x |g|_\sigma & \lesssim \| |g|_\sigma \|_{H^{\frac{7}{4}}}, \text{ and } \sup_x |g|_2 \lesssim \sqrt{\mathcal{E}_{2,2,0}(g)} \\
\sup_x |g|_{\sigma,w} & \lesssim \sum_{|\gamma|\leq 2} ||\partial^\gamma g||_{\sigma,w(\alpha,\beta)} \lesssim \sum_{|\gamma|\leq 2} ||\partial^\gamma g||_{\sigma,w(\gamma,0)} \lesssim \sqrt{\mathcal{D}_{2,l,q}(g)} \\
\sup_x |g|_{2,w} & \lesssim \sum_{|\gamma|\leq 2} ||\partial^\gamma g||_{2,w(\alpha,\beta)} \lesssim \sum_{|\gamma|\leq 2} ||\partial^\gamma g||_{2,w(\gamma,0)} \lesssim \sqrt{\mathcal{E}_{2,l,q}(g)}. \qquad (53)
\end{aligned}$$

By Lemma 3 and (53), we bound (37) by

$$\begin{aligned}
&\int_{\mathbf{T}^3} |\langle w^2 \partial_\beta^\alpha \Gamma[g_1, g_2], \partial_\beta^\alpha g_2\rangle| \\
\lesssim\ & C_l \left\| |g_1|_\sigma \right\|_{H^{\frac{7}{4}}} \sqrt{\mathcal{D}_{2,l,q}(g_2)}\sqrt{\mathcal{E}_{2,l,q}(g_2)} \\
&+\sqrt{\mathcal{E}_{2,2,0}(g_1)}\mathcal{D}_{2,l,q}(g_2) + \left\| |g_1|_\sigma \right\|_{H^{\frac{7}{4}}} \sqrt{\mathcal{E}_{2,2,0}(g_2)}\sqrt{\mathcal{D}_{2,l,q}(g_2)} \\
\lesssim\ & C_l \left\| |g_1|_\sigma \right\|_{H^{\frac{7}{4}}} \sqrt{\mathcal{D}_{2,l,q}(g_2)}\sqrt{\mathcal{E}_{2,l,q}(g_2)} + \sqrt{\mathcal{E}_{2,2,0}(g_1)}\mathcal{D}_{2,l,q}(g_2).
\end{aligned}$$

We note

$$\left\| |g_1|_\sigma \right\|_{H^{\frac{7}{4}}} \sqrt{\mathcal{D}_{2,l,q}(g_2)}\sqrt{\mathcal{E}_{2,l,q}(g_2)} \leq \eta \mathcal{D}_{2,l,q}(g_2) + C_\eta \left\| |g_1|_\sigma \right\|_{H^{\frac{7}{4}}}^2 \mathcal{E}_{2,l,q}(g_2). \quad (54)$$

We conclude (38) if (51) is valid.

If $(\alpha_1, \bar{\beta}) \neq 0$ and $(\alpha - \alpha_1, \beta - \beta_1) \neq 0$. Now $|\alpha_1| + |\bar{\beta}| = 1$ and $|\alpha - \alpha_1| + |\beta - \beta_1| = 1$ since $|\alpha| + |\beta| = 2$. We take $L^4 - L^4 - L^2$ to get:

$$\begin{aligned}
&\langle w^2 \partial_\beta^\alpha \Gamma[g_1, g_2], \partial_\beta^\alpha g_2\rangle \\
\lesssim\ & C_l \left\| |\partial_{\bar{\beta}}^{\alpha_1} g_1|_\sigma \right\|_{H^{\frac{3}{4}}} ||\partial^\gamma \partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2||_{\sigma,w} ||\partial_\beta^\alpha g_2||_{2,w} \\
&+ [||\partial^\gamma \partial_{\bar{\beta}}^{\alpha_1} g_1||_2 ||\partial^\gamma \partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2||_{\sigma,w} + \left\| |\partial_{\bar{\beta}}^{\alpha_1} g_1|_\sigma \right\|_{H^{\frac{3}{4}}} ||\partial^\gamma \partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2||_{2,w}] ||\partial_\beta^\alpha g_2||_{\sigma,w} \\
\lesssim\ & C_l \left\| |\partial_{\bar{\beta}}^{\alpha_1} g_1|_\sigma \right\|_{H^{\frac{3}{4}}} \sqrt{\mathcal{D}_{2,l,q}(g_2)}\sqrt{\mathcal{E}_{2,l,q}(g_2)} + \sqrt{\mathcal{E}_{2,2,0}(g_1)}\mathcal{D}_{2,l,q}(g_2),
\end{aligned}$$

where $|\gamma| \leq 1$. This concludes step 2 by (54).

Step 3. Proof of (39).

In the case of $|\alpha_1| + |\bar{\beta}| \leq \frac{|\alpha|+|\beta|}{2}$, the estimate again follows exactly as in Lemma 10 of [SG2] for (43) to (46).

The case of $|\alpha_1| + |\bar{\beta}| \geq \frac{|\alpha|+|\beta|}{2}$ is most delicate as we need to avoid a contribution of $|\partial_\beta^\alpha g_1|_\sigma |g_2|_{2,w} |\partial_\beta^\alpha g_2|_{\sigma,w}$ in Lemma 10 of [SG2]. Our goal is to 'move' the weight w out of $|g_2|_{2,w}$ to $|\partial_\beta^\alpha g_1|_\sigma$. To accomplish this, we following exactly the proof of Lemma 10 of [SG2] but with a different splitting of the phase space v, v', depending on l and m :

$$\{|w| \leq 2\}, \ \left\{ \left| \frac{|v'|}{|v|} - 1 \right| \leq \varepsilon_{l,m}, \ \ |w| \geq 2 \right\} \text{ and } \left\{ \left| \frac{|v'|}{|v|} - 1 \right| \geq \varepsilon_{l,m}, \ \ |w| \geq 2 \right\}$$

with $\varepsilon_{l,m} < 1$ satisfying ($l \geq |\alpha| + |\beta|$ from (23)):

$$\frac{q\varepsilon_{l,m}}{(1-\varepsilon_{l,m})^2} < \frac{1}{4} \text{ and } (1-\varepsilon_{l,m})^{-2(l-|\alpha|-|\beta|)} \leq 2. \quad (55)$$

The estimate in the first region $\{|w| \leq 2\}$ follows the proof of case 1 in Lemma 10 in [SG2] with an upper bound of terms (43) to (46) for $|\alpha_1| + |\bar{\beta}| \geq \frac{|\alpha|+|\beta|}{2}$ as

$$\begin{aligned}
&[|\partial_{\bar{\beta}}^{\alpha_1} g_1|_2 |\partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2|_{\sigma,w} + |\partial_{\bar{\beta}}^{\alpha_1} g_1|_\sigma |\partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2|_{2,w}] |\partial_\beta^\alpha g_2|_{\sigma,w} \quad (56) \\
\lesssim\ & |\partial_{\bar{\beta}}^{\alpha_1} g_1|_{2,w} |\partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2|_\sigma |\partial_\beta^\alpha g_2|_{\sigma,w} + |\partial_{\bar{\beta}}^{\alpha_1} g_1|_{\sigma,w} |\partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2|_2 |\partial_\beta^\alpha g_2|_{\sigma,w},
\end{aligned}$$

where we have used the important fact $w \leq 2$ and the property

$$w(\alpha, \beta) \geq 1.$$

We remark even if $q > 0$ and $l = 0$ in (23):

$$w(\alpha, \beta) = e^{\frac{q}{2}|v|^2} \langle v \rangle^{-2|\alpha|-2|\beta|}, w \geq c_{\alpha,\beta} \to 0$$

as $|\alpha|+|\beta| \to \infty$.A large constant of $\frac{1}{c_{\alpha,\beta}}$ appears in front of $|\partial_{\bar{\beta}}^{\alpha_1} g_1|_{\sigma,w} |\partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2|_2 |\partial_\beta^\alpha g_2|_{\sigma,w}$ in (56), when we bound $|\partial_{\bar{\beta}}^{\alpha_1} g_1|_\sigma$ by $|\partial_{\bar{\beta}}^{\alpha_1} g_1|_{\sigma,w}$. In particular

$$\frac{1}{c_{\alpha,\beta}} |\partial_\beta^\alpha g_1|_{\sigma,w} |g_2|_2 |\partial_\beta^\alpha g_2|_{\sigma,w}$$

can not be absorbed by the dissipation rate (25). This illustrates the importance of the choice of $l \geq |\alpha| + |\beta|$ to guarantee $w \geq 1$.

For the second region $\left\{ \left| \frac{|v'|}{|v|} - 1 \right| \leq \varepsilon_{l,m}, \quad |w| \geq 2 \right\}$ we shall swap the weight $w(v)$ by $w(v')$. Since $(1 - \varepsilon_{l,m})|v| \leq |v'| \leq (1 + \varepsilon_{l,m})|v|$, we deduce that

$$\begin{aligned} \frac{w(v)}{w(v')} &= e^{\frac{q}{2}|v|^2 - \frac{q}{2}|v'|^2} \left(\frac{1 + |v|^2}{1 + |v'|^2} \right)^{l-|\alpha|-|\beta|} \\ &\leq e^{\frac{q}{2}|v'|^2 [\frac{1}{(1-\varepsilon_{l,m})^2} - 1]} \left(\frac{1 + \frac{|v'|^2}{(1-\varepsilon_{l,m})^2}}{1 + |v'|^2} \right)^{l-|\alpha|-|\beta|} \\ &\leq e^{\frac{q\varepsilon_{l,m}}{(1-\varepsilon_{l,m})^2}|v'|^2} (1 - \varepsilon_{l,m})^{-2(l-|\alpha|-|\beta|)} \\ &\leq 2e^{\frac{1}{4}|v'|^2} \end{aligned} \tag{57}$$

by (55). In the Case 2 of the proof of Lemma 10 of [SG2], due to exponential decay of $v'e^{\frac{1}{2}|v'|^2}$ in the Landau kernel which dominates $2e^{\frac{1}{4}|v'|^2}$, by (57) we can move weight function $w(v) = w(v') \times \frac{w(v)}{w(v')}$ to g_1 to obtain bounds ($w \geq 2$) for (43) to (46) as:

$$[|\partial_{\bar{\beta}}^{\alpha_1} g_1|_{2,w} |\partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2|_\sigma + |\partial_{\bar{\beta}}^{\alpha_1} g_1|_{\sigma,w} |\partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2|_2] |\partial_\beta^\alpha g_2|_{\sigma,w}.$$

For the third region of $\left\{ \left| \frac{|v'|}{|v|} - 1 \right| \geq \varepsilon_{l,m}, \quad |w| \geq 2 \right\}$, we note

$$|v' - v| \geq ||v'| - |v|| \geq \varepsilon_{l,m}|v| \quad \text{and } |v| \geq \theta_{l,m} > 0.$$

We can repeat the same argument as in Case 3 of the proof for Lemma 10 [SG2], upon further integration by parts in v' to bring down the v' derivative in $|\partial_{\bar{\beta}}^{\alpha_1} g_1|_\sigma$. This process creates a constant depending on (large) constant $C_{l,m}$ and we obtain a desired upper bound of

$$C_{l,m} |\partial_{\bar{\beta}}^{\alpha_1} g_1|_2 |\partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2|_{\sigma,w} |\partial_\beta^\alpha g_2|_{\sigma,w}$$

to conclude step 3.

Step 4. Proof of (40).

We shall separate three cases in (39). The first case is either $\alpha_1 = \alpha, \beta = \bar{\beta}$ or $\alpha - \alpha_1 = \alpha$, and $\beta - \beta_1 = \beta$. Indeed we now have the coefficients $C_\alpha^{\alpha_1} C_\beta^{\beta_1} = 1$. Recall (51) and (52) in this case. We note that $m \geq 3$ so that we can apply L_x^∞ estimate to terms without derivatives, and then take corresponding $L^\infty - L^2 - L^2$ for the x integration. By (53), we obtain an upper bound of (39)

$$\begin{aligned}
& C_l[\||g_1|_\sigma\|_{H^{\frac{7}{4}}} ||\partial_\beta^\alpha g_2||_{\sigma,w} + ||\partial_\beta^\alpha g_1||_\sigma \||g_2|_\sigma\|_{H^{\frac{7}{4}}}]\sqrt{\mathcal{E}_{m;l,q}(g_2)} \\
& +[\sqrt{\mathcal{E}_{2;2,0}(g_1)}||\partial_\beta^\alpha g_2||_{\sigma,w} + \||g_1|_\sigma\|_{H^{\frac{7}{4}}} \sqrt{\mathcal{E}_{m;l,q}(g_2)}]||\partial_\beta^\alpha g_2||_{\sigma,w} \\
& +[\sqrt{\mathcal{E}_{m;l,q}(g_1)} \||g_2|_\sigma\|_{H^{\frac{7}{4}}} + ||\partial_\beta^\alpha g_1||_{\sigma,w}\sqrt{\mathcal{E}_{2;2,0}(g_2)}]||\partial_\beta^\alpha g_2||_{\sigma,w} \\
& +C_{l,m}\sqrt{\mathcal{E}_{m;l,0}(g_1)} \||g_2|_{\sigma,w}\|_{H^{\frac{7}{4}}} ||\partial_\beta^\alpha g_2||_{\sigma,w} \\
\lesssim \; & \{\sqrt{\mathcal{E}_{2;2,0}(g_1)} + \sqrt{\mathcal{E}_{2;2,0}(g_2)} + \eta\}\{||\partial_\beta^\alpha g_1||_{\sigma,w}^2 + ||\partial_\beta^\alpha g_2||_{\sigma,w}^2\} \\
& +C_{l,m,\eta}\{\||g_1|_{\sigma,w}\|_{H^{\frac{7}{4}}}^2 + \||g_2|_{\sigma,w}\|_{H^{\frac{7}{4}}}^2\}\{\mathcal{E}_{m;l,q}(g_1) + \mathcal{E}_{m;l,q}(g_2)\}.
\end{aligned}$$

We have used the fact $w \geq 1$ and $l \geq 2$, $q \geq 0$. We note that

$$\begin{aligned}
& \||g_1|_{\sigma,w}\|_{H^{\frac{7}{4}}}^2 + \||g_2|_{\sigma,w}\|_{H^{\frac{7}{4}}}^2 \\
\lesssim \; & \sum_{|\alpha'|+|\beta'|\leq 1} \left\||\partial_{\beta'}^{\alpha'} g_1|_{\sigma,w(\alpha,\beta)}\right\|_{H^{\frac{3}{4}}}^2 + \left\||\partial_{\beta'}^{\alpha'} g_2|_{\sigma,w(\alpha,\beta)}\right\|_{H^{\frac{3}{4}}}^2 \\
\lesssim \; & \sum_{|\alpha'|+|\beta'|\leq [\frac{m}{2}]} \left\||\partial_{\beta'}^{\alpha'} g_1|_{\sigma,\frac{w(\alpha',\beta')}{\langle v\rangle^2}}\right\|_{H^{\frac{3}{4}}}^2 + \left\||\partial_{\beta'}^{\alpha'} g_2|_{\sigma,\frac{w(\alpha',\beta')}{\langle v\rangle^2}}\right\|_{H^{\frac{3}{4}}}^2, \qquad (58)
\end{aligned}$$

as $[\frac{m}{2}] \geq 1$ and $w(\alpha,\beta) \leq \frac{w(\alpha',\beta')}{\langle v\rangle^2}$ for $|\alpha| + |\beta| \geq |\alpha'| + |\beta'| + 1$ from (23). We thus conclude the first case.

The second case is when $2 \leq |\alpha_1| + |\bar{\beta}| \leq m-2$ and $2 \leq |\alpha - \alpha_1| + |\beta - \beta_1| \leq m-2$. In this case, we shall take $L^4 - L^4 - L^2$ and by Lemma 3 to find an upper bound of (39):

$$\begin{aligned}
& C_{l,m}\left\||\partial_{\bar{\beta}}^{\alpha_1} g_1|_\sigma\right\|_{H^{\frac{3}{4}}} \left\||\partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2|_{\sigma,w}\right\|_{H^{\frac{3}{4}}} ||\partial_\beta^\alpha g_2||_{2,w} \\
& +C_{l,m}[\left\||\partial_{\bar{\beta}}^{\alpha_1} g_1|_2\right\|_{H^{\frac{3}{4}}} \left\||\partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2|_{\sigma,w}\right\|_{H^{\frac{3}{4}}} + \left\||\partial_{\bar{\beta}}^{\alpha_1} g_1|_\sigma\right\|_{H^{\frac{3}{4}}} \left\||\partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2|_{2,w}\right\|_{H^{\frac{3}{4}}}]||\partial_\beta^\alpha g_2||_{\sigma,w} \\
& +C_{l,m}[\left\||\partial_{\bar{\beta}}^{\alpha_1} g_1|_{2,w}\right\|_{H^{\frac{3}{4}}} \left\||\partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2|_\sigma\right\|_{H^{\frac{3}{4}}} + \left\||\partial_{\bar{\beta}}^{\alpha_1} g_1|_{\sigma,w}\right\|_{H^{\frac{3}{4}}} \left\||\partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2|_2\right\|_{H^{\frac{3}{4}}}]||\partial_\beta^\alpha g_2||_{\sigma,w} \\
& +C_{l,m}\left\||\partial_{\bar{\beta}}^{\alpha_1} g_1|_2\right\|_{H^{\frac{3}{4}}} \left\||\partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2|_{\sigma,w}\right\|_{H^{\frac{3}{4}}} ||\partial_\beta^\alpha g_2||_{\sigma,w}. \qquad (59)
\end{aligned}$$

For the last three terms, we use $H^1 \subset H^{\frac{3}{4}}$ and the facts

$$\begin{aligned}
\left\||\partial_{\beta-\beta_1}^{\alpha-\alpha_1} g|_{2,w}\right\|_{H^1} + \left\||\partial_{\bar{\beta}}^{\alpha_1} g|_{2,w}\right\|_{H^1} & \lesssim \mathcal{E}_{m-1;l,q}(g) \\
\left\||\partial_{\beta-\beta_1}^{\alpha-\alpha_1} g|_{\sigma,w}\right\|_{H^1} + \left\||\partial_{\bar{\beta}}^{\alpha_1} g|_{\sigma,w}\right\|_{H^1} & \lesssim \mathcal{D}_{m-1;l,q}(g)
\end{aligned}$$

to obtain a desired upper bound of

$$\eta||\partial^\alpha_\beta g_2||^2_{\sigma,w} + C_{l,m,\eta}\{\mathcal{E}_{m-1;l,q}(g_1)+\mathcal{E}_{m-1;l,q}(g_2)\}\{\mathcal{D}_{m-1;l,q}(g_1)+\mathcal{D}_{m-1;l,q}(g_2)\}.$$

For the first term in (59), we combine the factor of smaller total derivatives (less then $[\frac{m}{2}]$) with $||\partial^\alpha_\beta g_2||_{2,w}$ to obtain an upper bound:

$$\begin{aligned}
&\eta \sum_{[\frac{m}{2}]\le|\alpha'|+|\beta'|\le m-1}\{\left\||\partial^{\alpha'}_{\beta'}g_1|_{\sigma,w}\right\|^2_{H^{\frac{3}{4}}}+\left\||\partial^{\alpha'}_{\beta'}g_2|_{\sigma,w}\right\|^2_{H^{\frac{3}{4}}}\}\\
&+C_{l,m,\eta}\sum_{|\alpha'|+|\beta'|\le[\frac{m}{2}]}\{\left\||\partial^{\alpha'}_{\beta'}g_1|_{\sigma,w}\right\|^2_{H^{\frac{3}{4}}}+\left\||\partial^{\alpha'}_{\beta'}g_2|_{\sigma,w}\right\|^2_{H^{\frac{3}{4}}}\}||\partial^\alpha_\beta g_2||^2_{2,w}\\
\le\ &\eta\sum_{|\alpha'|+|\beta'|=m}\{||\partial^{\alpha'}_{\beta'}g_1||^2_{\sigma,w}+||\partial^{\alpha'}_{\beta'}g_2||^2_{\sigma,w}\}+\eta\{\mathcal{D}_{m-1;l,q}(g_1)+\mathcal{D}_{m-1;l,q}(g_2)\}\\
&+C_{l,m,\eta}\sum_{|\alpha'|+|\beta'|\le[\frac{m}{2}]}\{\left\||\partial^{\alpha'}_{\beta'}g_1|_{\sigma,\frac{w'}{\langle v\rangle^2}}\right\|^2_{H^{\frac{3}{4}}}+\left\||\partial^{\alpha'}_{\beta'}g_2|_{\sigma,\frac{w'}{\langle v\rangle^2}}\right\|^2_{H^{\frac{3}{4}}}\}||\partial^\alpha_\beta g_2||^2_{2,w}
\end{aligned}\tag{60}$$

where $w'=w(\alpha',\beta')$ and $w(\alpha,\beta)\le\frac{w'(\alpha',\beta')}{\langle v\rangle^2}$ for $|\alpha|+|\beta|\ge|\alpha'|+|\beta'|+1$ by (23). We therefore conclude the second case.

We now consider the last case of following possibilities: $1=|\alpha_1|+|\bar\beta|$, $|\alpha_1|+|\bar\beta|=m-1$, $1=|\alpha-\alpha_1|+|\beta-\beta_1|$, or $|\alpha-\alpha_1|+|\beta-\beta_1|=m-1$. Note that in this case the weight function satisfies:

$$w(\alpha,\beta)\le w(\alpha_1+\gamma,\bar\beta),\ w(\alpha,\beta)\le w(\alpha-\alpha_1+\gamma,\beta_1),$$

for $|\gamma|\le 1$. We bound (59) by taking $L^4-L^4-L^2$ as

$$\begin{aligned}
&C_{l,m}\sum\left\||\partial^{\alpha_1}_{\bar\beta}g_1|_\sigma\right\|_{H^{\frac{3}{4}}}\left\||\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|_{\sigma,w}\right\|_{H^{\frac{3}{4}}}||\partial^\alpha_\beta g_2||_{2,w}\\
&+C_{l,m}\sum_{|\alpha|+|\bar\beta|\le1}[\left\||\partial^{\alpha_1}_{\bar\beta}g_1|_2\right\|_{H^{\frac{3}{4}}}\left\||\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|_{\sigma,w}\right\|_{H^{\frac{3}{4}}}+\\
&+\left\||\partial^{\alpha_1}_{\bar\beta}g_1|_\sigma\right\|_{H^{\frac{3}{4}}}\left\||\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|_{2,w}\right\|_{H^{\frac{3}{4}}}]||\partial^\alpha_\beta g_2||_{\sigma,w}\\
&+C_{l,m}\sum_{|\alpha-\alpha_1|+|\beta-\beta_1|\le1}[\left\||\partial^{\alpha_1}_{\bar\beta}g_1|_{2,w}\right\|_{H^{\frac{3}{4}}}\left\||\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|_\sigma\right\|_{H^{\frac{3}{4}}}+\\
&+\left\||\partial^{\alpha_1}_{\bar\beta}g_1|_{\sigma,w}\right\|_{H^{\frac{3}{4}}}\left\||\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|_2\right\|_{H^{\frac{3}{4}}}]||\partial^\alpha_\beta g_2||_{\sigma,w}\\
&+C_{l,m}\sum_{|\alpha-\alpha_1|+|\beta-\beta_1|\le1}\left\||\partial^{\alpha_1}_{\bar\beta}g_1|_2\right\|_{H^{\frac{3}{4}}}\left\||\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|_{\sigma,w}\right\|_{H^{\frac{3}{4}}}||\partial^\alpha_\beta g_2||_{\sigma,w}.
\end{aligned}\tag{61}$$

The first term above can be estimated exactly as (60). Note that from (25)

$$\sum_{|\alpha|+|\bar\beta|\le1}\left\||\partial^{\alpha_1}_{\bar\beta}g|_2\right\|_{H^{\frac{3}{4}}}\lesssim\mathcal{E}_{2;2,0}(g),\tag{62}$$

by Lemma 3, the second term in (61) is bounded by a desired upper bound:

$$
\begin{aligned}
& \eta||\partial_\beta^\alpha g_2||^2_{\sigma,w} + C_{l,m,\eta}\mathcal{E}_{2;2,0}(g_1)\left\| |\partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2|_{\sigma,w}\right\|^2_{H^{\frac{3}{4}}} + C_{l,m,\eta}\sum_{|\alpha|+|\bar\beta|\le 1}\left\||\partial^{\alpha_1}_{\bar\beta} g_1|_\sigma\right\|^2_{H^{\frac{3}{4}}}\left\||\partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2|_{2,w}\right\|^2_{H^{\frac{3}{4}}} \\
\lesssim\ & \eta||\partial_\beta^\alpha g_2||^2_{\sigma,w} + C_{l,m,\eta}\eta_1\mathcal{E}_{2;2,0}(g_1)\sum_{|\alpha'|+|\beta'|=m}||\partial^{\alpha'}_{\beta'} g_2||^2_{\sigma,w(\alpha',\beta')} + C_{l,m,\eta,\eta_1}\mathcal{E}_{2;2,0}(g_1)\mathcal{D}_{m-1;l,q}(g_2) \\
& + C_{l,m,\eta}\sum_{|\alpha|+|\bar\beta|\le 1}\left\||\partial^{\alpha_1}_{\bar\beta} g_1|_\sigma\right\|^2_{H^{\frac{3}{4}}}\mathcal{E}_{m;l,q}(g_2) \\
\lesssim\ & \eta||\partial_\beta^\alpha g_2||^2_{\sigma,w} + \eta\mathcal{E}_{2;2,0}(g_1)\sum_{|\alpha'|+|\beta'|=m}||\partial^{\alpha'}_{\beta'} g_2||^2_{\sigma,w(\alpha',\beta')} + C_{l,m,\eta}\sum_{|\alpha|+|\bar\beta|\le 1}\left\||\partial^{\alpha_1}_{\bar\beta} g_1|_\sigma\right\|^2_{H^{\frac{3}{4}}}\mathcal{E}_{m;l,q}(g_2) \\
& + C_{l,m,\eta}\mathcal{E}_{2;2,0}(g_1)\mathcal{D}_{m-1;l,q}(g_2),
\end{aligned}
$$

with η_1 further small.

Similarly, the second and the third terms in (61) are bounded by a desired upper bound of ($w = w(\alpha,\beta)$)

$$
\begin{aligned}
\eta\quad & ||\partial_\beta^\alpha g_2||^2_{\sigma,w} + C_{l,m,\eta}\left\{\sum_{|\alpha'|+|\beta'|\le 1}\left\||\partial^{\alpha'}_{\beta'} g_2|_{\sigma,w}\right\|^2_{H^{\frac{3}{4}}}\mathcal{E}_{m;l,q}(g_1) + \sum_{|\alpha-\alpha_1|+|\beta-\beta_1|\le 1}\left\||\partial^{\alpha_1}_{\bar\beta} g_1|_{\sigma,w}\right\|^2_{H^{\frac{3}{4}}}\mathcal{E}_{2;2,0}(g_2)\right\} \\
\lesssim\quad & \eta||\partial_\beta^\alpha g_2||^2_{\sigma,w} + C_{l,m,\eta}\sum_{|\alpha'|+|\beta'|\le 1}\left\||\partial^{\alpha'}_{\beta'} g_2|_{\sigma,w}\right\|^2_{H^{\frac{3}{4}}}\mathcal{E}_{m;l,q}(g_1) \\
& + \eta\sum_{|\alpha'|+|\beta'|=m}||\partial^{\alpha'}_{\beta'} g_1||^2_{\sigma,w(\alpha',\beta')} + C_{l,m,\eta}\mathcal{E}_{2;2,0}(g_2)\mathcal{D}_{m-1;l,q}(g_1),
\end{aligned}
$$

where we have applied compact imbedding again to $\left\||\partial^{\alpha'}_{\beta'} g_1|_{\sigma,w}\right\|^2_{H^{\frac{3}{4}}}$. By (58), we complete the proof of the proposition. ■

In order to obtain decay for the electric field, we now treat pure spatial derivatives in more details of $\partial^\alpha\Gamma_\pm(f,f)\partial^\alpha f_\pm$.

Lemma 7 *(1) If $|\alpha| = 0,1,2$, then*

$$\int\langle\partial^\alpha\Gamma_\pm(f,f),\partial^\alpha f_\pm\rangle dx \lesssim \sqrt{\mathcal{E}_{2;2,0}(f)}\sum_{|\alpha'|\le|\alpha|}||\partial^{\alpha'} f||^2_\sigma.$$

(2) If $|\alpha| = m \ge 3$, then for any $\eta > 0$ there exists $C_{m,\eta} > 0$ such that

$$
\begin{aligned}
\int|\langle\partial^\alpha\Gamma_\pm(f,f),\partial^\alpha f_\pm\rangle|dx \quad\lesssim\quad & [\sqrt{\mathcal{E}_{2;2,0}(f)}+\eta]||\partial^\alpha f||^2_\sigma + \eta[\sqrt{\mathcal{E}_{2;2,0}(f)}+1]\sum_{|\alpha'|=m}||\partial^{\alpha'} f||^2_\sigma \\
& + C_{m,\eta}[\mathcal{D}_{2;2,0}(f)\mathcal{E}_{m;l,q}(f) + \mathcal{E}_{m-1;l,q}(f)\mathcal{D}_{m-1;l,q}(f)].
\end{aligned}
$$

Proof. Since there is no weight w, we use Theorem 3 in [G1] with $w = 1$:

$$\int\langle\partial^\alpha\Gamma_\pm(f,f),\partial^\alpha f_\pm\rangle dx \lesssim \sum_{\alpha_1\le\alpha} C^{\alpha_1}_\alpha\int|\partial^{\alpha-\alpha_1} f|_2|\partial^{\alpha_1} f|_\sigma|\partial^\alpha f|_\sigma dx. \qquad (63)$$

We now estimate case by case according to $|\alpha|$.

Assume $|\alpha|=0$. We can take $\sup_x |f|_2 \lesssim \sqrt{\mathcal{E}_{2;2,0}}$ by (24) and $L^\infty - L^2 - L^2$ in (63) to obtain an upper bound of

$$\int |f|_2 |f|_\sigma |f|_\sigma dx \lesssim \sqrt{\mathcal{E}_{2;2,0}} ||f||_\sigma^2.$$

Assume $|\alpha|=1$. We take $L^4 - L^4 - L^2$ in (63) if $\alpha_1 = 0$, and $L^\infty - L^2 - L^2$ in (63) if $|\alpha_1| = 1$. By Lemma 3 with $\eta = 1$ and $w = 1$, we obtain an upper bound of

$$\sum_{\alpha_1 \le \alpha} \int |\partial^{\alpha-\alpha_1} f|_2 |\partial^{\alpha_1} f|_\sigma |\partial^\alpha f|_\sigma dx \lesssim \sqrt{\mathcal{E}_{2;2,0}} \sum_{|\alpha| \le 1} ||\partial^\alpha f||_\sigma^2.$$

Assume $|\alpha| = 2$. If $\alpha_1 = 0$, we take $L^2 - L^\infty - L^2$; if $|\alpha_1| = 1$ we $L^4 - L^4 - L^2$; if $|\alpha_1| = 2$, we take $L^\infty - L^2 - L^2$ in (63) respectively. By Lemma 3 with $w = 1$ and $\eta = 1$, we obtain:

$$\sum_{\alpha_1 \le \alpha} \int |\partial^{\alpha-\alpha_1} f|_2 |\partial^{\alpha_1} f|_\sigma |\partial^\alpha f|_\sigma dx \lesssim \sqrt{\mathcal{E}_{2;2,0}} \sum_{|\alpha| \le 2} ||\partial^\alpha f||_\sigma^2.$$

Combining with the cases $|\alpha| = 0, 1, 2$ we conclude part (1) of the lemma.

Assume $|\alpha| = m \ge 3$. We need to be careful with the (large) constant $C_\alpha^{\alpha_1}$. We single out the terms with highest order derivatives with either $\alpha_1 = 0$ or $\alpha_1 = \alpha$. By taking L^∞ on the term without any derivatives with $\sup_x |f|_\sigma \lesssim \sqrt{\mathcal{D}_{2;2,0}}$, $\sup_x |f|_\sigma \lesssim \sqrt{\mathcal{E}_{2;2,0}}$, we apply Lemma 3 with $w = 1$ to find an upper bound of (63):

$$\begin{aligned} & \int [|f|_2 |\partial^\alpha f|_\sigma^2 + |\partial^\alpha f|_2 |f|_\sigma |\partial^\alpha f|_\sigma] dx \\ \le\ & \sqrt{\mathcal{E}_{2;2,0}} ||\partial^\alpha f||_\sigma^2 + \sqrt{\mathcal{D}_{2;2,0}} ||\partial^\alpha f||_2 ||\partial^\alpha f||_\sigma \\ \le\ & [\sqrt{\mathcal{E}_{2;2,0}} + \eta] ||\partial^\alpha f||_\sigma^2 + C_\eta \mathcal{D}_{2;2,0} \mathcal{E}_{m;l,q}(f). \end{aligned}$$

For the remaining cases of $1 \le |\alpha_1| \le |\alpha| - 1$, we always take $L^4 - L^4 - L^2$ and use Lemma 3 with small η (depending on α). By singling out the cases of $|\alpha_1| = m-1$ and $|\alpha_1| = 1$, and combining the rest of lower order terms together, we obtain

$$\begin{aligned} & \eta C_m \sum_{|\gamma|=1} ||\partial^{\alpha-\alpha_1} \partial^\gamma f||_2 ||\partial^{\alpha_1} \partial^\gamma f||_\sigma ||\partial^\alpha f||_\sigma + C_{m,\eta} ||\partial^{\alpha-\alpha_1} f||_2 ||\partial^{\alpha_1} f||_\sigma ||\partial^\alpha f||_\sigma \\ \le\ & \eta C_m [\sqrt{\mathcal{E}_{2;2,0}(f)} \sum_{|\gamma|=1, |\alpha_1|=m-1} ||\partial^\gamma \partial^{\alpha_1} f||_\sigma^2 + \sqrt{\mathcal{D}_{2;2,0}(f)} \sum_{|\gamma|=1, |\alpha_1|=1} ||\partial^\gamma \partial^{\alpha-\alpha_1} f||_2 ||\partial^\alpha f||_\sigma] \\ & + C_{m,\eta} \sqrt{\mathcal{E}_{m-1;l,q}(f)} \sqrt{\mathcal{D}_{m-1;l,q}(f)} ||\partial^\alpha f||_\sigma \\ \le\ & \eta [C_m \sqrt{\mathcal{E}_{2;2,0}(f)} + 1] \sum_{|\alpha|=m} ||\partial^\alpha f||_\sigma^2 + C_{m,\eta} [\mathcal{D}_{2;2,0}(f) \mathcal{E}_{m;l,q}(f) + \mathcal{E}_{m-1;l,q}(f) \mathcal{D}_{m-1;l,q}(f)]. \end{aligned}$$

We thus conclude the lemma by further adjusting η (depending on m). ■

Next we estimate the other nonlinear terms with the electric field.

Lemma 8 *Let $|\alpha|+|\beta|=m\geq 1$. For $\alpha_1<\alpha$ with $w=w(\alpha,\beta)$ defined in (23)*

$$\begin{aligned}&\int |w^2\partial_\beta^\alpha f_\pm \partial^{\alpha-\alpha_1}\nabla_x\phi\cdot\nabla_v\partial_\beta^{\alpha_1}f_\pm| + \int |w^2\partial_\beta^\alpha f_\pm\partial^{\alpha-\alpha_1}\nabla_x\phi\cdot\partial_\beta[v\partial^{\alpha_1}f_\pm]| \\ \lesssim\;& \eta||\partial_\beta^\alpha f_\pm||_{\sigma,w}^2 + C_\eta\left\||f_\pm|_{\sigma,\frac{w(0,0)}{\langle v\rangle^2}}\right\|_{H^{\frac34}}^2 ||\partial^\alpha\nabla_x^2\phi||_2^2 + C_\eta[||\nabla_x^2\phi||_{H^{m-1}}^2 + ||\nabla_x^4\phi||_2]\mathcal{D}_{m-1;l,q}(f).\end{aligned}$$

Proof. Note $\alpha_1<\alpha$,

$$w(\alpha,\beta)=w(\alpha_1,\beta)\langle v\rangle^{-2|\alpha|+2|\alpha_1|}=w(\alpha_1,\beta)\langle v\rangle^{-2}\langle v\rangle^{-2[|\alpha|-|\alpha_1|-1]}, \tag{64}$$

from (19), we obtain:

$$\begin{aligned}&-\int w^2\partial_\beta^\alpha f_\pm\partial^{\alpha-\alpha_1}\nabla_x\phi\cdot\nabla_v\partial_\beta^{\alpha_1}f_\pm \\ \lesssim\;& \int |w\langle v\rangle^{-1/2}\partial_\beta^\alpha f_\pm\partial^{\alpha-\alpha_1}\nabla_x\phi\cdot\langle v\rangle^{-2[|\alpha|-|\alpha_1|-1]}w(\alpha_1,\beta)\langle v\rangle^{-3/2}\nabla_v\partial_\beta^{\alpha_1}f_\pm| \\ \lesssim\;& \int |\partial_\beta^\alpha f_\pm|_{\sigma,w}|\partial^{\alpha-\alpha_1}\nabla_x\phi||\partial_\beta^{\alpha_1}f_\pm|_{\sigma,\frac{w(\alpha_1,\beta)}{\langle v\rangle^{2[|\alpha|-|\alpha_1|-1]}}}dx \end{aligned}\tag{65}$$

We now separate three cases.

Case 1. $|\alpha_1|+|\beta|=m-1$ *or* $m=1$. Since $|\alpha_1|+|\beta|+|\alpha-\alpha_1|=m$ and $\alpha_1>\alpha$, we have $|\alpha|-|\alpha_1|=1$ in this case. Taking $L^2-L^\infty-L^2$ in (65) yields a desired upper bound:

$$||\partial_\beta^\alpha f_\pm||_{\sigma,w}||\nabla_x^4\phi||_2\left\||\partial_\beta^{\alpha_1}f_\pm|_{\sigma,w(\alpha_1,\beta)}\right\|_2\lesssim\eta||\partial^\alpha f_\pm||_{\sigma,w}^2+C_\eta||\nabla_x^4\phi||_2^2\mathcal{D}_{m-1;l,q}(f).$$

For the rest of the cases, we take $L^2-L^4-L^4$ of (65) to get an upper bound:

$$\lesssim ||\partial_\beta^\alpha f_\pm||_{\sigma,w}||\partial^{\alpha-\alpha_1}\nabla_x\phi||_{H^{\frac34}}\left\||\partial_\beta^{\alpha_1}f_\pm|_{\sigma,\frac{w(\alpha_1,\beta)}{\langle v\rangle^{2[|\alpha|-|\alpha_1|-1]}}}\right\|_{H^{\frac34}}. \tag{66}$$

Case 2. $|\alpha-\alpha_1|=m\geq 2$. We have $\alpha_1=\beta=0$ now. Since $|\alpha|-|\alpha_1|-1\geq 1$ now, we further estimate (66) by ()

$$\begin{aligned}&||\partial^\alpha f_\pm||_{\sigma,w}||\partial^\alpha\nabla_x\phi||_{H^{\frac34}}\left\||f_\pm|_{\sigma,\frac{w(0,0)}{\langle v\rangle^2}}\right\|_{H^{\frac34}} \\ \lesssim\;& \eta||\partial^\alpha f_\pm||_{\sigma,w}^2+C_\eta||\partial^\alpha\nabla_x^2\phi||_2^2\left\||f_\pm|_{\sigma,\frac{w(0,0)}{\langle v\rangle^2}}\right\|_{H^{\frac34}}^2 \\ \lesssim\;& \eta||\partial^\alpha f_\pm||_{\sigma,w}^2+C_\eta\left\||f_\pm|_{\sigma,\frac{w(0,0)}{\langle v\rangle^2}}\right\|_{H^{\frac34}}^2||\partial^\alpha\nabla_x^2\phi||_2^2.\end{aligned}$$

Case 3. $|\alpha_1|+|\beta|\leq m-2$ *and* $|\alpha-\alpha_1|\leq m-1$. By $|\alpha_1|+|\alpha-\alpha_1|+|\beta|=m$, we deduce $|\alpha-\alpha_1|\geq 2$. Hence $\langle v\rangle^{-2[|\alpha|-|\alpha_1|-1]}\leq\langle v\rangle^{-2}$. We estimate (66) as

$$\begin{aligned}
&||\partial^\alpha_\beta f_\pm||_{\sigma,w}||\partial^{\alpha-\alpha_1}\nabla_x\phi||_{H^{\frac34}}\left\|\left|\partial^{\alpha_1}_\beta f_\pm\right|_{\sigma,\frac{w(\alpha_1,\beta)}{\langle v\rangle^2}}\right\|_{H^{\frac34}}\\
\lesssim\;& \eta||\partial^\alpha_\beta f_\pm||^2_{\sigma,w}+C_\eta||\partial^{\alpha-\alpha_1}\nabla_x\phi||^2_{H^1}\left\|\left|\partial^{\alpha_1}_\beta f_\pm\right|_{\sigma,\frac{w(\alpha_1,\beta)}{\langle v\rangle^2}}\right\|_{H^1}\\
\lesssim\;& \eta||\partial^\alpha_\beta f_\pm||^2_{\sigma,w}+C_\eta||\partial^{\alpha-\alpha_1}\nabla^2\phi||^2_2\mathcal{D}_{m-1;l,q}(f)\\
\lesssim\;& \eta||\partial^\alpha_\beta f_\pm||^2_{\sigma,w}+C_\eta||\nabla^2\phi||^2_{H^{m-1}}\mathcal{D}_{m-1;l,q}(f).
\end{aligned}$$

Here we have used the fact $\frac{w(\alpha_1,\beta)}{\langle v\rangle^2}\leq w(\alpha_1+\gamma,\beta)$ for $|\gamma|\leq 1$ from (23) so that

$$\begin{aligned}
\left\|\left|\partial^{\alpha_1}_\beta f_\pm\right|_{\sigma,\frac{w(\alpha_1,\beta)}{\langle v\rangle^2}}\right\|_{H^1} &= \sum_{|\gamma|\leq 1}\left\|\left|\partial^\gamma\partial^{\alpha_1}_\beta f_\pm\right|_{\sigma,\frac{w(\alpha_1,\beta)}{\langle v\rangle^2}}\right\|_2\lesssim\sum_{|\gamma|\leq 1}\left\|\left|\partial^\gamma\partial^{\alpha_1}_\beta f_\pm\right|_{\sigma,w(\alpha_1+\gamma,\beta)}\right\|_2\\
&\lesssim \mathcal{D}_{m-1;l,q}(f).
\end{aligned}$$

We now turn to the second term. Similarly, by (64) $w(\alpha,\beta)=w(\alpha_1,\beta-\mathbf{e}_i)\langle v\rangle^{-4}\langle v\rangle^{-2[|\alpha|-|\alpha_1|-1]}$, we have from (19):

$$\begin{aligned}
&\int w^2\partial^\alpha_\beta f_\pm\partial^{\alpha-\alpha_1}\nabla_x\phi\cdot\partial_\beta[v\partial^{\alpha_1}f_\pm]\\
\leq\;& |\int w^2\partial^\alpha_\beta f_\pm\partial^{\alpha-\alpha_1}\nabla_x\phi\cdot v\partial^{\alpha_1}_\beta f_\pm]|+C_\beta|\int w^2\partial^\alpha_\beta f_\pm\partial^{\alpha-\alpha_1}\nabla_x\phi\cdot\delta^{\mathbf{e}_i}_\beta\partial^{\alpha_1}_{\beta-\mathbf{e}_i}f_\pm]|\\
\leq\;& \int|w\langle v\rangle^{-1/2}\partial^{\alpha_1}_\beta f_\pm\partial^{\alpha-\alpha_1}\nabla_x\phi\cdot\frac{w(\alpha_1,\beta)}{\langle v\rangle^{2[|\alpha|-|\alpha_1|-1]}}\langle v\rangle^{-\frac32}\partial^{\alpha_1}_\beta f_\pm|\\
&+C_\beta\int|w\langle v\rangle^{-\frac12}\partial^\alpha_\beta f_\pm\partial^{\alpha-\alpha_1}\nabla_x\phi\cdot\frac{w(\alpha_1,\beta-\mathbf{e}_i)}{\langle v\rangle^{2[|\alpha|-|\alpha_1|]}}\langle v\rangle^{-\frac32}\delta^{\mathbf{e}_i}_\beta\partial^{\alpha_1}_{\beta-\mathbf{e}_i}f_\pm|\\
\leq\;& C_m\int|\partial^\alpha_\beta f_\pm|_{\sigma,w(\alpha,\beta)}|\partial^{\alpha-\alpha_1}\nabla_x\phi||\partial^{\alpha_1}_\beta f_\pm|_{\sigma,\frac{w(\alpha_1,\beta)}{\langle v\rangle^{2[|\alpha|-|\alpha_1|-1]}}}\\
&C_m\int|\partial^\alpha_\beta f_\pm|_{\sigma,w(\alpha,\beta)}|\partial^{\alpha-\alpha_1}\nabla_x\phi||\delta^{\mathbf{e}_i}_\beta\partial^{\alpha_1}_{\beta-\mathbf{e}_i}f_\pm|_{\sigma,\frac{w(\alpha_1,\beta-\mathbf{e}_i)}{\langle v\rangle^{2[|\alpha|-|\alpha_1|]}}}.
\end{aligned}$$

The first term is estimated exactly as in (66). For the second term, we note that $|\beta|\geq 1$ so that $|\alpha-\alpha_1|\leq m-1$, and $|\alpha_1|+|\beta|-1\leq m-2$. Since $\alpha_1<\alpha$

$$|\delta^{\mathbf{e}_i}_\beta\partial^{\alpha_1}_{\beta-\mathbf{e}_i}f_\pm|_{\sigma,\frac{w(\alpha_1,\beta-\mathbf{e}_i)}{\langle v\rangle^{2[|\alpha|-|\alpha_1|]}}}\leq|\delta^{\mathbf{e}_i}_\beta\partial^{\alpha_1}_{\beta-\mathbf{e}_i}f_\pm|_{\sigma,\frac{w(\alpha_1,\beta-\mathbf{e}_i)}{\langle v\rangle^2}},$$

we thus can apply case 3 above to complete the proof. ■

We also need more precise estimate for purely spatial derivatives of $E\cdot\nabla_v f_\pm$.

Lemma 9 *Let* $-\Delta\phi=\int\sqrt{\mu}[f_+-f_-]dv$ *with* $\int\phi=0$.

(1) For $|\alpha|=1,2$*, then for* $\alpha_1<\alpha$*,*

$$\begin{aligned}&\int\left|\int\partial^\alpha f_\pm\partial^{\alpha-\alpha_1}\nabla_x\phi\cdot\nabla_v\partial^{\alpha_1}f_\pm dv\right|dx+\int\left|\int\partial^\alpha f_\pm\partial^{\alpha-\alpha_1}\nabla_x\phi\cdot v\partial^{\alpha_1}f_\pm dv\right|dx\\ \lesssim\ &\sqrt{\mathcal{E}_{2;2,0}(f)}\sum_{|\alpha'|\le|\alpha|}||\partial^\alpha f||_\sigma^2.\end{aligned}$$

(2) For $|\alpha|=m\ge 3$*, then for* $\alpha_1<\alpha$*,any* $\eta>0$*,*

$$\begin{aligned}&\int\left|\int\partial^\alpha f_\pm\partial^{\alpha-\alpha_1}\nabla_x\phi\cdot\nabla_v\partial^{\alpha_1}f_\pm dv\right|dx+\int\left|\int\partial^\alpha f_\pm\partial^{\alpha-\alpha_1}\nabla_x\phi\cdot v\partial^{\alpha_1}f_\pm dv\right|dx\\ \lesssim\ &\sqrt{\mathcal{E}_{2;2,0}(f)}||\partial^\alpha f||_\sigma^2+\eta\sum_{|\alpha'|=|\alpha|}||\partial^{\alpha'}f||_\sigma^2+C_{m,\eta}\mathcal{D}_{m-1;l,0}(f)\mathcal{E}_{m-1;l,0}(f).\end{aligned}$$

Proof. We first perform integration by part in v to get:

$$\begin{aligned}&\int\left|\int\partial^\alpha f_\pm\partial^{\alpha-\alpha_1}\nabla_x\phi\cdot\nabla_v\partial^{\alpha_1}f_\pm dv\right|+\int\left|\int\partial^\alpha f_\pm\partial^{\alpha-\alpha_1}\nabla_x\phi\cdot v\partial^{\alpha_1}f_\pm dv\right| \quad (67)\\ =\ &\int\left|\int\nabla_v\partial^\alpha f_\pm\cdot\partial^{\alpha-\alpha_1}\nabla_x\phi\partial^{\alpha_1}f_\pm dv\right|dx+\int\left|\int\partial^\alpha f_\pm\partial^{\alpha-\alpha_1}\nabla_x\phi\cdot v\partial^{\alpha_1}f_\pm dv\right|dx\\ \le\ &\int[|\langle v\rangle^{-\frac32}\partial^\alpha\nabla_v f_\pm|+|\langle v\rangle^{-\frac12}\partial^\alpha f_\pm|]|\partial^{\alpha-\alpha_1}\nabla_x\phi[\langle v\rangle^{\frac32}\partial^{\alpha_1}f_\pm]|\\ \lesssim\ &\int|\partial^\alpha f_\pm|_\sigma|\partial^{\alpha-\alpha_1}\nabla_x\phi||\langle v\rangle^{3/2}\partial^{\alpha_1}f_\pm|_2dx.\end{aligned}$$

Here we have used norm (19) for $\partial^\alpha f_\pm$.

If $|\alpha|=1$, then since $\alpha_1<\alpha$ so $\alpha_1=0$. From the elliptic estimate:

$$||\partial^\alpha\nabla_x\phi||_4\lesssim||\partial^\alpha\nabla_x^2\phi||_2=||\int\sqrt{\mu}\partial^\alpha f_\pm||_2\lesssim||\partial^\alpha f||_\sigma$$

and $\sum_{|\gamma|\le 1}||\langle v\rangle^{3/2}\partial^\gamma f_\pm||_2\lesssim\sqrt{\mathcal{E}_{2;2,0}(f)}$. Since $|\langle v\rangle^{-3/2}\partial^\alpha\nabla_v f_\pm|_2\lesssim|\partial^\alpha f_\pm|_\sigma$ in (67), we take $L^2-L^4-L^4$ to get

$$\begin{aligned}(67)\ \lesssim\ &||\partial^\alpha f||_\sigma||\partial^\alpha\nabla_x\phi||_4\sum_{|\gamma|\le1}||\langle v\rangle^{3/2}\partial^\gamma f_\pm||_2 \quad (68)\\ \lesssim\ &||\partial^\alpha f||_\sigma^2\times\sum_{|\gamma|\le1}||\langle v\rangle^{3/2}\partial^\gamma f_\pm||_2\lesssim\sqrt{\mathcal{E}_{2;2,0}(f)}||\partial^\alpha f||_\sigma^2.\end{aligned}$$

If $|\alpha|=2$, the case $\alpha_1=0$ is treated as in (68) and we only need to treat the case of $|\alpha_1|=1$. Note $||\langle v\rangle^{3/2}\partial^{\alpha_1}f_\pm||_2\lesssim\sqrt{\mathcal{E}_{2;2,0}(f)}$. We take $L^2-L^\infty-L^2$ in (67) to obtain

$$\begin{aligned}(67)\ \lesssim\ &\int|\partial^\alpha f_\pm|_\sigma|\partial^{\alpha-\alpha_1}\nabla_x\phi||\langle v\rangle^{3/2}\partial^{\alpha_1}f_\pm|_2dx\\ \lesssim\ &||\partial^\alpha f_\pm||_\sigma||\nabla_x\partial^{\alpha-\alpha_1}\phi||_\infty||\langle v\rangle^{3/2}\partial^{\alpha_1}f_\pm||_2\\ \lesssim\ &\sum_{|\alpha'|\le|\alpha|}||\partial^{\alpha'}f_\pm||_\sigma^2\sqrt{\mathcal{E}_{2;2,0}(f)}. \quad (69)\end{aligned}$$

Here we have used the elliptic estimate ($|\alpha_1| = 1$):

$$||\nabla_x \partial^{\alpha-\alpha_1}\phi||_\infty \lesssim ||\nabla_x^2\phi||_\infty \lesssim \sum_{|\alpha'|\leq|\alpha|} ||\partial^{\alpha'} f||_\sigma.$$

This completes the proof of the first part of the lemma.

If $|\alpha| = m \geq 3$, the case $|\alpha_1| = 0$ is again treated in (68). When $|\alpha_1| = m-1$, as in (69), by (24), $||\langle v\rangle^{3/2}\partial^{\alpha_1} f_\pm||_2 \lesssim \mathcal{E}_{m-1;l,q}(f)$, and $||\nabla_x^2\phi||_\infty \lesssim \sqrt{\mathcal{D}_{2;2,0}(f)}$. We take $L^2 - L^\infty - L^2$ in (67) to get

$$\begin{aligned}(67) \quad &\lesssim \quad ||\partial^\alpha f_\pm||_\sigma ||\nabla_x^2\phi||_\infty ||\langle v\rangle^{3/2}\partial^{\alpha_1} f_\pm||_2 \\ &\lesssim \quad \eta||\partial^\alpha f_\pm||_\sigma^2 + C_\eta \mathcal{D}_{2;2,0}(f)\mathcal{E}_{m-1;l,q}(f).\end{aligned}$$

When $1 \leq |\alpha_1| \leq m-2$,

$$\begin{aligned}||\partial^{\alpha-\alpha_1}\nabla_x\phi||_4 \quad &\lesssim \quad ||\partial^{\alpha-\alpha_1}\nabla_x^2\phi||_2 \lesssim ||\partial^{\alpha-\alpha_1} f||_2 \lesssim \sqrt{\mathcal{D}_{m-1;l,0}(f)}, \\ \sum_{|\gamma|\leq 1} ||\langle v\rangle^{3/2}\partial^{\alpha_1+\gamma} f_\pm||_2 \quad &\lesssim \quad \sqrt{\mathcal{E}_{m-1;l,0}(f)}.\end{aligned}$$

By (69), we obtain by taking $L^2 - L^4 - L^4$ in (67):

$$\begin{aligned}&||\partial^\alpha f_\pm||_\sigma ||\partial^{\alpha-\alpha_1}\nabla_x\phi||_4 \sum_{|\gamma|\leq 1} ||\langle v\rangle^{3/2}\partial^{\alpha_1+\gamma} f_\pm||_2 \\ \lesssim \quad &\eta||\partial^\alpha f_\pm||_\sigma^2 + C_{m,\eta}\mathcal{D}_{m-1;l,0}(f)\mathcal{E}_{m-1;l,0}(f).\end{aligned}$$

This completes the proof of the lemma. ■

3 Local Solutions

Our goal is to construct a unique local-in time solution to the Vlasov-Poisson-Landau system (8) and (9) if $\mathcal{E}_{2;2,0}(f_0)$ is sufficiently small. The construction is based on an uniform energy estimate for a sequence of iterating approximate solutions. We first note, by direct computations $\sum_{ij}\partial_{ij}\Phi^{ij} = 8\pi\delta$ and the Landau collision operator has the following non-divergent form of [G1]:

$$Q(G_1, G_2) = \{\Phi^{ij} * G_1\}\partial_{ij}G_2 + 8\pi G_1 G_2$$

We start with

$$F^0(t,x,v) = \mu \quad \text{or} \quad f^0 \equiv 0. \tag{70}$$

To preserve the positivity for F^{n+1}, we design the following iterating sequence of $F_\pm^{n+1}$ as [SG3]:

$$\begin{aligned}[\partial_t + v\cdot\nabla_x \mp \nabla_x\phi^n\cdot\nabla_v]F_\pm^{n+1} \quad &= \quad Q(F_\pm^n, F_\pm^{n+1}) - 8\pi F_\pm^n(F_\pm^{n+1} - F_\pm^n) \\ &\quad\ +Q(F_\mp^n, F_\pm^{n+1}) - 8\pi F_\mp^n(F_\pm^{n+1} - F_\pm^n) \\ &= \quad \Phi^{ij} * [F_\pm^n + F_\mp^n]\partial_{ij}F_\pm^{n+1} + 8\pi\{F_\pm^n\}^2 + 8\pi F_\mp^n F_\pm^n, \\ \Delta\phi^{n+1} \quad &= \quad -\int (F_+^{n+1} - F_-^{n+1})dv \end{aligned} \tag{71}$$

with $\int_{\mathbf{T}^3}\phi^{n+1}=0$. We note that $F^n_\pm\geq 0$ implies $F^{n+1}_\pm\geq 0$ from (71). We now rewrite the above iteration in the perturbation form of $F^{n+1}=\mu+\sqrt{\mu}f^{n+1}$:

$$\begin{aligned}
& [\partial_t+v\cdot\nabla_x\mp\nabla_x\phi^n\cdot\nabla_v]f^{n+1}_\pm-Af^{n+1}_\pm\pm\nabla_x\phi^n\cdot vf^{n+1}_\pm \\
= & \mp 2\nabla_x\phi^n\cdot v\sqrt{\mu}+K_\pm f^n+\Gamma_\pm(f^n,f^{n+1}) \\
& -8\pi(f^n_\pm+f^n_\mp)\sqrt{\mu}(f^{n+1}_\pm-f^n_\pm)-16\pi\mu(f^{n+1}_\pm-f^n_\pm) \\
& -\Delta\phi^{n+1}=\int(f^{n+1}_+-f^{n+1}_-)dv. \qquad (72)
\end{aligned}$$

with $f^{n+1}|_{t=0}=f_0$. Here for $g=\binom{g_+}{g_-}$, we denote as in [G1] [SG1-2]:

$$\begin{aligned}
Ag_\pm &= \frac{2}{\sqrt{\mu}}Q(\mu,\sqrt{\mu}g_\pm), \\
K_\pm g &= \frac{1}{\sqrt{\mu}}Q(\sqrt{\mu}[g_\pm+g_\mp],\mu).
\end{aligned}$$

To solve such f^{n+1}, we can add an artificial dissipation

$$\varepsilon\{A^1f^{n+1}_\pm+\Delta_x(1+|v|^2)f^{n+1}\}$$

with

$$A^1=\frac{2}{\sqrt{\mu}}Q(\mu,\sqrt{\mu}g_\pm)\qquad\text{with }\Phi^1(u)=|u|\left(I-\frac{u\otimes u}{|u|^2}\right)\text{ in (3).}$$

This choice makes the problem strongly parabolic in both x and v with strong bound in v which justifies the moments estimates [SG2]. We shall construct f^{n+1} as $\varepsilon\to 0$ with uniform bound in ε. The procedure is standard and for notational brevity, we ignore such a regularization. We take ∂^α_β of the Landau-Poisson system (setting $\varepsilon=0$):

$$\begin{aligned}
& [\partial_t+v\cdot\nabla_x\mp\nabla_x\phi^n\cdot\nabla_v]\partial^\alpha_\beta f^{n+1}_\pm\pm[\nabla_x\phi^n\cdot v\partial^\alpha_\beta f^{n+1}_\pm]-\partial^\alpha_\beta Af^{n+1}_\pm \\
= & -\delta^{\mathbf{e}_i}_\beta\partial^{\alpha+\mathbf{e}_i}_{\beta-\mathbf{e}_i}f^{n+1}_\pm\pm\sum_{\alpha_1<\alpha}C^{\alpha_1}_\alpha\{\partial^{\alpha-\alpha_1}\nabla_x\phi^n\cdot\nabla_v\partial^{\alpha_1}_\beta f^{n+1}_\pm-\partial^{\alpha-\alpha_1}\nabla_x\phi^n\cdot\partial_\beta[v\partial^{\alpha_1}f^{n+1}_\pm]\} \\
& \mp 2\partial^\alpha_\beta[\nabla_x\phi^n\cdot v\sqrt{\mu}]+\partial^\alpha_\beta K_\pm f^n+\partial^\alpha_\beta\Gamma_\pm(f^n,f^{n+1}) \\
& -8\pi\partial^\alpha_\beta[\sqrt{\mu}(f^n_\pm+f^n_\mp)(f^{n+1}_\pm-f^n_\pm)]-16\pi\partial^\alpha_\beta[\mu(f^{n+1}_\pm-f^n_\pm)].
\end{aligned}$$

We multiplying with $e^{(2q\pm2)\phi_n}w^2$ with $w=w(\alpha,\beta)$ in (23) to get:

$$
\begin{aligned}
& [e^{\pm2(q+1)\phi^n}w^2\partial^\alpha_\beta f^{n+1}_\pm]\times\ \{[\partial_t+v\cdot\nabla_x\mp\nabla_x\phi^n\cdot\nabla_v]\partial^\alpha_\beta f^{n+1}_\pm\pm[\nabla_x\phi^n\cdot v\partial^\alpha_\beta f^{n+1}_\pm]\}\\
= & \frac{d}{dt}\{\frac{e^{\pm2(q+1)\phi^n}w^2(\partial^\alpha_\beta f^{n+1}_\pm)^2}{2}\}\mp(q+1)\phi^n_t e^{\pm2(q+1)\phi^n}w^2(\partial^\alpha_\beta f^{n+1}_\pm)^2\\
& +v\cdot\nabla_x\{\frac{e^{\pm2(q+1)\phi^n}w^2(\partial^\alpha_\beta f^{n+1}_\pm)^2}{2}\}\mp(q+1)v\cdot\nabla_x\phi^n e^{\pm2(q+1)\phi_n}(\partial^\alpha_\beta f^{n+1}_\pm)^2w^2\\
& \mp\nabla_x\phi^n\cdot\nabla_v\{\frac{e^{\pm2(q+1)\phi^n}w^2(\partial^\alpha_\beta f^{n+1}_\pm)^2}{2}\}\\
& \pm q\nabla_x\phi^n\cdot v(\partial^\alpha_\beta f^{n+1}_\pm)^2e^{\pm2(q+1)\phi^n}w^2\\
& \pm\left\{\frac{2(l-|\alpha|-|\beta|)}{1+|v|^2}\right\}\nabla_x\phi^n\cdot ve^{\pm2(q+1)\phi^n}w^2(\partial^\alpha_\beta f^{n+1}_\pm)^2\\
& \pm\nabla_x\phi^n\cdot v(\partial^\alpha_\beta f^{n+1}_\pm)^2w^2e^{\pm2(q+1)\phi^n}
\end{aligned}
\tag{73}
$$

where we have used (42). Our weight function is so designed such that there is an exact cancellation for the high momentum contributions:

$$
\begin{aligned}
& \mp(q+1)v\cdot\nabla_x\phi^n e^{\pm2(q+1)\phi_n}(\partial^\alpha_\beta f^{n+1}_\pm)^2w^2\\
& \pm q\nabla_x\phi^n\cdot ve^{\pm2(q+1)\phi^n}w^2(\partial^\alpha_\beta f^{n+1}_\pm)^2\\
& \pm\nabla_x\phi^n\cdot v(\partial^\alpha_\beta f^{n+1}_\pm)^2w^2e^{\pm2(q+1)\phi^n}\\
= & \ 0.
\end{aligned}
\tag{74}
$$

Therefore, we can rewrite (73) as:

$$
\begin{aligned}
& \frac{d}{dt}\{\frac{e^{\pm2(q+1)\phi^n}w^2(\partial^\alpha_\beta f^{n+1}_\pm)^2}{2}\}\mp(q+1)\phi^n_t e^{\pm2(q+1)\phi^n}w^2(\partial^\alpha_\beta f^{n+1}_\pm)^2\\
& +v\cdot\nabla_x\{\frac{e^{\pm2(q+1)\phi^n}w^2(\partial^\alpha_\beta f^{n+1}_\pm)^2}{2}\}\\
& \mp\nabla_x\phi^n\cdot\nabla_v\{\frac{e^{\pm2(q+1)\phi^n}w^2(\partial^\alpha_\beta f^{n+1}_\pm)^2}{2}\}\\
& \pm\left\{\frac{2(l-|\alpha|-|\beta|)}{1+|v|^2}\right\}\nabla_x\phi^n\cdot ve^{\pm2(q+1)\phi^n}w^2(\partial^\alpha_\beta f^{n+1}_\pm)^2.
\end{aligned}
\tag{75}
$$

Upon integration over $\mathbf{T}^3 \times \mathbf{R}^3$, and combining terms we obtain:

$$
\begin{aligned}
& \frac{d}{dt}\left\{\int \frac{e^{\pm 2(q+1)\phi^n} w^2 (\partial_\beta^\alpha f_\pm^{n+1})^2}{2}\right\} - \int \langle w^2 \partial_\beta^\alpha A f^{n+1}, \partial_\beta^\alpha f^{n+1}\rangle && (76)\\
= \quad & -\int e^{\pm 2(q+1)\phi^n} w^2 \delta_\beta^{\mathbf{e}_i} \partial_{\beta-\mathbf{e}_i}^{\alpha+\mathbf{e}_i} f_\pm^{n+1} \partial_\beta^\alpha f_\pm^{n+1} && (77)\\
& \pm \sum_{\alpha_1<\alpha} C_\alpha^{\alpha_1} \int e^{\pm 2(q+1)\phi^n} w^2 \partial_\beta^\alpha f_\pm^{n+1} \partial^{\alpha-\alpha_1} \nabla_x \phi^n \cdot \nabla_v \partial_\beta^{\alpha_1} f_\pm^{n+1} && (78)\\
& \mp \sum_{\alpha_1<\alpha} C_\alpha^{\alpha_1} \int e^{\pm 2(q+1)\phi^n} w^2 \partial_\beta^\alpha f_+^{n+1} \partial^{\alpha-\alpha_1} \nabla_x \phi^n \cdot \partial_\beta [v \partial^{\alpha_1} f_\pm^{n+1}] && (79)\\
& \mp \int [\frac{2(l-|\alpha|-|\beta|)}{1+|v|^2} \nabla_x \phi^n \cdot v - (q+1)\phi_t^n] e^{\pm 2(q+1)\phi^n} w^2 (\partial_\beta^\alpha f_\pm^{n+1})^2 && (80)\\
& + \int w^2 (e^{\pm 2(q+1)\phi^n} - 1) \partial_\beta^\alpha f_\pm^{n+1} \partial_\beta^\alpha A f_\pm^{n+1} && (81)\\
& \mp 2 \int e^{\pm 2(q+1)\phi^n} w^2 \nabla_x \partial^\alpha \phi^n \cdot \partial_\beta [v\sqrt{\mu}] \partial_\beta^\alpha f_\pm^{n+1} && (82)\\
& + \int w^2 e^{(2q\pm 2)\phi_n} \partial_\beta^\alpha K_\pm f^n \partial_\beta^\alpha f_\pm^{n+1} && (83)\\
& + \int w^2 e^{\pm 2(q+1)\phi^n} \partial_\beta^\alpha \Gamma_\pm (f^n, f^{n+1}) \partial_\beta^\alpha f_\pm^{n+1} && (84)\\
& -8\pi \int w^2 e^{\pm 2(q+1)\phi^n} \partial_\beta^\alpha [\sqrt{\mu}(f_\pm^n + f_\mp^n)(f_\pm^{n+1} - f_\pm^n)] \partial_\beta^\alpha f_\pm^{n+1} && (85)\\
& -16\pi \int w^2 e^{\pm 2(q+1)\phi^n} \partial_\beta^\alpha [\mu (f_\pm^{n+1} - f_\pm^n)] \partial_\beta^\alpha f_\pm^{n+1}. && (86)
\end{aligned}
$$

We now derive estimates for (76) to (86).

Lemma 10 *Assume for M sufficiently small,*

$$\mathcal{E}_{2;2,0}(f^n) \leq M.$$

(1) We have

$$
\begin{aligned}
&\mathcal{E}_{2;l,q}(f^{n+1})+\int_0^t\mathcal{D}_{2;l,q}(f^{n+1})ds\\
\le\;&\frac14\int_0^t\mathcal{D}_{2;l,q}(f^n)ds+C_l[\mathcal{E}_{2;l,q}(f_0)+\int_0^t\sum_{|\alpha|\le 2}\{||\mu^{\frac{q-1}{8}}\partial^\alpha f^n||_2^2+||\mu^{\frac{q-1}{8}}\partial^\alpha f^{n+1}||_2^2\}]\\
&+C_l\int_0^t[\sum_{|\alpha'|+|\beta'|\le 1}\left\||\partial^{\alpha'}_{\beta'}f^n|_{\sigma,\frac{w(\alpha',\beta')}{\langle v\rangle^2}}\right\|^2_{H^{\frac34}}+||\nabla_x\phi^n||_\infty+||\partial_t\phi^n||_\infty]\mathcal{E}_{2;l,q}(f^{n+1})\\
&+C_l\int_0^t\sum_{|\alpha'|+|\beta'|\le 1}\left\||\partial^{\alpha'}_{\beta'}f^{n+1}|_{\sigma,\frac{w(\alpha',\beta')}{\langle v\rangle^2}}\right\|^2_{H^{\frac34}}\mathcal{E}_{2;l,q}(f^n)\\
&+C_l\int_0^t\sqrt{\mathcal{E}_{2;l,q}(f^{n+1}-f^n)}[1+||f^n||_{H^{\frac74}}]\sqrt{\mathcal{E}_{2;l,q}(f^{n+1})}\\
&+C_l\int_0^t||f^{n+1}-f^n||_{H^{\frac74}}\sqrt{\mathcal{E}_{2;l,q}(f^n)}\sqrt{\mathcal{E}_{2;l,q}(f^{n+1})}.
\end{aligned}\tag{87}
$$

(2) For $m\ge 3$, we have

$$
\begin{aligned}
&\mathcal{E}_{m;l,q}(f^{n+1},\phi^n)+\int_0^t\mathcal{D}_{m;l,q}(f^{n+1})ds\\
\le\;&\frac14\int_0^t\mathcal{D}_{m;l,q}(f^n)ds+C_l\mathcal{E}_{m;l,q}(f_0)+C_{l,m}\int_0^t\sum_{|\alpha|=m}\{||\mu^{\frac{q-1}{8}}\partial^\alpha f^n||_2^2+||\mu^{\frac{q-1}{8}}\partial^\alpha f^{n+1}||_2^2\}\\
&+C_{l,m}\int_0^t[\sum_{|\alpha'|+|\beta'|\le[\frac m2]}\left\{\left\||\partial^{\alpha'}_{\beta'}f^n|_{\sigma,\frac{w(\alpha',\beta')}{\langle v\rangle^2}}\right\|^2_{H^{\frac34}}+\left\||\partial^{\alpha'}_{\beta'}f^{n+1}|_{\sigma,\frac{w(\alpha',\beta')}{\langle v\rangle^2}}\right\|^2_{H^{\frac34}}\right\}+||\nabla\phi^n||_\infty+||\partial_t\phi^n||]\\
&\times[\mathcal{E}_{m;l,q}(f^n)+\mathcal{E}_{m;l,q}(f^{n+1})]\\
&+C_{l,m}\int_0^t[1+||f^n||_{H^{[\frac m2]+\frac34}}]\sqrt{\mathcal{E}_{m;l,q}(f^{n+1}-f^n)}\sqrt{\mathcal{E}_{m;l,q}(f^{n+1})}\\
&+C_{l,m}\int_0^t||f^{n+1}-f^n||_{H^{[\frac m2]+\frac34}}\sqrt{\mathcal{E}_{m;l,q}(f^n)}\sqrt{\mathcal{E}_{m;l,q}(f^{n+1})}\\
&+C_{l,m}\int_0^t[\mathcal{D}_{m-1;l,q}(f^n)+\mathcal{D}_{m-1;l,q}(f^{n+1})][\mathcal{E}_{m-1;l,q}(f^n)+\mathcal{E}_{m-1;l,q}(f^{n+1})+1].
\end{aligned}\tag{88}
$$

We remark the exponential factor $\mu^{\frac{q-1}{8}}$ is chosen for convenience.
Proof. First note that $\Delta\phi^n=\int\sqrt{\mu}(f^n_+-f^n_-)dv$ and $\int\phi^n=0$ so that $||\phi^n||_\infty\lesssim M$ by the elliptic estimate so that

$$e^{\pm 2(q+1)\phi^n}\lesssim 1.$$

We shall estimate term by term in (76) to (86). In the second term of (76), we apply both Lemma 8 and 9 of [SG2]. For $\beta = 0$,

$$-\int \langle w^2(\alpha,0)\partial^\alpha A f_\pm^{n+1}, \partial^\alpha f_\pm^{n+1}\rangle dx \gtrsim ||\partial^\alpha f^{n+1}||^2_{\sigma,w(\alpha,0)} - C_m||\chi\partial^\alpha f^{n+1}||_2^2; \tag{89}$$

$\chi(v)$ being a general cutoff function in v. For $\beta \neq 0$, for any $\eta > 0$, we have

$$\begin{aligned} & -\int \langle w^2(\alpha,\beta)\partial^\alpha_\beta A f_\pm^{n+1}, \partial^\alpha_\beta f_\pm^{n+1}\rangle dx \\ \gtrsim \ & ||\partial^\alpha_\beta f^{n+1}||^2_{\sigma,w(\alpha,\beta)} - \eta D_{m;l,q}(f^{n+1}) - C_{l,\eta}\sum_{\beta'<\beta}||\partial^\alpha_{\beta'} f^{n+1}||^2_{\sigma,w(\alpha,\beta')}. \end{aligned} \tag{90}$$

From Lemma 5, we have for any $\eta > 0$ and $\beta \geq \mathbf{e}_i$

$$\begin{aligned} (77) \ & \lesssim \ ||\partial^\alpha_{\beta-\mathbf{e}_i} f^{n+1}||_{\sigma,w(\alpha,\beta-\mathbf{e}_i)}||\partial^{\alpha+\mathbf{e}_i}_{\beta-\mathbf{e}_i} f_\pm^{n+1}||_{\sigma,w(\alpha+\mathbf{e}_i,\beta-\mathbf{e}_i)} \\ & \leq \ \eta\mathcal{D}_{m;l,q}(f^{n+1}) + C_\eta||\delta^{\mathbf{e}_i}_\beta\partial^\alpha_{\beta-\mathbf{e}_i} f_\pm^{n+1}||_{\sigma,w(\alpha,\beta-\mathbf{e}_i)}. \end{aligned} \tag{91}$$

To estimate (78) and (79), from the elliptic estimate and our assumption,

$$\begin{aligned} ||\nabla^4\phi^n||_2 \ & \lesssim \ ||f^n||_{H^2} \lesssim \sqrt{M}, \\ ||\nabla^2\phi^n||_{H^{m-1}} \ & \lesssim \ ||\int\sqrt{\mu}f^n||_{H^{m-1}} \lesssim \mathcal{E}_{m-1;m-1,0}(f^n). \end{aligned}$$

We deduce from Lemma 8 that

$$\begin{aligned} & (78) + (79) \\ \lesssim \ & \eta||\partial^\alpha_\beta f^{n+1}||^2_{\sigma,w} + C_\eta\left\||f_\pm^{n+1}|_{\sigma,\frac{w(0,0)}{\langle v\rangle^2}}\right\|^2_{H^{\frac{3}{4}}}\mathcal{E}_{m;m,0}(f^n) + C_\eta[\mathcal{E}_{m-1;m-1,0}(f^n) + M]\mathcal{D}_{m-1;l,q}(f^{n+1}). \end{aligned} \tag{92}$$

In particular, if $m = 2$, we have $\mathcal{E}_{m;m,0}(f^n) \leq M$ and by (23)

$$\left\||f_\pm^{n+1}|_{\sigma,\frac{w(0,0)}{\langle v\rangle^2}}\right\|^2_{H^1} \lesssim \sum_{|\gamma|\leq 1}||\partial^\gamma f_\pm^{n+1}||^2_{\sigma,w(\gamma,0)} \lesssim \sum_{|\alpha|+|\beta|\leq 2}||\partial^\alpha_\beta f_\pm^{n+1}||^2_{\sigma,w(\alpha,\beta)},$$

so that

$$(78) + (79) \lesssim (\eta + C_\eta M)\sum_{|\alpha|+|\beta|\leq 2}||\partial^\alpha_\beta f_\pm^{n+1}||^2_{\sigma,w}. \tag{93}$$

Next, we easily control

$$(80) \leq C_{l,m}\int_0^t\{||\phi^n_t||_\infty + ||\nabla_x\phi^n||_\infty\}w^2(\partial^\alpha_\beta f_\pm^{n+1})^2. \tag{94}$$

To estimate (81), since $(e^{\pm 2(q+1)\phi^n} - 1) \lesssim \sqrt{M}$, we control from Lemmas 8-9 of [SG2]: for any $\eta > 0$,

$$\begin{aligned} (81) \ \lesssim \ & \sqrt{M}||\partial^\alpha_\beta f^{n+1}||^2_{\sigma,w(\alpha,\beta)} + C_{l,m}\sqrt{M}\sum_{|\alpha|\leq m}||\partial^\alpha f^{n+1}\chi||_2^2 \\ & +\eta D_{m;l,q}(f^{n+1}) + C_{l,m,\eta}\sum_{\beta'<\beta}||\partial^\alpha_{\beta'} f^{n+1}||^2_{\sigma,w(\alpha,\beta')}. \end{aligned} \tag{95}$$

In (82), since $0 \le q < 1$ in (23), via repeated integration by part in v, we can move all the v derivatives ∂_β out of $\partial_\beta^\alpha f^{n+1}$ to the factor $\partial_\beta[v\sqrt{\mu}w^2]$, so that

$$(82) \lesssim C_{l,m} \int \mu^{\frac{q-1}{8}} [|\partial^\alpha f^{n+1}|^2 + |\partial^\alpha f^n|^2] dv dx. \tag{96}$$

By Lemma 8 in [SG2], we have

$$\begin{aligned}(83) &\lesssim \{\eta\sqrt{\mathcal{D}_{m;l,q}(f^n)} + C_{l,m,\eta} \sum_{|\alpha|\le m} ||\chi\partial^\alpha f^n||_2\}\sqrt{\mathcal{D}_{m;l,q}(f^{n+1})} \\ &\lesssim \eta[\mathcal{D}_{m;l,q}(f^n) + \mathcal{D}_{m;l,q}(f^{n+1})] + C_{l,m,\eta} \sum_{|\alpha|\le m} ||\chi\partial^\alpha f^n||_2^2.\end{aligned} \tag{97}$$

We now turn to (85) and (86). If $|\alpha| + |\beta| \le 2$, due to the decay of $\sqrt{\mu}$ and the fact $q < 1$ in (23), by product rule and Sobolev imbedding (Lemma 3) in $\mathbf{T}^3$:

$$\begin{aligned}&||w\partial_\beta^\alpha[\sqrt{\mu}(f_\pm^n + f_\mp^n)(f_\pm^{n+1} - f_\pm^n)] + \partial_\beta^\alpha[\mu(f_\pm^{n+1} - f_\pm^n)]||_2 \\ \lesssim\;& C\{\sqrt{\mathcal{E}_{2;l,q}(f^{n+1} - f^n)}[1 + ||f^n||_{H^{\frac{7}{4}}}] + \sqrt{\mathcal{E}_{2;l,q}(f^n)}||f^{n+1} - f^n||_{H^{\frac{7}{4}}}\},\end{aligned}$$

so that for $|\alpha| + |\beta| \le 2$, we have

$$\begin{aligned}&(85) + (86) \\ \lesssim\;& C\{\sqrt{\mathcal{E}_{2;l,q}(f^{n+1} - f^n)}[1 + ||f^n||_{H^{\frac{7}{4}}}]\sqrt{\mathcal{E}_{2;l,q}(f^{n+1})} + \sqrt{\mathcal{E}_{2;l,q}(f^n)}||f^{n+1} - f^n||_{H^{\frac{7}{4}}}\}.\end{aligned} \tag{98}$$

For $|\alpha| + |\beta| = m \ge 3$, due to the decay of $\sqrt{\mu}$, we have

$$\begin{aligned}&||w\partial_\beta^\alpha[\sqrt{\mu}(f_\pm^n + f_\mp^n)(f_\pm^{n+1} - f_\pm^n)] + \partial_\beta^\alpha[\mu(f_\pm^{n+1} - f_\pm^n)]||_2 \\ \lesssim\;& C_m\{\sqrt{\mathcal{E}_{m;l,q}(f^{n+1} - f^n)}[1 + ||f^n||_{H^{[\frac{m}{2}]+\frac{3}{4}}}] + ||f^{n+1} - f^n||_{H^{[\frac{m}{2}]+\frac{3}{4}}}\sqrt{\mathcal{E}_{m;l,q}(f^n)}\}.\end{aligned} \tag{99}$$

We now prove the first part (1). Applying Proposition 6 with $g_1 = f^n$, $g_2 = f^{n+1}$, we choose η small and M further small such that $C_\eta M << 1$. By (23) and (25), $|\partial_{\beta'}^{\alpha'} f|_\sigma \lesssim |\partial_{\beta'}^{\alpha'} f|_{\sigma, \frac{w(\alpha',\beta')}{\langle v\rangle^2}}$ for $|\alpha'| + |\beta'| \le 1$. We collect terms to get

$$
\begin{aligned}
&||\partial_\beta^\alpha f^{n+1}||_{2,w}^2 + \int_0^t ||\partial_\beta^\alpha f^{n+1}||_{\sigma,w}^2 \\
\lesssim\ & C_l ||\partial_\beta^\alpha f^{n+1}(0)||_w^2 + \eta \int_0^t ||\partial_\beta^\alpha f^n||_{\sigma,w}^2 \\
&+ C_{l,\eta} \int_0^t \sum_{|\alpha|\le 2} [||\mu^{\frac{1-q}{8}} \partial^\alpha f^{n+1}||_2^2 + ||\mu^{\frac{1-q}{8}} \partial^\alpha f^n||_2^2] \\
&+ C_l \int_0^t \left\{ \sum_{|\alpha'|+|\beta'|\le 1} \left\| |\partial_{\beta'}^{\alpha'} f^n|_{\sigma, \frac{w(\alpha',\beta')}{\langle v\rangle^2}} \right\|_{H^{\frac{3}{4}}}^2 + ||\nabla_x \phi^n||_\infty + ||\partial_t \phi^n||_\infty \right\} \mathcal{E}_{2;l,q}(f^{n+1}) \\
&+ C_l \int_0^t \sum_{|\alpha'|+|\beta'|\le 1} \left\| |\partial_{\beta'}^{\alpha'} f^{n+1}|_{\sigma, \frac{w(\alpha',\beta')}{\langle v\rangle^2}} \right\|_{H^{\frac{3}{4}}}^2 \mathcal{E}_{2;l,q}(f^n) \\
&+ C_{l,\eta} \int_0^t \sum_{|\alpha'|\le 2, |\beta'|<|\beta|} ||\partial_{\beta'}^{\alpha} f^{n+1}||_{\sigma,w(\alpha',\beta')}^2 \\
&+ C_m \int_0^t \{\sqrt{\mathcal{E}_{2;l,q}(f^{n+1}-f^n)}[1+||f^n||_{H^{\frac{7}{4}}}] + \sqrt{\mathcal{E}_{2;l,q}(f^n)}||f^{n+1}-f^n||_{H^{\frac{7}{4}}}\}\sqrt{\mathcal{E}_{2;l,q}(f^{n+1})}.
\end{aligned}
$$

Here we have bounded the cutoff function χ by $\mu^{\frac{1-q}{4}}$. As in [G1], we can get rid of the contribution of $C_{l,\eta}\int_0^t \sum_{|\alpha'|\le 2,|\beta'|<|\beta|} ||\partial_{\beta'}^{\alpha} f^{n+1}||_{\sigma,w(\alpha',\beta')}^2$ which has less $v-$derivatives. Upon an induction starting from $|\beta| = 0, 1, 2$ by choosing different small value of η each time, we conclude that for any $\eta > 0$ (different value of C_l) that

$$
\begin{aligned}
&\sum_{|\alpha|+|\beta|\le 2} ||\partial_\beta^\alpha f^{n+1}||_{2,w}^2 + \int_0^t \sum_{|\alpha|+|\beta|\le m} ||\partial_\beta^\alpha f^{n+1}||_{\sigma,w}^2 \\
\lesssim\ & \eta \int_0^t \sum_{|\alpha|+|\beta|\le 2} ||\partial_\beta^\alpha f^{n+1}||_{\sigma,w}^2 + C_{l,\eta} \sum_{|\alpha|+|\beta|\le m} ||\partial_\beta^\alpha f^{n+1}(0)||_{2,w}^2 \\
&+ C_l \int_0^t \left\{ \sum_{|\alpha'|+|\beta'|\le 1} \left\| |\partial_{\beta'}^{\alpha'} f^n|_{\sigma, \frac{w(\alpha',\beta')}{\langle v\rangle^2}} \right\|_{H^{\frac{3}{4}}}^2 + ||\nabla_x \phi^n||_\infty + ||\partial_t \phi^n||_\infty \right\} \mathcal{E}_{2;l,q}(f^{n+1}) \\
&+ C_{l,\eta} \int_0^t \sum_{|\alpha'|+|\beta'|\le 1} \left\| |\partial_{\beta'}^{\alpha'} f^{n+1}|_{\sigma, \frac{w(\alpha',\beta')}{\langle v\rangle^2}} \right\|_{H^{\frac{3}{4}}}^2 \mathcal{E}_{2;l,q}(f^n) \\
&+ C_{l,\eta} \int_0^t \sum_{|\alpha|\le 2} [||\mu^{\frac{1-q}{8}} \partial^\alpha f^{n+1}||_2^2 + ||\mu^{\frac{1-q}{8}} \partial^\alpha f^n||_2^2] \\
&+ C_m \int_0^t \{\sqrt{\mathcal{E}_{2;l,q}(f^{n+1}-f^n)}[1+\sqrt{||f^n||_{H^{\frac{7}{4}}}}] + \sqrt{\mathcal{E}_{2;l,q}(f^n)}||f^{n+1}-f^n||_{H^{\frac{7}{4}}}\}\sqrt{\mathcal{E}_{2;l,q}(f^{n+1})}.
\end{aligned}
$$

We sum over $|\alpha|+|\beta| \le 2$ then choose η sufficiently small (but fixed) to conclude the first part of our lemma. We remark that M is also small but fixed.

For $|\alpha|+|\beta| = m \geq 3$, we simply put terms like $\sum_{\beta'<\beta} ||\partial_{\beta'}^{\alpha} f^{n+1}||^2_{\sigma,w(\alpha,\beta')}$ into $\mathcal{D}_{m-1;l,q}(f^{n+1})$. We apply Proposition 6 and collect terms to get

$$
\begin{aligned}
& ||\partial_\beta^\alpha f^{n+1}||^2_{2,w} + \int_0^t ||\partial_\beta^\alpha f^{n+1}||^2_{\sigma,w} \\
\lesssim \; & \eta \int_0^t [\mathcal{D}_{m;l,q}(f^n) + \mathcal{D}_{m;l,q}(f^{n+1})] + C_{l,m,\eta} ||\partial_\beta^\alpha f^{n+1}(0)||^2_{2,w} \\
& + C_{l,m,\eta} \int_0^t \sum_{|\alpha|=m} [||\mu^{\frac{1-q}{8}} \partial^\alpha f^{n+1}||_2^2 + ||\mu^{\frac{1-q}{8}} \partial^\alpha f^{n+1}||_2^2] \\
& + C_{l,m} \int_0^t [\sum_{|\alpha'|+|\beta'| \leq [\frac{m}{2}]} \left\{ \left\| |\partial_{\beta'}^{\alpha'} f^n|_{\sigma, \frac{w(\alpha',\beta')}{\langle v\rangle^2}} \right\|^2_{H^{\frac{3}{4}}} + \left\| |\partial_{\beta'}^{\alpha'} f^{n+1}|_{\sigma, \frac{w(\alpha',\beta')}{\langle v\rangle^2}} \right\|^2_{H^{\frac{3}{4}}} \right\} + ||\nabla\phi^n||_\infty + ||\partial_t \phi^n||] \\
& \times \{\mathcal{E}_{m;l,q}(f^n) + \mathcal{E}_{m;l,q}(f^{n+1})\} \\
& + C_{l,m,\eta} \int_0^t [1 + ||f^n||_{H^{[\frac{m}{2}]+\frac{3}{4}}}] \sqrt{\mathcal{E}_{m;l,q}(f^{n+1}-f^n)} \sqrt{\mathcal{E}_{m;l,q}(f^{n+1})} \\
& + C_{l,m,\eta} \int_0^t ||f^{n+1}-f^n||_{H^{[\frac{m}{2}]+\frac{3}{4}}} \sqrt{\mathcal{E}_{m;l,q}(f^{n+1})} \sqrt{\mathcal{E}_{m;l,q}(f^n)} \\
& + C_{l,m} \int_0^t \{\mathcal{E}_{m-1;l,q}(f^n) + \mathcal{E}_{m-1;l,q}(f^{n+1}) + 1\} \{\mathcal{D}_{m-1;l,q}(f^n) + \mathcal{D}_{m-1;l,q}(f^{n+1})\}.
\end{aligned}
$$

We therefore conclude the lemma by setting η sufficiently small. ■

We are now ready to construct local in time solutions by showing uniform bounds for f^n which requires the use of fractional Sobolev norms.

Lemma 11 *Assume $f_0 \in C_c^\infty$ such that $F_0 = \mu + \sqrt{\mu} f_0 > 0$ with (70).*

(1) There exist small constants $0 < T \leq 1$ and $M > 0$, such that if $\mathcal{E}_{2;2,0}(f_0, \phi_0)$ sufficiently small,

$$\mathcal{E}_{2;2,0}(f^{n+1}) + \int_0^t \mathcal{D}_{2;2,0}(f^{n+1})(s) ds \leq M. \tag{100}$$

(2) $\{f^n\}$ is Cauchy in $L^\infty([0,T], L^2_{x,v})$.

(3) There exists $C_l > 0$ such that for $0 \leq t \leq T$:

$$\mathcal{E}_{2;l,q}(f^{n+1})(t) + \int_0^t \mathcal{D}_{2;l,q}(f^{n+1})(s) ds \leq C_l \mathcal{E}_{2;l,q}(0). \tag{101}$$

(4) Assume (2) is valid. For $m \geq 3$, there exists an increasing continuous function $P_{m,l}$ with $P_{m,l}(0) = 0$ such that for $0 \leq t \leq T$:

$$\mathcal{E}_{m;l;q}(f^{n+1}) + \int_0^t \mathcal{D}_{m;l,q}(f^{n+1}) ds \leq P_{m,l}(\mathcal{E}_{m;l,q}(f_0)). \tag{102}$$

(5) $\{F^n \geq 0\}$.

Proof. To prove (1), we apply (87) with an induction over n. We assume (100) is valid for $k = 0, 1, 2, ..n$. Recall $\rho^n = \int \sqrt{\mu}[f_+^n - f_-^n]dv$ and $j^n = \int v\sqrt{\mu}[f_+^n - f_-^n]dv$. We now note that from the continuity equation of

$$\rho_t^n + \nabla_x \cdot j^n = 0,$$

we have

$$-\Delta \nabla_x \phi^n = \nabla_x \rho^n, \quad \Delta \partial_t \phi^n = \nabla_x j^n \tag{103}$$

and

$$||\partial_t \phi^n||_\infty + ||\nabla_x \phi^n||_\infty \lesssim |\partial_t \int [f_+^n - f_-^n]\sqrt{\mu}dv|_2 + |\partial_x \int [f_+^n - f_-^n]\sqrt{\mu}dv|_2 \lesssim \sqrt{M}. \tag{104}$$

We therefore deduce that for $t \leq T$

$$\int_0^T \{||\phi_t^n||_\infty + ||\nabla_x \phi^n||_\infty\}ds \lesssim \sqrt{M}.$$

It follows that

$$\int_0^t \sum \{||\mu^{\frac{1-q}{8}} \partial^\alpha f^{n+1}||_2^2 + ||\mu^{\frac{1-q}{8}} \partial^\alpha f^n||_2^2\} \lesssim \int_0^t [\mathcal{E}_{2;l,q}(f^n) + \mathcal{E}_{2;l,q}(f^{n+1})].$$

We then summarize from Lemma 10 with $l = 2$, by $||f||_{H^{\frac{7}{4}}} \lesssim \sqrt{\mathcal{E}_{2;2,0}(f^n)}$, and by (41):

$$\begin{aligned}
& \mathcal{E}_{2;2,0}(f^{n+1}) + \int_0^t \mathcal{D}_{2;2,0}(f^{n+1})ds \\
\leq\ & \frac{1}{4}\int_0^t \mathcal{D}_{2;2,0}(f^n)ds \\
& + C\mathcal{E}_{2;2,0}(f_0) + C\int_0^t [\mathcal{D}_{2;2,0}(f^n) + \sqrt{\mathcal{E}_{2;2,0}(f^n)} + 1]\mathcal{E}_{2;2,0}(f^{n+1}) \\
& + C\int_0^t [\mathcal{D}_{2;2,0}(f^{n+1}) + \sqrt{\mathcal{E}_{2;2,0}(f^n)} + 1]\mathcal{E}_{2;2,0}(f^n)] \\
\leq\ & \frac{2M}{3} + C\{\mathcal{E}_{2;l,q}(f_0) + M^{3/2}T + MT + [M + \sqrt{M} + T]\sup_{0\leq t\leq T} \mathcal{E}_{2;2,0}(f^{n+1}) + M\int_0^t \mathcal{D}_{2;2,0}(f^{n+1})\}.
\end{aligned}$$

We have used the induction hypothesis for f^n. For M and T both small, we have

$$\mathcal{E}_{2;2,0}(f^{n+1}) + \int_0^t \mathcal{D}_{2;2,0}(f^{n+1})ds \leq \frac{4M}{5} + C\mathcal{E}_{2;2,0}(f_0) < M$$

We thus deduce part (1) of the lemma for $\mathcal{E}_{2;2,0}(f_0)$ sufficiently small.

We now prove part (2). It is standard to prove $\{f^n\}$ is Cauchy in light of strong bound obtained in part (1). In particular, $\sup |vf_\pm^n|$ and $\sup |\nabla_v f_\pm^n|$ are bounded (part of $\mathcal{E}_{2;2,0}(f^n)$). We take difference in (72) to obtain

$$
\begin{aligned}
& [\partial_t + v\cdot\nabla_x \mp \nabla_x\phi^n\cdot\nabla_v][f_\pm^{n+1} - f_\pm^n] - A_\pm[f^{n+1} - f^n] \pm \nabla_x\phi^n\cdot v[f_\pm^{n+1} - f_\pm^n] \\
& \mp[\nabla_x\phi^n - \nabla_x\phi^{n-1}]\nabla_v f_\pm^n \pm [\nabla_x\phi^n - \nabla_x\phi^{n-1}]\cdot v f_\pm^n \\
= \; & \mp 2[\nabla_x\phi^n - \nabla_x\phi^{n-1}]\cdot v\sqrt{\mu} + K_\pm[f^n - f^{n-1}] + [\Gamma_\pm(f^{n+1}, f^n) - \Gamma_\pm(f^n, f^{n-1})] \\
& -8\pi\mu[f_\pm^n(f_\pm^{n+1} - f_\pm^n) - f_\pm^{n-1}(f_\pm^n - f_\pm^{n-1})] - 16\pi\sqrt{\mu}([f_\pm^{n+1} - f_\pm^n] - [f_\pm^n - f_\pm^{n-1}]) \\
& \Delta[\phi^{n+1} - \phi^n] = -\int([f_+^{n+1} - f_+^n] - [f_-^{n+1} - f_+^n])dv,
\end{aligned}
$$

By multiplying with $e^{\pm 2\phi^n}(f^{n+1} - f^n)$, ($f^{n+1}$ and f^n has the same initial value)

$$
\begin{aligned}
& \sum_\pm ||e^{\pm\phi^n}(f_\pm^{n+1} - f_\pm^n)||_2^2(t) + \int_0^t ||e^{\pm\phi^n}(f_\pm^{n+1} - f_\pm^n)||_\sigma^2(s)ds \\
\lesssim \; & \int_0^t \sum_\pm ||e^{\pm\phi^n}(f_\pm^{n+1} - f_\pm^n)||_2^2(s)ds + \int_0^t \sum_\pm ||e^{\pm\phi^n}(f_\pm^n - f_\pm^{n-1})||_2^2(s)ds.
\end{aligned}
$$

Since ϕ^n is uniformly bounded, by Lemma 4, we can repeat this process to obtain

$$
||f^{n+1} - f^n||_2^2(t) \lesssim \int_0^t ||f^n - f^{n-1}||_2^2(s)ds \lesssim \frac{t^n}{n!}.
$$

Hence $\{f^n\}$ is Cauchy and we can take the limit as $n \to \infty$ to obtain H^m solutions for all m. We denote f to be the limit of f^n. We remark, however, unlike [G1], it is impossible to establish f^n is Cauchy with respect to $\sqrt{\mathcal{E}_{2;2,0}}$, due to the presence of the electric field E.

We now turn to part (3) for which we have to make use of the sharper estimates of $H^{\frac{4}{3}}$ norms. Collecting terms, we rewrite (87) as

$$
\begin{aligned}
& \mathcal{E}_{2;l,q}(f^{n+1}) + \int_0^t \mathcal{D}_{2;l,q}(f^{n+1})ds \qquad (105) \\
\le \; & \frac{1}{4}\int_0^t \mathcal{D}_{2;l,q}(f^n)ds + C_l\mathcal{E}_{2;l,q}(f_0) + C_l\int_0^t A_n(s)[\mathcal{E}_{2;l,q}(f^{n+1}) + \mathcal{E}_{2;l,q}(f^n)]
\end{aligned}
$$

where

$$
\begin{aligned}
A_n(s) \;\equiv\; & \sum_{|\alpha'|+|\beta'|\le 1}\left\{\left\| |\partial_{\beta'}^{\alpha'} f^n|_{\sigma,\frac{w(\alpha',\beta')}{\langle v\rangle^2}}\right\|_{H^{\frac{3}{4}}}^2 + \left\| |\partial_{\beta'}^{\alpha'} f^{n+1}|_{\sigma,\frac{w(\alpha',\beta')}{\langle v\rangle^2}}\right\|_{H^{\frac{3}{4}}}^2\right\} \\
& + ||f^{n+1}||_{H^{\frac{7}{4}}} + ||f^n||_{H^{\frac{7}{4}}} + ||\nabla_x\phi^n||_\infty + ||\partial_t\phi^n||_\infty + 1
\end{aligned}
$$

The key difficulty to prove (101) is to replace $A_n(s)$ by a fixed, integrable function. First of all, in light of parts (1) and (2), we shall prove

$$
\begin{aligned}
\lim_{n\to\infty}\int_0^T A_n(s)ds \;=\; & \int_0^T [2\sum_{|\alpha'|+|\beta'|\le 1}\left\| |\partial_{\beta'}^{\alpha'} f|_{\sigma,\frac{w(\alpha',\beta')}{\langle v\rangle}}\right\|_{H^{\frac{3}{4}}}^2 + 2||f||_{H^{\frac{7}{4}}} + ||\nabla_x\phi||_\infty + ||\partial_t\phi||_\infty + 1] \\
\equiv \; & \int_0^T A(s)ds < \infty. \qquad (106)
\end{aligned}
$$

In fact, since $f^n \to f$ in L^2 from part (1), we deduce from (100):

$$\mathcal{E}_{2;2,0}(f) + \int_0^T \mathcal{D}_{2;2,0}(f) \le M.$$

Again from (100), $\sup_{0\le t\le T} \mathcal{E}_{2;2,0}(f^n)$ is uniformly bounded so that

$$\max_{0\le t\le T} ||f^n - f||_{H^{\frac{7}{4}}} \to 0 \text{ and } \int_0^T ||f^n - f||_{H^{\frac{7}{4}}} ds \to 0$$

by compact imbedding. Moreover, by (103) and (104),

$$\begin{aligned} & \max_{0\le t\le} ||\partial_t \phi^n - \partial_t \phi||_\infty + ||\nabla_x \phi^n - \nabla_x \phi||_\infty \\ \lesssim \; & [||\nabla_x j^n - \nabla_x j||_2 + ||\nabla_x \rho^n - \nabla_x \rho||_2] \\ \lesssim \; & ||f^n - f||_{H^1} \to 0, \end{aligned} \tag{107}$$

from (100) and $f^n \to f$ in L^2.

We separate $|v| \ge R$ and $|v| \le R$ to get, for any $\eta > 0$, by (23) and (25)

$$\begin{aligned} & \int_0^T \sum_{|\alpha'|+|\beta'|\le 1} \left\| |\partial_{\beta'}^{\alpha'} f^n - \partial_{\beta'}^{\alpha'} f|_{\sigma, \frac{w(\alpha',\beta')}{\langle v\rangle^2}} \right\|_{H^{\frac{3}{4}}}^2 \\ \le \; & \eta \int_0^T \sum_{|\alpha'|+|\beta'|\le 1} \left\| |\partial_{\beta'}^{\alpha'} f^n - \partial_{\beta'}^{\alpha'} f|_{\sigma, \frac{w(\alpha',\beta')}{\langle v\rangle^2}} \right\|_{H^1}^2 + C_\eta \int_0^T \sum_{|\alpha'|+|\beta'|\le 1} \left\| |\partial_{\beta'}^{\alpha'} f^n - \partial_{\beta'}^{\alpha'} f|_{\sigma, \frac{w(\alpha',\beta')}{\langle v\rangle^2}} \right\|_2^2 \\ \le \; & \eta \int_0^T \sum_{\substack{|\alpha'|+|\beta'|\le 1 \\ |\gamma|\le 1}} ||\partial^\gamma \partial_{\beta'}^{\alpha'} f^n - \partial^\gamma \partial_{\beta'}^{\alpha'} f||_{\sigma, w(\alpha'+\gamma,\beta')}^2 + C_\eta \int_0^T \sum_{|\alpha'|+|\beta'|\le 1} \left\| |\partial_{\beta'}^{\alpha'} f^n - \partial_{\beta'}^{\alpha'} f|_{\sigma, \frac{w(\alpha',\beta')}{\langle v\rangle^2}} \right\|_2^2 \\ \lesssim \; & \eta [\int_0^T \mathcal{D}_{2;2,0}(f^n) + \int_0^T \mathcal{D}_{2;2,0}(f)] + \frac{C_\eta}{R^4} \int_0^T \sum_{|\alpha'|+|\beta'|\le 1} \left\| |\partial_{\beta'}^{\alpha'} f^n - \partial_{\beta'}^{\alpha'} f|_{\sigma, w(\alpha',\beta')} \right\|_2^2 \\ & + C_\eta \int_0^T \sum_{|\alpha'|+|\beta'|\le 1} \left\| |\partial_{\beta'}^{\alpha'} f^n - \partial_{\beta'}^{\alpha'} f|_{\sigma, \frac{w(\alpha',\beta')}{\langle v\rangle^2}} \chi_{(|v|\le R)} \right\|_2^2 \\ \lesssim \; & \eta [\int_0^T \mathcal{D}_{2;2,0}(f^n) + \int_0^T \mathcal{D}_{2;2,0}(f)] + \frac{C_\eta}{R^4} [\int_0^T \mathcal{D}_{2;2,0}(f^n) + \int_0^T \mathcal{D}_{2;2,0}(f)] \\ & + C_{\eta,R} [\int_0^1 ||[\nabla_v f^n - \nabla_v f] \chi_{(|v|\le R)}||_{H^1}^2] \end{aligned} \tag{108}$$

where χ is a cutoff function. In light of $\int_0^T \mathcal{D}_{2;2,0}(f^n) + \int_0^T \mathcal{D}_{2;2,0}(f) < \infty$, and by (25), $\int_0^T ||[\nabla_v f^n - \nabla_v f]\chi_{(|v|\le R)}||_{H^2}^2$ is uniformly bounded. Hence

$$\lim_{n\to\infty} \int_0^T ||[\nabla_v f^n - \nabla_v f]\chi_{(|v|\le R)}||_{H^1}^2 \to 0$$

from $f^n \to f$ in L^2. We thus deduce that

$$\lim_{n\to\infty}\int_0^T \sum_{|\alpha'|+|\beta'|\le 1} \left\| |\partial^{\alpha'}_{\beta'} f^n - \partial^{\alpha'}_{\beta'} f|_{\sigma,\frac{w(\alpha',\beta')}{\langle v\rangle^2}} \right\|^2_{H^{\frac{3}{4}}} = 0$$

by first choosing η sufficiently small, then R sufficiently large and letting $n \to \infty$. We thus complete the proof of (106).

Now from (106), for any ε small, we can find $N(\varepsilon)$ such that for $n \ge N$,

$$\int_0^T [A_n(s) - A(s)]ds < \varepsilon. \tag{109}$$

Our strategy of establishing (101) is to separate two cases of $n \ge N$ and $n \le N$. For $n \le N$, by (105),

$$\begin{aligned} &\mathcal{E}_{2;l,q}(f^{n+1}) + \int_0^t \mathcal{D}_{2;l,q}(f^{n+1})ds \\ \le\ & \frac{1}{4}\int_0^t \mathcal{D}_{2;l,q}(f^n)ds + C_l\mathcal{E}_{2;l,q}(f_0) + C_l\int_0^t A_n(s)[\mathcal{E}_{2;l,q}(f^{n+1}) + \mathcal{E}_{2;l,q}(f^n)] \end{aligned}$$

for $1 \le n \le N-1$. We apply the Gronwall Lemma 4 to $\mathcal{E}_{2;l,q}(f^{n+1})(t)$ with $A = A_n(s)$ and

$$B = -\int_0^s \mathcal{D}_{2;l,q}(f^{n+1})d\tau + \frac{1}{4}\int_0^s \mathcal{D}_{2;l,q}(f^n)d\tau + C_l\mathcal{E}_{2;l,q}(f_0) + C_l\int_0^t A_n(s)\mathcal{E}_{2;l,q}(f^n)$$

to obtain

$$\begin{aligned} &\mathcal{E}_{2;l,q}(f^{n+1})(t) \\ \le\ & e^{\int_0^t A_n(s)ds}\int_0^s A_n(s)\times \\ & \times\left\{-\int_0^s \mathcal{D}_{2;l,q}(f^{n+1})d\tau + \frac{1}{4}\int_0^s \mathcal{D}_{2;l,q}(f^n)d\tau + C_l[\mathcal{E}_{2;l,q}(f_0) + \int_0^s A_n(s)\mathcal{E}_{2;l,q}(f^n)]\right\} \\ & -\int_0^t \mathcal{D}_{2;l,q}(f^{n+1})d\tau + \frac{1}{4}\int_0^t \mathcal{D}_{2;l,q}(f^n)d\tau + C_l\mathcal{E}_{2;l,q}(f_0) + C_l\int_0^t A_n(s)\mathcal{E}_{2;l,q}(f^n). \end{aligned}$$

By the boundedness of $\int_0^T A_n(s)$, $\int_0^t A_n(s)\mathcal{E}_{2;l,q}(f^n) \le C\sup_{0\le t\le T}\mathcal{E}_{2;l,q}(f^n)$. Collecting terms, dropping $-\int_0^s \mathcal{D}_{2;l,q}(f^{n+1})d\tau$ above, we use induction over n

to get

$$
\begin{aligned}
& \sup_{0\le t\le T}\mathcal{E}_{2;l,q}(f^{n+1})(t)+\int_0^T\mathcal{D}_{2;l,q}(f^{n+1})d\tau \\
\le\ & C_l\left\{\sup_{0\le t\le T}\mathcal{E}_{2;l,q}(f^{n})+\int_0^T\mathcal{D}_{2;l,q}(f^{n})d\tau\right\}+C_l\mathcal{E}_{2;l,q}(f_0) \\
\le\ & C_l^2\left\{\sup_{0\le t\le T}\mathcal{E}_{2;l,q}(f^{n-1})+\int_0^T\mathcal{D}_{2;l,q}(f^{n-1})d\tau\right\}+[C_l+C_l^2]\mathcal{E}_{2;l,q}(f_0) \\
& \dots \\
\le\ & C_l^{n+1}\left\{\sup_{0\le t\le T}\mathcal{E}_{2;l,q}(f^{0})+\int_0^T\mathcal{D}_{2;l,q}(f^{0})d\tau\right\}+[C_l+...C_l^{n+1}]\mathcal{E}_{2;l,q}(f_0) \\
\le\ & C_l^{N+2}\mathcal{E}_{2;l,q}(f_0), \qquad (110)
\end{aligned}
$$

as $f^0\equiv 0$ from (70). This conclude the case for $n\le N$.

For $n\ge N$, we define for $k\ge N$:

$$
\mathcal{X}_k(t)=\max_{N\le n\le k}\mathcal{E}_{2;l,q}(f^n)
$$

and our goal is to show uniform bound for $\mathcal{X}_k(t)$. By (105) and (109), for $n+1\le k$, we replace A_n by A to get

$$
\begin{aligned}
& \mathcal{E}_{2;l,q}(f^{n+1})+\int_0^t\mathcal{D}_{2;l,q}(f^{n+1})ds \\
\le\ & \frac{1}{4}\int_0^t\mathcal{D}_{2;l,q}(f^{n})ds+C_l\mathcal{E}_{2;l,q}(f_0)+C_l\int_0^t A_n(s)\mathcal{X}_k(t) \\
\le\ & \frac{1}{4}\int_0^t\mathcal{D}_{2;l,q}(f^{n})ds+C_l\mathcal{E}_{2;l,q}(f_0)+C_l\int_0^t A(s)\mathcal{X}_k(t)+C_l\varepsilon\sup_{0\le t\le T}\mathcal{X}_k(t).
\end{aligned}
$$

We iterate such an inequality $n+1$ back to $n...,N$ to get

$$
\begin{aligned}
& \mathcal{E}_{2;l,q}(f^{n+1})+\int_0^t\mathcal{D}_{2;l,q}(f^{n+1})ds \\
\le\ & \frac{1}{4}\int_0^t\mathcal{D}_{2;l,q}(f^{n})ds+C_l\{\mathcal{E}_{2;l,q}(f_0)+\int_0^t A(s)\mathcal{X}_k(t)+\varepsilon\sup_{0\le t\le T}\mathcal{X}_k(t)\} \\
\le\ & \frac{1}{4^2}\int_0^t\mathcal{D}_{2;l,q}(f^{n-1})ds+[1+\frac{1}{4}]C_l\left\{\mathcal{E}_{2;l,q}(f_0)+\int_0^t A(s)\mathcal{X}_k(t)+\varepsilon\sup_{0\le t\le T}\mathcal{X}_k(t)\right\} \\
& \dots \\
\le\ & \frac{1}{4^{n-N}}\int_0^t\mathcal{D}_{2;l,q}(f^{N})ds+\sum_1^{n+1}\frac{1}{4^j}C_l\left\{\mathcal{E}_{2;l,q}(f_0)+\int_0^t A(s)\mathcal{X}_k(t)+\varepsilon\sup_{0\le t\le T}\mathcal{X}_k(t)\right\} \\
\le\ & C_{l,N}\mathcal{E}_{2;l,q}(f_0)+C_l\left\{\int_0^t A(s)\mathcal{X}_k(t)+\varepsilon\sup_{0\le t\le T}\mathcal{X}_k(t)\right\}, \qquad (111)
\end{aligned}
$$

from the estimate for $\int_0^t \mathcal{D}_{2;l,q}(f^N)ds$. We now take $\max_{N\le n\le k}$ to get

$$\mathcal{X}_k(t) \le C_{l,N}\mathcal{E}_{2;l,q}(f_0) + C_l\left\{\int_0^t A(s)\mathcal{X}_k(t) + \varepsilon \sup_{0\le t\le T}\mathcal{X}_k(t)\right\}.$$

From the Gronwall Lemma 4, we obtain from $\int_0^T A(s)ds < +\infty$:

$$\mathcal{X}_k(t) \le C_{l,N}\mathcal{E}_{2;l,q}(f_0) + C_l\varepsilon \sup_{0\le t\le T}\mathcal{X}_k(t).$$

With ε sufficiently small we obtain

$$\sup_{0\le t\le T}\mathcal{X}_k(t) \le C_{l,N}\mathcal{E}_{2;l,q}(f_0).$$

Plugging this into (111) yields

$$\int_0^t \mathcal{D}_{2;l,q}(f^{n+1})ds \le C_{l,N}\mathcal{E}_{2;l,q}(f_0)$$

and we complete part (3) and the proof of (101).

We now turn to part (4) and (102). We shall use an induction over m. We assume that (102) is valid for $m-1$ and all l :

$$\sup_{0\le t\le T}\mathcal{E}_{m-1;l;q}(f^{n+1}(t)) + \int_0^T \mathcal{D}_{m-1;l,q}(f^{n+1})ds \le P_{m-1,l}(\mathcal{E}_{m-1;l,q}(f_0)) \quad (112)$$

for some increasing function $P_{m-1,l}(0) = 0$. Clearly, this is valid for $m-1 = 2$ in light of (101).

We follow the same argument as in the proof of (101). Collecting terms and using the induction hypothesis (112), we summarize (88) as

$$\begin{aligned}
&\mathcal{E}_{m;l,q}(f^{n+1}) + \int_0^t \mathcal{D}_{m;l,q}(f^{n+1})ds \\
\le\ & \frac{1}{4}\int_0^t \mathcal{D}_{m;l,q}(f^n)ds + \mathcal{E}_{m;l,q}(f_0) + C_{l,m}\int_0^t A_{n,m}(s)[\mathcal{E}_{m;l,q}(f^n) + \mathcal{E}_{m;l,q}(f^{n+1})] \\
& + C_{l,m}\int_0^t [\mathcal{D}_{m-1;l,q}(f^n) + \mathcal{D}_{m-1;l,q}(f^{n+1})][\mathcal{E}_{m-1;l,q}(f^n) + \mathcal{E}_{m-1;l,q}(f^{n+1}) + 1] \\
\le\ & \frac{1}{4}\int_0^t \mathcal{D}_{m;l,q}(f^n)ds + \mathcal{E}_{m;l,q}(f_0) + C_{l,m}\int_0^t A_{n,m}(s)[\mathcal{E}_{m;l,q}(f^n) + \mathcal{E}_{m;l,q}(f^{n+1})] \\
& + C_{l,m}[P^2_{m-1,l}(\mathcal{E}_{m-1;l,q}(f_0)) + P_{m-1,l}(\mathcal{E}_{m-1;l,q}(f_0))] \\
\le\ & \frac{1}{4}\int_0^t \mathcal{D}_{m;l,q}(f^n)ds + P_{m,l}(\mathcal{E}_{m;l,q}(f_0)) + C_{l,m}\int_0^t A_{n,m}(s)[\mathcal{E}_{m;l,q}(f^n) + \mathcal{E}_{m;l,q}(f^{n+1})].
\end{aligned}$$

where $P_{m,l}(z) \equiv C_{l,m}[P^2_{m-1,l}(z) + P_{m-1,l}(z)] + z$ and

$$\begin{aligned}
A_{n,m} \equiv & \sum_{|\alpha'|+|\beta'|\le[\frac{m}{2}]}\left\{\left\||\partial^{\alpha'}_{\beta'}f^n|_{\sigma,\frac{w(\alpha',\beta')}{\langle v\rangle^2}}\right\|^2_{H^{\frac{3}{4}}} + \left\||\partial^{\alpha'}_{\beta'}f^{n+1}|_{\sigma,\frac{w(\alpha',\beta')}{\langle v\rangle^2}}\right\|^2_{H^{\frac{3}{4}}}\right\} \\
& + ||\nabla\phi^n||_\infty + ||\partial_t\phi^n||_\infty + ||f^n||_{H^{[\frac{m}{2}]+\frac{3}{4}}} + ||f^{n+1}||_{H^{[\frac{m}{2}]+\frac{3}{4}}} + 1.
\end{aligned}$$

The major step is to show that

$$\lim_{n\to\infty}\int_0^T A_{n,m}(s)ds \to \int_0^T A_m(s)ds \tag{113}$$

with

$$A_m \equiv 2\sum_{|\alpha'|+|\beta'|\le[\frac{m}{2}]}\left\||\partial_{\beta'}^{\alpha'}f|_{\sigma,\frac{w(\alpha',\beta')}{\langle v\rangle^2}}\right\|_{H^{\frac34}}^2+||\nabla\phi||_\infty+||\partial_t\phi||_\infty+||f||_{H^{[\frac{m}{2}]+\frac34}}+||f||_{H^{[\frac{m}{2}]+\frac34}}+1.$$

Note $\int_0^T A_m<\infty$ by (112). Moreover, $f^n\to f$ in L^2, by $m-1>[\frac{m}{2}]+\frac34$ for $m\ge 3$,

$$\max_{0\le t\le T}||f^n-f||_{H^{[\frac{m}{2}]+\frac34}}\le \eta\max_{0\le t\le T}||f^n-f||_{H^{m-1}}+C_\eta\max_{0\le t\le T}||f^n-f||_{L^2}\to 0$$

by choosing first η small then $n\to\infty$. Similarly, as in (108),

$$\begin{aligned}&\int_0^T\sum_{|\alpha'|+|\beta'|\le[\frac{m}{2}]}\left\||\partial_{\beta'}^{\alpha'}f^n-\partial_{\beta'}^{\alpha'}f|_{\sigma,\frac{w(\alpha',\beta')}{\langle v\rangle^2}}\right\|_{H^{\frac34}}^2\\ \le\;&\eta[\int_0^T\mathcal{D}_{m-1;l,q}(f^n)+\int_0^T\mathcal{D}_{m-1;l,q}(f)]+\frac{C_\eta}{R^4}[\int_0^T\mathcal{D}_{m-1;l,q}(f^n)+\int_0^T\mathcal{D}_{m-1;l,q}(f)]\\ &+C_{\eta,R}[\int_0^T||[\nabla_vf^n-\nabla_vf]\chi_{(|v|\le R)}||_{H^{[\frac{m}{2}]}}^2].\end{aligned}$$

We note that $[\frac{m}{2}]\le m-2$ for $m\ge 3$, from (112), $\int_0^T||[\nabla_vf^n-\nabla_vf]\chi_{(|v|\le R)}||_{H^{m-1}}^2$ is bounded for fixed R and

$$\lim_{n\to\infty}\int_0^T||[\nabla_vf^n-\nabla_vf]\chi_{(|v|\le R)}||_{H^{[\frac{m}{2}]}}^2\to 0.$$

We therefore conclude that (113) is valid and (102) is proven exactly as in part (3).

We note that from the maximum principle for the original (71) $F^n>0$ for all $n=0,1,2....$ ■

We summarize the local well-posedness as $n\to\infty$.

Theorem 12 *Assume that $\mathcal{E}_{2;2,0}(f_0)$ is sufficiently small. Then there exist $0<T\le 1$ and $M>0$ small such that there is a unique solution $F=\mu+\sqrt{\mu}f>0$ with*

$$\mathcal{E}_{2;2,0}(f)(t)+\int_0^t\mathcal{D}_{2;2,0}(f)(s)ds\lesssim\mathcal{E}_2(0)\le M.$$

In general, if $0\le t\le T$, there exists an increasing continuous function $P_{m,l}(\cdot)$ with $P_{m,l}(0)=0$ such that

$$\mathcal{E}_{m;l,q}(f)(t)+\int_0^t\mathcal{D}_{m;l,q}(f)(s)ds\le P_{m,l}(\mathcal{E}_{m;l,q}(f_0))$$

The uniqueness is standard with the strong bound $\mathcal{E}_{2;2,0}(f)(t) < \infty$. To establish Theorem 12 for $\mathcal{E}_{m;l,q}(f_0) < \infty$ and $\mu + \sqrt{\mu} f_0 \geq 0$, we first choose a velocity cutoff function χ such that $(1 - \frac{1}{R}) f_0 \chi(\frac{|v|}{R})$ has compact support in v and $\mathcal{E}_{m;l,q}(f_0 \chi(\frac{|v|}{R})) < \infty$. We then choose a smooth approximation of $(1 - \frac{1}{R}) f_0 \chi(\frac{|v|}{R})$ as f_0^k. For fixed R, we can choose f_0^k such that

$$\mathcal{E}_{m;l,q}\left((1 - \frac{1}{R}) f_0 \chi(\frac{|v|}{R}) - f_0^k)\right) \to 0. \tag{114}$$

We therefore can construct a solution for the system with initial condition f_0^k thanks to Lemma 11. We finally take limits as $k \to \infty$, and $R \to \infty$ to construct a solution for the desired initial datum f_0.

4 Time Decay and Global Solution

In this section, we establish our main theorem. We first summarize the mixed x and v derivative estimates by applying Lemma 10 with $f^n = f^{n+1}$ and $\phi^n = \phi$:

Lemma 13 *Let $f_0 \in C_c^\infty$ and assume f is the solution constructed in Theorem 12. Assume for M sufficiently small,*

$$\mathcal{E}_{2;2,0}(f) \leq M,$$

(1) We have

$$\begin{aligned} & \mathcal{E}_{2;l,q}(f) + \int_0^t \mathcal{D}_{2;l,q}(f) ds \qquad (115) \\ \leq \quad & C_l \{\mathcal{E}_{2;l,q}(f_0) + \int_0^t \sum_{|\alpha| \leq 2} ||\partial^\alpha f||_\sigma^2 + \int_0^t [\mathcal{D}_{2;2,0}(f) + ||\nabla \phi||_\infty + ||\partial_t \phi||_\infty] \mathcal{E}_{2;l,q}(f) ds\}. \end{aligned}$$

(2) If $m \geq 3$, we have

$$\begin{aligned} & \mathcal{E}_{m;l,q}(f) + \int_0^t \mathcal{D}_{m;l,q}(f) ds \qquad (116) \\ \leq \quad & C_{l,m} \{\mathcal{E}_{2;l,q}(f_0) + \int_0^t [\mathcal{D}_{m-1;l,q}(f) + ||\nabla_x \phi||_\infty + ||\partial_t \phi||_\infty] \mathcal{E}_{m;l,q}(f) + \int_0^t \sum_{|\alpha| = m} ||\partial^\alpha f||_\sigma^2 \\ & + \int_0^t [\mathcal{E}_{m-1;l,q}(f) + 1] \mathcal{D}_{m-1;l,q}(f)\}. \end{aligned}$$

We note that since $f_0 \in C_c^\infty$, the sequence f_n in Lemma 11 satisfies $\sup_n \mathcal{E}_{m;l,q}(f^n) < +\infty$ for any l and any m (the bound may depends on l, m). This implies that for f^n is compact in all $\mathcal{E}_{m;l,q}$ norm so that we can take $n \to \infty$ in Lemma 10. We also have used (41) and the fact $\sum_{|\alpha| \leq 2} ||\mu^{\frac{q-1}{8}} \partial^\alpha f||^2 \lesssim ||\partial^\alpha f||_\sigma^2$.

We next investigate the pure x derivatives to control $\int_0^t ||\partial^\alpha f||_\sigma^2$ above. We take ∂^α of the Vlasov-Poisson-Landau system (8):

$$
\begin{aligned}
&[\partial_t + v\cdot\nabla_x \mp \nabla_x\phi\cdot\nabla_v]\partial^\alpha f_\pm \pm 2\nabla_x\partial^\alpha\phi\cdot v\sqrt{\mu} + L_\pm\partial^\alpha f \\
= \;& \mp\nabla_x\phi\cdot v\partial^\alpha f_\pm \mp \sum_{\alpha_1<\alpha} C_\alpha^{\alpha_1}\partial^{\alpha-\alpha_1}\nabla_x\phi\cdot v\partial^{\alpha_1} f_\pm \\
& \pm\sum_{\alpha_1<\alpha} C_\alpha^{\alpha_1}\partial^{\alpha-\alpha_1}\nabla_x\phi\cdot\nabla_v\partial^{\alpha_1} f_\pm + \partial^\alpha\Gamma_\pm(f,f).
\end{aligned}
$$

Lemma 14 *Assume Let $f_0\in C_c^\infty$ and assume f is the solution constructed in Theorem 12 with $\mathcal{E}_{2;2,0}(f)\le M$. Then for $|\alpha| = 0,1,2$:*

$$
\begin{aligned}
&\frac{d}{dt}\left[\int\sum_\pm \frac{e^{\pm 2\phi}(\partial^\alpha f_\pm)^2}{2} + \int|\nabla\partial^\alpha\phi|^2\right] + \int\langle L\partial^\alpha f,\partial^\alpha f\rangle \\
\lesssim\;& \sum_\pm\int|\phi_t|(\partial^\alpha f_\pm)^2 + \sqrt{M}\sum_{|\alpha'|\le|\alpha|}||\partial^{\alpha'} f||_\sigma^2. \qquad (117)
\end{aligned}
$$

For $|\alpha| = m\ge 3$, we have for any $\eta>0$,

$$
\begin{aligned}
&\frac{d}{dt}\left[\int\sum_\pm \frac{e^{\pm 2\phi}(\partial^\alpha f_\pm)^2}{2} + \int|\nabla\partial^\alpha\phi|^2\right] + \int\langle L\partial^\alpha f,\partial^\alpha f\rangle \\
\lesssim\;& \sum_\pm\int|\phi_t||\partial^\alpha f_\pm|^2 + \sqrt{M}\int|\partial^\alpha f_\pm|_\sigma^2 + \eta\sum_{|\alpha|=m}||\partial^{\alpha'} f||_\sigma^2 \\
&+C_{m,\eta}[\mathcal{D}_{2;2,0}(f)\mathcal{E}_{m;l,q}(f) + \{1+\mathcal{E}_{m-1;l,q}(f)\}\mathcal{D}_{m-1;l,q}(f)]. \qquad (118)
\end{aligned}
$$

Proof. We sum over $\pm$ of (76) to (86) with $f^n = f^{n+1}$ and with $w = 1$ $(l = |\alpha|+|\beta|)$. Note we can combine $-A-K = L$. From the continuity equation $\partial^\alpha\rho_t + \nabla_x\cdot\partial^\alpha j = 0$ (with $\rho = \int\sqrt{\mu}[f_+ - f_-]dv$ and $j = \int v\sqrt{\mu}[f_+ - f_-]dv$)

$$
-2\int\nabla_x\partial^\alpha\phi\cdot v\sqrt{\mu}[\partial^\alpha f_+ - \partial^\alpha f_-] = \frac{d}{dt}\int|\nabla\partial^\alpha\phi|^2,
$$

and we deduce:

$$
\begin{aligned}
& \frac{d}{dt}\left[\int \sum_{\pm} \frac{e^{\pm 2\phi}(\partial^\alpha f_\pm)^2}{2} + \int |\nabla \partial^\alpha \phi|^2\right] + \int \langle L\partial^\alpha f, \partial^\alpha f\rangle \\
= \; & \sum_{\pm} \int e^{\pm 2\phi} \phi_t (\partial^\alpha f_\pm)^2 && (119) \\
& +2\sum_{\pm} \int e^{\pm 2\phi} \nabla_x \partial^\alpha \phi \cdot v\sqrt{\mu}\partial^\alpha f_\pm (e^{\pm 2\phi} - 1) && (120) \\
& +\sum_{\pm} \int (1 - e^{\pm 2\phi})\partial^\alpha f_\pm L_\pm \partial^\alpha f && (121) \\
& +\sum_{\pm} \int e^{\pm 2\phi} \partial^\alpha f_\pm \partial^\alpha \Gamma_\pm(f,f) && (122) \\
& +\sum_{\pm,\alpha_1<\alpha} C_\alpha^{\alpha_1} \int e^{\pm 2\phi}\partial^\alpha f_\pm \partial^{\alpha-\alpha_1}\nabla_x\phi \cdot \nabla_v \partial^{\alpha_1} f_\pm && (123) \\
& -\sum_{\pm,\alpha_1<\alpha} C_\alpha^{\alpha_1} \int e^{\pm 2\phi}\partial^\alpha f_\pm \partial^{\alpha-\alpha_1}\nabla_x\phi \cdot v \partial^{\alpha_1} f_\pm. && (124)
\end{aligned}
$$

We estimate each term of (120) to (124). From $|1 - e^{\pm\phi}| \lesssim ||\phi||_\infty \lesssim \sqrt{\mathcal{E}_{2;2,0}(f)} \lesssim \sqrt{M}$, $||\nabla_x \partial^\alpha \phi||_2 \lesssim ||\partial^\alpha f||_\sigma$ and $||v\mu^{1/4}\partial^\alpha f||_2 \lesssim ||\partial^\alpha f||_\sigma$, then clearly

$$(120) \lesssim \sqrt{M}||\partial^\alpha f||_\sigma^2.$$

By Lemma 5 of [G1], we have

$$(121) \lesssim \sqrt{M}||\partial^\alpha f||_\sigma^2.$$

We apply Lemma 7 to estimate (122) and Lemma 9 to estimate (123) and (124) to conclude the proof. ■

We now establish a positivity of L in a 'differential form' [G3]. Recalling (8), we rewrite

$$
\begin{aligned}
\{\partial_t + v\cdot\nabla_x\} f_\pm \mp 2\{E\cdot v\}\sqrt{\mu} + L_\pm f &= N_\pm(f) && (125) \\
&\equiv \mp E\cdot \nabla_v f_\pm \pm \{E\cdot v\} f_\pm + \Gamma_\pm(f,f).
\end{aligned}
$$

Proposition 15 *Assume that for $0 \le t \le T$, f is the solution to the Vlasov-Poisson-Landau system (8) and (9) with*

$$\mathcal{E}_{2;2,0}(t) \le M,$$

sufficiently small. Then for $m \ge 0$, there exists a function $G(t)$ with

$$G(t) \lesssim \sqrt{\sum_{|\alpha|=m} ||\partial^\alpha f||^2}\sqrt{\sum_{|\alpha|=m} ||\partial^\alpha \nabla_x P f||^2} \qquad (126)$$

such that

$$\sum_{|\alpha|=m}[||\nabla_x\partial^\alpha Pf||_2^2+||\partial^\alpha\nabla_x E||_2^2] \tag{127}$$
$$\lesssim \frac{d}{dt}G(t)+\sum_{|\alpha|=m}[||\nabla_x\partial^\alpha(I-P)f||_\sigma^2+||\partial^\alpha(I-P)f||_\sigma^2+||\partial^\alpha N_{||}||^2].$$

Here $\partial^\alpha N_{||}$ *denotes the* L^2_v *projection of* $\partial^\alpha N_\pm(f)$ *with respect to the subspace generated by* $[\sqrt{\mu}, v_i\sqrt{\mu}, v_iv_j\sqrt{\mu}, v_i|v|^2\sqrt{\mu}]$*. Furthermore, for* ε *small,*

$$\sum_{|\alpha|=m}\int\langle L\nabla_x\partial^\alpha f,\nabla_x\partial^\alpha f\rangle dx \gtrsim \varepsilon\sum_{|\alpha|=m}\{||\nabla_x\partial^\alpha f||_\sigma^2-||(I-P)\partial^\alpha f||_\sigma^2-||\partial^\alpha N_\pm(f)||_2^2\}-\varepsilon\frac{dG}{dt}. \tag{128}$$

Proof. The proof of this lemma is now standard in light of methods developed in [G2], [G3] (with $\varepsilon=1$). We denote the kernel of L as

$$Pf=\begin{pmatrix}a_+(t,x)\sqrt{\mu}\\ a_-(t,x)\sqrt{\mu}\end{pmatrix}+\begin{pmatrix}b(t,x)\cdot v\sqrt{\mu}\\ b(t,x)\cdot v\sqrt{\mu}\end{pmatrix}+\begin{pmatrix}c(t,x)|v|^2\sqrt{\mu}\\ c(t,x)|v|^2\sqrt{\mu}\end{pmatrix},$$

so that $||\nabla_x\partial^\alpha b||_2^2+||\nabla_x\partial^\alpha c||_2^2+||\nabla_x\partial^\alpha a_\pm||_2^2\backsim||P\nabla_x\partial^\alpha f||_2^2$.

The first step is to use local conservation laws to estimate the temporal derivatives of $P\nabla_x\partial^\alpha f$ in terms of spatial derivatives. Recalling (8) and (125) we denote

$$\{\partial_t+v\cdot\nabla_x\}f_\pm+L_\pm f=\pm 2\{E\cdot v\}\sqrt{\mu}+N_\pm(f).$$

Upon taking vector inner product with $\sqrt{\mu}\binom{1}{0}, \sqrt{\mu}\binom{0}{1}, v\sqrt{\mu}\binom{1}{1}$ and $|v|^2\sqrt{\mu}\binom{1}{1}$ (the null space of L, see (34)), we obtain local conservations of masses, total momentum and total energy as in Eq. (6.5) in [G3]:

$$\begin{aligned}(\rho_0\partial_t a_\pm+\rho_2\partial_t c)+\frac{\rho_2\nabla_x\cdot b}{3} &= \langle N_\pm,\sqrt{\mu}\rangle, &(129)\\ \frac{2\rho_2\partial_t b}{3}+\frac{\rho_2\nabla_x[a_++a_-]}{3}+\frac{2\rho_4\nabla_x c}{3} &= \langle -v\cdot\nabla_x(I-P)f, v\sqrt{\mu}\begin{pmatrix}1\\1\end{pmatrix}\rangle+\langle N, v\sqrt{\mu}\begin{pmatrix}1\\1\end{pmatrix}\rangle,\\ \partial_t(\rho_2[a_++a_-]+2\rho_4 c)+\frac{2\rho_4\nabla_x\cdot b}{3} &= \langle N-v\cdot\nabla_x(I-P)f,\sqrt{\mu}\begin{pmatrix}|v|^2\\|v|^2\end{pmatrix}\rangle. &(130)\end{aligned}$$

Here $\rho_i=\int|v|^i\mu dv$ and $\pm E\cdot v\sqrt{\mu}$ makes no contribution in the process. Subtract the $+$ from $-$ parts in Eqs. (129), then take

$$\rho_4\times[\text{Eq.}(129)_++\text{Eq.}(129)_-]-\rho_2\times\text{Eq.}(130)$$

to get:

$$\begin{aligned}\rho_0\partial_t[a_+-a_-] &= \langle N_+-N_-,\sqrt{\mu}\rangle,\\ (\rho_0\rho_4-\rho_2^2)\partial_t[a_++a_-] &= \langle N_++N_-,[\rho_4-\rho_2|v|^2]\sqrt{\mu}\rangle+\rho_2\langle v\cdot\nabla_x(I-P)f,|v|^2\sqrt{\mu}\begin{pmatrix}1\\1\end{pmatrix}\rangle.\end{aligned}$$

Since $\rho_0\rho_4 - \rho_2^2 > 0, \rho_0 > 0$, we solve for $\partial_t a_\pm$ and $\partial_t\partial^\alpha a_\pm$. We then take ∂^α of Eqs. (129) to (130) to get:

$$\begin{aligned}
||\partial_t\partial^\alpha a||_2 &\lesssim ||(I-P)\nabla_x\partial^\alpha f||_2 + ||\partial^\alpha N_\pm||_2, && (131)\\
||\partial_t\partial^\alpha c||_2 &\lesssim ||\nabla_x\partial^\alpha b||_2 + ||(I-P)\nabla_x\partial^\alpha f||_\sigma + ||\partial^\alpha N_\pm||_2, && (132)\\
||\partial_t\partial^\alpha b||_2 &\lesssim ||\nabla_x\partial^\alpha a||_2 + ||\nabla_x\partial^\alpha c||_2 + ||(I-P)\nabla_x\partial^\alpha f||_\sigma + ||\partial^\alpha N_\pm||_2 && (133)
\end{aligned}$$

The next step is to use so-called macroscopic equations to estimates $a_\pm, b$ and c. In fact, following the same procedures in Lemma 6.1 [G3] (with $\varepsilon = 1$), for b and c in (132) and (133), we obtain for any $\eta > 0$:

$$\begin{aligned}
||\nabla_x\partial^\alpha b||_2^2 + ||\nabla_x\partial^\alpha c||_2^2 \le\ & \frac{dG_{bc}}{dt} + \eta||P\nabla_x\partial^\alpha f||_2^2 \qquad (134)\\
& + C_\eta[||\nabla\partial^\alpha(I-P)f||_\sigma^2 + ||\partial^\alpha(I-P)f||_\sigma^2 + ||\partial^\alpha N_\pm||_2],
\end{aligned}$$

with some $G_{bc} \lesssim ||\partial^\alpha f|| \cdot ||\partial^\alpha\nabla_x Pf||_2$. In the macroscopic equation (6.10) of [G3], $\nabla_x a$ should be replaced by $\nabla_x a_\pm \mp E$. Taking $\nabla_x\cdot$ and inner product with $a_\pm$ with revised Eq. (6.10) in [G3], and using the fact $\partial^\alpha E = -\nabla\partial^\alpha\phi$ [G4]

$$-\Delta\partial^\alpha\phi = \int[\partial^\alpha f_+ - \partial^\alpha f_-]\sqrt{\mu}dv = \langle\partial^\alpha f, \sqrt{\mu}\begin{pmatrix}1\\-1\end{pmatrix}\rangle = [\partial^\alpha a_+ - \partial^\alpha a_-]\rho_0,$$

we deduce that

$$\begin{aligned}
& ||\nabla_x\partial^\alpha a_\pm||_2^2 + ||\partial^\alpha E||_2^2 \qquad (135)\\
\le\ & \frac{dG_a}{dt} + C[||\nabla_x\partial^\alpha b||_2^2 + ||\nabla\partial^\alpha(I-P)f||_\sigma^2 + ||\partial^\alpha(I-P)f||_\sigma^2 + ||\partial^\alpha N_\pm||_2],
\end{aligned}$$

for some $G_a \lesssim ||\partial^\alpha f|| \cdot ||\partial^\alpha\nabla_x Pf||_2$. Assume $C \ge 1$. We take $2C \times (134) + (135)$ to absorb $C||\nabla_x\partial^\alpha b||_2^2$ from the right hand side:

$$\begin{aligned}
& C||\nabla_x\partial^\alpha b||_2^2 + 2C||\nabla_x\partial^\alpha c||_2^2 + ||\nabla_x\partial^\alpha a_\pm||_2^2 + ||\partial^\alpha E||_2^2\\
\le\ & \frac{d[2CG_{bc} + G_a]}{dt} + 2\eta C||P\nabla_x\partial^\alpha f||_2^2\\
& + (2CC_\eta + C)[||\nabla\partial^\alpha(I-P)f||_\sigma^2 + ||\partial^\alpha(I-P)f||_\sigma^2 + ||\partial^\alpha N_\pm||_2].
\end{aligned}$$

By choosing η sufficiently small, we obtain

$$\begin{aligned}
& ||\nabla_x\partial^\alpha b||_2^2 + ||\nabla_x\partial^\alpha c||_2^2 + ||\nabla_x\partial^\alpha a_\pm||_2^2 + ||\partial^\alpha E||_2^2\\
\lesssim \frac{d[2CG_{bc} + G_a]}{dt}\quad & + ||\nabla\partial^\alpha(I-P)f||_\sigma^2 + ||\partial^\alpha(I-P)f||_\sigma^2 + ||\partial^\alpha N_\pm||_2^2.
\end{aligned}$$

We therefore deduce the lemma by summing over $|\alpha| = m$.

The proof of (128) follows from (35) and (127):

$$\begin{aligned}
&\sum_{|\alpha|=m}\int\langle L\nabla_x\partial^\alpha f,\nabla_x\partial^\alpha f\rangle dx\\
\gtrsim\ &\sum_{|\alpha|=m}||(I-P)\nabla_x\partial^\alpha f||_\sigma^2\\
=\ &(1-\varepsilon)\sum_{|\alpha|=m}||(I-P)\nabla_x\partial^\alpha f||_\sigma^2+\varepsilon\sum_{|\alpha|=m}||(I-P)\nabla_x\partial^\alpha f||_\sigma^2\\
\gtrsim\ &(1-\varepsilon)\sum_{|\alpha|=m}||(I-P)\nabla_x\partial^\alpha f||_\sigma^2+\varepsilon\sum_{|\alpha|=m}||P\nabla_x\partial^\alpha f||_\sigma^2-\varepsilon\frac{dG}{dt}-\varepsilon\sum_{|\alpha|=m}||(I-P)\partial^\alpha f||_\sigma^2\\
&-\varepsilon\sum_{|\alpha|=m}||\partial^\alpha N_\pm(f)||_2^2\\
\gtrsim\ &\varepsilon\sum_{|\alpha|=m}||\nabla_x\partial^\alpha f||_\sigma^2-\varepsilon\frac{dG}{dt}-\varepsilon\sum_{|\alpha|=m}||(I-P)\partial^\alpha f||_\sigma^2-\varepsilon\sum_{|\alpha|=m}||\partial^\alpha N_\pm(f)||_2^2
\end{aligned}$$

for $1-\varepsilon\geq\varepsilon$. ∎

The following proposition establish the crucial decay to obtain global solution. It is important to only use up to first order derivatives of f to extract strong decay with a $\mathcal{E}_{2;2,0}(t)$ bound.

Proposition 16 *Assume that for* $0\leq t\leq T$, $\sup_{0\leq t\leq T}\mathcal{E}_{2;2,0}(t)\leq M$ *sufficiently small, and*

$$\int_0^T||\phi_t(s)||_\infty ds\leq 1. \tag{136}$$

Assume conservations laws (12), (13) and (14) are valid. Then there exists $C_l>0$ *such that*

$$\begin{aligned}
||\nabla_{t,x}\phi(t)||_\infty+\sum_{|\alpha|\leq 1}||\partial^\alpha f(t)||_2 &\lesssim C_l\left\{1+\frac{t}{4l-4}\right\}^{-2l+2}\sup_{0\leq s\leq T}\sqrt{\mathcal{E}_{2;l,0}(f(s))}\\
||\nabla_{t,x}\phi(t)||_\infty+\sum_{|\alpha|\leq 1}||\partial^\alpha f(t)||_2 &\lesssim e^{-C_l t^{\frac{2}{3}}}\sup_{0\leq s\leq T}\sqrt{\mathcal{E}_{2;l,q}(f(s))}.
\end{aligned}\tag{137}$$

Proof. Summing over $|\alpha|\leq 1$ in (117), by the Gronwall's inequality, we have

$$\begin{aligned}
&\frac{d}{dt}\left[e^{-C\int_0^t||\phi_t||_\infty(s)ds}\{\int\sum_{|\alpha|\leq 1}\frac{e^{\pm 2\phi}(\partial^\alpha f_\pm)^2}{2}+\sum_{|\alpha|\leq 1}\int|\nabla\partial^\alpha\phi|^2\}\right]\\
&+e^{-C\int_0^t||\phi_t||_\infty(s)ds}\sum_{|\alpha|\leq 1}\int\langle L\partial^\alpha f,\partial^\alpha f\rangle dx\\
\lesssim\ &\sqrt{M}e^{-C\int_0^t||\phi_t||_\infty(s)ds}\sum_{|\alpha|\leq 1}||\partial^\alpha f||_\sigma^2.
\end{aligned}$$

for some constant $C > 0$. Since $e^{-\int_0^t ||\phi_t||_\infty(s)ds} \backsim 1$, Applying (35) for $\alpha = 0$ and (128) for $|\alpha| = 0$ to L, we obtain for small $\varepsilon > 0$

$$
\begin{aligned}
\sqrt{M} \sum_{|\alpha|\leq 1} ||\partial^\alpha f||_\sigma^2 &\gtrsim \frac{d}{dt}\left[e^{-\int_0^t |\phi_t|_\infty(s)ds}\{\int \sum_{|\alpha|\leq 1} \frac{e^{\pm 2\phi}(\partial^\alpha f_\pm)^2}{2} + \sum_{|\alpha|\leq 1}\int |\nabla\partial^\alpha\phi|^2\}\right] \\
&+||(I-P)f||_\sigma^2 + \varepsilon\sum_{|\alpha|=1}||\partial^\alpha f||_\sigma^2 - \varepsilon||(I-P)f||_\sigma^2 - \varepsilon\frac{dG}{dt} - \varepsilon||N_\pm||_2^2 \\
\geq \quad & \frac{d}{dt}\left[e^{-\int_0^t ||\phi_t||_\infty(s)ds}\{\int \sum_{|\alpha|\leq 1} \frac{e^{\pm 2\phi}(\partial^\alpha f_\pm)^2}{2} + \sum_{|\alpha|\leq 1}\int |\nabla\partial^\alpha\phi|^2\} - \varepsilon G(t)\right] \\
&+\varepsilon\{||(I-P)f||_\sigma^2 + \sum_{|\alpha|=1}||\partial^\alpha f||_\sigma^2\} - \varepsilon||N_{||}||_2^2, \qquad (138)
\end{aligned}
$$

for some ε small. It is clear from Lemma 7 of [G1] that $||N_{||}||_2^2 \lesssim M||f||_\sigma^2$, and from standard arguments in [G2],

$$
||Pf||_\sigma \lesssim ||\nabla_x Pf||_\sigma \qquad (139)
$$

thanks to the conservation laws (12), (13), (14) with $\left|\int Pf\right| \lesssim \sqrt{M}||Pf||_2$ and the Poincare inequality. Since $e^{\pm 2\phi} \lesssim 1$ and $||\nabla\partial^\alpha\phi||_2 \lesssim ||\partial^\alpha f||_2$, by (126), we can choose ε small but fixed such that

$$
\begin{aligned}
Y(t) \quad &\equiv \quad e^{-\int_0^t ||\phi_t||_\infty ds}\sum_{|\alpha|\leq 1}\{\int \frac{e^{\pm 2\phi}(\partial^\alpha f_\pm)^2}{2} + \int |\nabla\partial^\alpha\phi|^2\} - \varepsilon G(t) \quad (140) \\
&\backsim \quad \sum_{|\alpha|\leq 1}||\partial^\alpha f||_2^2.
\end{aligned}
$$

By (103) and (104) with $\phi^n = \phi$ and $f^n = f$, we have

$$
||\partial_t\phi(t)||_\infty + ||\nabla_x\phi(t)||_\infty + \sum_{|\alpha|\leq 1}||\partial^\alpha f(t)||_2 \lesssim Y(t). \qquad (141)
$$

With such ε, we therefore conclude from (138) that

$$
\dot{Y} + 2\delta\sum_{|\alpha|\leq 1}||\partial^\alpha f||_\sigma^2 \lesssim \sqrt{M}\sum_{|\alpha|\leq 1}||\partial^\alpha f||_\sigma^2
$$

where $\delta = \delta(\varepsilon)$. For M sufficiently small, we finally have

$$
\dot{Y} + \delta\sum_{|\alpha|\leq 1}||\partial^\alpha f||_\sigma^2 \leq 0. \qquad (142)
$$

We now establish polynomial decay by applying the interpolation method developed in [SG1]. We observe from Holder's inequality with $p = \frac{4l-3}{4l-4}$ and

$q = 4l-3$:

$$\left\{\int |\partial^\alpha f|^2\right\} = \left\{\int \frac{1}{\langle v\rangle^{\frac{4l-4}{4l-3}}}\langle v\rangle^{\frac{4l-4}{4l-3}}|\partial^\alpha f|^2\right\}$$
$$\leq \left\{\int \frac{1}{\langle v\rangle}|\partial^\alpha f|^2\right\}^{\frac{4l-4}{4l-3}}\left\{\int \langle v\rangle^{4l-4}|\partial^\alpha f|^2\right\}^{\frac{1}{4l-3}}.$$

From (19) and an interpolation, we obtain:

$$\begin{aligned}||\partial^\alpha f||_\sigma^2 &\gtrsim \int \frac{1}{\langle v\rangle}|\partial^\alpha f|^2 dv\\ &\gtrsim \left\{\int |\partial^\alpha f|^2\right\}^{\frac{4l-3}{4l-4}}\left\{\int \langle v\rangle^{4l-4}|\partial^\alpha f|^2\right\}^{-\frac{1}{4l-4}}\\ &\gtrsim \left\{\int |\partial^\alpha f|^2\right\}^{\frac{4l-3}{4l-4}}\left\{\sup_{0\leq s\leq T}\mathcal{E}_{2;l,0}(f(s))\right\}^{-\frac{1}{4l-4}}\end{aligned}$$

as $4l-4>0$. We therefore have for some other $\delta > 0$,

$$\frac{d}{dt}Y + \delta Y^{1+\frac{1}{4l-4}}\left\{\sup_{0\leq s\leq T}\mathcal{E}_{2;l,0}(f(s))\right\}^{-\frac{1}{4l-4}} \leq 0.$$

It thus follows

$$Y^{-1-\frac{1}{4l-4}}\frac{d}{dt}Y + \delta\left\{\sup_{0\leq s\leq T}\mathcal{E}_{2;l,0}(s)\right\}^{-\frac{1}{4l-4}} \leq 0$$

and by integrating over time, we obtain:

$$(4l-4)\{Y^{-\frac{1}{4l-4}}(0) - Y^{-\frac{1}{4l-4}}(t)\} \leq -\delta t\left\{\sup_{0\leq s\leq T}\mathcal{E}_{2;l,0}(f(s))\right\}^{-\frac{1}{4l-4}}.$$

But from (140), $Y(0)\lesssim \mathcal{E}_{2;l,0}(0)\leq \sup_{0\leq s\leq T}\mathcal{E}_{2;l,0}(f(s))$, we have

$$\begin{aligned}(4l-4)Y^{-\frac{1}{4l-4}}(t) &\gtrsim \delta t\left\{\sup_{0\leq s\leq T}\mathcal{E}_{2;l,0}(f(s))\right\}^{-\frac{1}{4l-4}} + (4l-4)\{Y^{-\frac{1}{4l-4}}(0)\}\\ &\gtrsim \{\delta t + (4l-4)\}\left\{\sup_{0\leq s\leq T}\mathcal{E}_{2;l,0}(f(s))\right\}^{-\frac{1}{4l-4}}.\end{aligned}$$

By (141), we conclude by taking $(4l-4)$-th power (positive) of both sides.

We now prove the stretched exponential decay (137) by the splitting method in [SR2]. For any $\theta > 0$,

$$\begin{aligned} ||\partial^\alpha f||_\sigma^2 &\gtrsim \int \frac{1}{\langle v\rangle}|\partial^\alpha f|^2 dv = \int_{|v|\le \theta t^{1/3}} + \int_{|v|\ge\theta t^{1/3}} \\ &\ge \frac{t^{-1/3}}{\theta}\int_{|v|\le\theta t^{1/3}} |\partial^\alpha f|^2 dv \\ &= \frac{t^{-1/3}}{\theta}\int |\partial^\alpha f|^2 dv - \frac{t^{-1/3}}{\theta}\int_{|v|\ge\theta t^{1/3}} |\partial^\alpha f|^2 dv \\ &\gtrsim \frac{t^{-1/3}}{\theta} Y(t) - \frac{t^{-1/3}}{\theta}\int_{|v|\ge\theta t^{1/3}} |\partial^\alpha f|^2 dv. \end{aligned}$$

By (142), for some other $\delta > 0$, $\frac{t^{-1/3}}{\theta} \le |v|$ and $e^{q|v|^2}e^{-q\theta^2 t^{2/3}} \ge 1$, so that

$$\begin{aligned} \frac{d}{dt}Y + \frac{\delta t^{-1/3}}{\theta}Y(t) &\le \frac{t^{-1/3}}{\theta}\int_{|v|\ge\theta t^{1/3}} |\partial^\alpha f|^2 dv \\ &\le \frac{t^{-1/3}}{\theta}\int e^{q|v|^2}e^{-q\theta^2 t^{2/3}}|\partial^\alpha f|^2 dv \\ &\le \frac{t^{-1/3}}{\theta}e^{-q\theta^2 t^{2/3}}\int e^{q|v|^2}|\partial^\alpha f|^2 dv \\ &\le \frac{t^{-1/3}}{\theta}e^{-q\theta^2 t^{2/3}}\sup_{0\le s\le T}\mathcal{E}_{2;l,q}(f(s)). \end{aligned}$$

We therefore have

$$\begin{aligned} \{e^{\frac{3\delta t^{2/3}}{2\theta}}Y(t)\}' &\le e^{\frac{3\delta t^{2/3}}{2\theta}}\frac{t^{-1/3}}{\theta}e^{-q\theta^2 t^{2/3}}\sup_{0\le s\le T}\mathcal{E}_{2;l,q}(f(s)) \\ &\le e^{\frac{3\delta t^{2/3}}{2\theta}-q\theta^2 t^{2/3}}\frac{t^{-1/3}}{\theta}\sup_{0\le s\le T}\mathcal{E}_{2;l,q}(f(s)). \end{aligned}$$

By (140), $Y(0) \lesssim \sup_{0\le s\le T}\mathcal{E}_{2;l,q}(f(s))$, we obtain by integrating from 0 to $t:$

$$\begin{aligned} Y(t) &\le \left\{1+e^{-\frac{3\delta t^{2/3}}{2\theta}}\int_0^t e^{\frac{3\delta s^{2/3}}{2\theta}-q\theta^2 s^{2/3}}\frac{s^{-1/3}}{\theta}ds\right\}\sup_{0\le s\le T}\mathcal{E}_{2;l,q}(f(s)) \\ &\le C_l e^{-\frac{3\delta t^{2/3}}{2\theta}}\sup_{0\le s\le T}\mathcal{E}_{2;l,q}(f(s)), \end{aligned}$$

if θ large that $\int_0^\infty e^{[\frac{3\delta}{2\theta}-q\theta^2]s^{2/3}}\frac{s^{-1/3}}{\theta}ds < \infty$. ■

We are now ready to prove the main Theorem 1.

Proof. We first choose smooth initial data $f_0 \in C_c^\infty$ and $F_0 = \mu + \sqrt{\mu}f_0 > 0$.

Step 1. Global Small $\mathcal{E}_{2;2,0}$ Solutions.

We denote

$$T_* = \sup_{t\ge0}\left\{\mathcal{E}_{2;2,0}(f)(t) + \int_0^t \mathcal{D}_{2;2,0}(f)(s)ds \le M \text{ and } \int_0^t ||\nabla_{t,x}\phi(s)||_\infty ds \le \sqrt{M}\right\}. \tag{143}$$

Clearly $T_* > 0$ if $\mathcal{E}_{2;2,0}(f_0)$ is sufficiently small from Theorem 12. Our goal is to show $T_* = \infty$ if we further choose $\mathcal{E}_{2;2,0}(f_0)$ small.

In (117), by $\int_0^{T_*} ||\partial_t\phi||_\infty(s)ds \leq 1$, we use the standard Gronwall lemma to get from (35):

$$\begin{aligned} &\sum_{|\alpha|\leq 2}\left\{|\partial^\alpha f|^2+\int|\nabla\partial^\alpha\phi|^2\right\}+\sum_{|\alpha|\leq 2}\int_0^t ||(I-P)\partial^\alpha f||_\sigma^2 \\ \lesssim\;& \mathcal{E}_{2;2,0}(f_0)+\sqrt{M}\int_0^t \mathcal{D}_{2;0,0}(f)ds. \end{aligned}$$

As in (138), we note $||\partial^\alpha N_\pm|| \lesssim \sqrt{M}\mathcal{D}_{2;2,0}(f)$ for $|\alpha|\leq 2$ from Lemma 7 in [G1] and $-\Delta\phi = \int[f_+ - f_-]\sqrt{\mu}dv$. We apply Proposition 15 for $1\leq|\alpha|\leq 2$ with a fixed and small ε in (128) so that

$$\sum_{|\alpha|\leq 2}||\partial^\alpha f||_2^2+\sum_{1\leq|\alpha|\leq 2}\int_0^t ||\partial^\alpha f||_\sigma^2+\int_0^t ||(I-P)f||_\sigma^2 \lesssim \mathcal{E}_{2;2,0}(f_0)+\sqrt{M}\int_0^t \mathcal{D}_{2;2,0}(f)ds.$$

Thanks to conservation laws (12), (13) and (14), $||Pf||_\sigma \lesssim ||\nabla_x Pf||_\sigma$ as in (139) and we deduce

$$\sum_{|\alpha|\leq 2}||\partial^\alpha f||_2^2+\sum_{|\alpha|\leq 2}\int_0^t ||\partial^\alpha f||_\sigma^2 \lesssim \mathcal{E}_{2;2,0}(f_0)+\sqrt{M}\int_0^t \mathcal{D}_{2;2,0}(f)ds.$$

For $\sqrt{M}$ sufficiently small,

$$\sum_{|\alpha|\leq 2}||\partial^\alpha f||_2^2+\sum_{|\alpha|\leq 2}\int_0^t ||\partial^\alpha f||_\sigma^2 \lesssim \mathcal{E}_{2;2,0}(f_0). \tag{144}$$

We take a large constant $C\times(144)+(115)$ to absorb $\int_0^t\sum_{|\alpha|\leq 2}||\partial^\alpha f||_\sigma^2 ds$:

$$\begin{aligned} &\mathcal{E}_{2;l,q}(f)+\int_0^t \mathcal{D}_{2;l,q}(f)ds \\ \lesssim\;& C_l[\mathcal{E}_{2;l,q}(f_0)+\int_0^t[||\partial_t\phi(s)||_\infty+||\nabla_x\phi(s)||_\infty+\mathcal{D}_{2;2,0}(f)]\mathcal{E}_{2;l,q}(f)ds]. \end{aligned}$$

Since $\int_0^{T_*}[||\partial_t\phi(s)||_\infty+||\nabla_x\phi(s)||_\infty+\mathcal{D}_{2;2,0}(f)]ds \leq 1$, Lemma 4 implies

$$\mathcal{E}_{2;l,q}(f)+\int_0^t \mathcal{D}_{2;l,q}(f)ds \lesssim C_l\mathcal{E}_{2;l,q}(f_0). \tag{145}$$

In particular, we choose $l=2$ and $q=0$ to conclude

$$\mathcal{E}_{2;2,0}(f)(t)+\int_0^t \mathcal{D}_{2;2,0}(f)ds \lesssim \mathcal{E}_{2;2,0}(f_0).$$

Combining this bound with Proposition 16, we obtain

$$\int_0^t \{||\nabla_x \phi(s)||_\infty + ||\partial_t \phi(s)||_\infty\} ds \lesssim \sqrt{\sup_{0\le s\le T_*} \mathcal{E}_{2;2,0}(f(s))} \int_0^t \frac{ds}{1+s^2} \lesssim \sqrt{\mathcal{E}_{2;2,0}(f_0)}.$$

Upon choosing the initial condition $\mathcal{E}_{2;2,0}(f_0)$ further small, we deduce that for $0 \le t \le T_*$,

$$\mathcal{E}_{2;2,0}(f(t)) + \int_0^t \mathcal{D}_{2;2,0}(f) ds \le \frac{M}{2} < M \text{ and } \int_0^{T_*} \{||\nabla_x \phi(s)||_\infty + ||\partial_t \phi(s)||_\infty\} ds \le \frac{M}{2} < M.$$

This implies that $T_* = \infty$ and the solution is global.

Step 2. Higher Moments and Higher Regularity.

We shall prove this via a induction of the total derivatives $|\alpha| + |\beta| = m$. By (145), clearly the theorem is valid for $m = 2$.

Assume $|\alpha| = m - 1$ is valid for (30). Summing over $|\alpha| = m$ in (118), by (35), we deduce

$$\begin{aligned} & \sum_{|\alpha|=m} \left\{ ||\partial^\alpha f||^2 + \int ||\nabla \partial^\alpha \phi||^2 \right\} + \sum_{|\alpha|=m} \int_0^t ||(I-P)\partial^\alpha f||_\sigma^2 \\ \lesssim \ & \mathcal{E}_{m;l,q}(f_0) + (\sqrt{M} + \eta) \int_0^t \sum_{|\alpha|=m} ||\partial^\alpha f||_\sigma^2 \\ & + \int_0^t \mathcal{D}_{2;2,0}(f) \mathcal{E}_{m;l,q}(f) + C_{m,\eta} \int_0^t \{1 + \mathcal{E}_{m-1;l,q}(f)\} \mathcal{D}_{m-1;l,q}(f). \end{aligned}$$

We now integrate (128) with $m-1$ from 0 to t. We note that from Lemma 7 of [G1]

$$\begin{aligned} \sum_{|\alpha|\le m-1} ||\partial^\alpha N_\pm||_2^2 & \lesssim C_{m,\eta}\{1 + \mathcal{E}_{m-1;l,q}(f)\} \mathcal{D}_{m-1;l,q}(f), \\ \sum_{|\alpha|\le m-1} ||\partial^\alpha (I-P) f||_\sigma^2 & \lesssim \mathcal{D}_{m-1;l,q}(f). \end{aligned}$$

From (126) with $m-1$, we have

$$\begin{aligned} G(t) & \lesssim \sum_{|\alpha|=m} ||\partial^\alpha f(t)||^2 + \sum_{|\alpha|=m-1} ||\partial^\alpha f(t)||^2 \\ & \lesssim \sum_{|\alpha|=m} ||\partial^\alpha f(t)||^2 + \mathcal{E}_{m-1;l,q}(f(t)) \\ & \lesssim \sum_{|\alpha|=m} ||\partial^\alpha f(t)||^2 + P_{m-1,l}(\mathcal{E}_{m-1;l,q}(f_0)) \end{aligned}$$

by the induction hypothesis. Choosing ε small in (128), we obtain:

$$
\begin{aligned}
& \sum_{|\alpha|=m} ||\partial^\alpha f||_2^2 + \sum_{|\alpha|=m} \int_0^t ||\partial^\alpha f||_\sigma^2 \\
\lesssim\ & \mathcal{E}_{m;l,q}(f_0) + P_{m-1,l}(\mathcal{E}_{m;l,q}(f_0)) + (\sqrt{M}+\eta) \int_0^t \sum_{|\alpha|=m} ||\partial^{\alpha'} f||_\sigma^2 + \int_0^t \mathcal{D}_{2;2,0}(f)\mathcal{E}_{m;l,q}(f) \\
& + C_{l,m,\eta} \int_0^t \{1 + \mathcal{E}_{m-1;l,q}(f)\} \mathcal{D}_{m-1;l,q}(f) \\
\lesssim\ & \mathcal{E}_{m;l,q}(f_0) + (\sqrt{M}+\eta) \sum_{|\alpha|=m} ||\partial^{\alpha'} f||_\sigma^2 + \int_0^t \mathcal{D}_{2,;2,0}(f)\mathcal{E}_{m;l,q}(f) \\
& + C_{l,m,\eta}[1 + P_{m-1,l}(\mathcal{E}_{m;l,q}(f_0))] P_{m-1,l}(\mathcal{E}_{m;l,q}(f_0))
\end{aligned}
$$

Here $P_{m-1,l}$ is a continuous, increasing function from the induction hypothesis. For M, η sufficiently small

$$
\begin{aligned}
& \sum_{|\alpha|=m} \left\{ |\partial^\alpha f|^2 + \int |\nabla \partial^\alpha \phi|^2 \right\} + \sum_{|\alpha|=m} \int_0^t ||\partial^\alpha f||_\sigma^2 \\
\lesssim\ & C_{l,m}[1 + P_{m-1,l}(\mathcal{E}_{m;l,q}(f_0))][\mathcal{E}_{m;l,q}(f_0) + P_{m-1,l}(\mathcal{E}_{m;l,q}(f_0))] \\
& + \int_0^t \mathcal{D}_{2;2,0}(f)\mathcal{E}_{m;l,q}(f),
\end{aligned} \tag{146}
$$

where we have used $\mathcal{E}_{m-1;l,q}(f_0) \leq \mathcal{E}_{m;l,q}(f_0), P_{m-1,l}(\mathcal{E}_{m-1;l,q}(f_0)) \leq P_{m-1,l}(\mathcal{E}_{m;l,q}(f_0))$.

Multiplying a large constant $C \times (146) + (116)$ to absorb $\int_0^t \sum_{|\alpha|=m} ||\partial^\alpha f||_\sigma^2$ in (116), we obtain:

$$
\begin{aligned}
\mathcal{E}_{m;l,q}(f) + \int_0^t \mathcal{D}_{m;l,q}(f) \quad \lesssim\quad & C_{l,m}[1 + P_{m-1,l}(\mathcal{E}_{m;l,q}(f_0))][\mathcal{E}_{m;l,q}(f_0) + P_{m-1,l}(\mathcal{E}_{m;l,q}(f_0))] \\
& + C_{l,m} \int_0^t \mathcal{D}_{m-1;l,q}(f)\mathcal{E}_{m;l,q}(f).
\end{aligned}
$$

We use Gronwall Lemma 4 with $\int_0^t \mathcal{D}_{m-1;l,q}(f) ds \lesssim P_{m-1,l}(\mathcal{E}_{m-1;l,q}(f_0))$ to get

$$
\begin{aligned}
& \mathcal{E}_{m;l,q}(f) + \int_0^t \mathcal{D}_{m;l,q}(f) \\
\lesssim\ & C_{l,m} e^{C_{l,m} P_{m-1}(\mathcal{E}_{m;l,q}(f_0))} [1 + P_{m-1,l}(\mathcal{E}_{m;l,q}(f_0))][\mathcal{E}_{m;l,q}(f_0) + P_{m-1,l}(\mathcal{E}_{m;l,q}(f_0))] \\
\equiv\ & P_{m,l}(\mathcal{E}_{m;l,q}(f_0)).
\end{aligned}
$$

This concludes the theorem for $f_0 \in C_c^\infty$. For a general datum $f_0 \in \mathcal{E}_{m;l,q}$ we can use a sequence of smooth approximation f_0^k as in (114) and take a limit. ■

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5 References

[AB] Arsen'ev, A. A., Buryak, O.E.: On the connection between a solution of the Boltzmann equation and a solution of the Landau-Fokker-Planck equation. Math. USSR. Sbornik 69 (2), 465-478 (1991).

[AV] Alexandre, R.; Villani, C. On the Landau approximation in plasma physics. Ann. Inst. H. Poincaré Anal. Non Linéaire 21 (2004), no. 1, 61–95.

[CDH] Chen, Y.; Desvillettes, L.; He, L.: Smoothing effects for classical solutions of the full Landau equation. Arch. Ration. Mech. Anal. 193 (2009), no. 1, 21–55,

[DL] Degond, P., Lemou, M.: Dispersion relations for the linearized Fokker-Planck equation. Arch. Rational Mech. Anal. 138 (2), 137-167 (1997).

[DV] Desvillettes, L.; Villani, C.: On the trend to global equilibrium for spatially inhomogeneous kinetic systems: the Boltzmann equation. Invent. Math. 159 (2005), no. 2, 245–316.

[G1] Guo, Y.: The Landau equation in a periodic box. Commun. Math. Phys., 231 (2002) 3, 391-434.

[G2] Guo, Y.: The Vlasov-Poisson-Boltzmann system near Maxwellians. Comm. Pure Appl. Math., Vol. LV, (2002)1104-1135.

[G3] Guo, Y.: Boltzmann diffusive limit beyond the Navier-Stokes approximation. Comm. Pure Appl. Math. 59 (2006), no. 5, 626–687. (Erratum) 60 (2007), no. 2, 291–293.

[G4] Guo, Y.: Classical solutions to the Boltzmann equation for molecules with an angular cutoff. Arch. Ration. Mech. Anal. 169 (2003), no. 4, 305–353.

[G5] Guo, Y. The Vlasov-Maxwell-Boltzmann system near Maxwellians. Invent. Math. 153 (2003), no. 3, 593–630.

[GH] Guo, Y., Hadzic, M.: Stability of Stefan problem with surface tension (I). Comm. PDE, 1532-4133, 35, 2, (2010), 201 – 244.

[GL] Glassey, R.:The Cauchy Problems in Kinetic Theory, SIAM, 1996.

[GT] Guo, Y., Tice, I.:Decay of viscous surface waves without surface tension. arXiv:1011.5179.

[GS] Guo, Y.; Strain, R.M.: Momentum Regularity and Stability of the Relativistic Vlasov-Maxwell-Boltzmann System. arXiv:1012.1158

[GrS1] Gressman, T. P., Strain, R. S.: Global Classical solutions of the Boltzmann equation without angular cut-off. arXiv:1011.5441v1.

[GrS2] Gressman, T. P., Strain, R. S.: Global Classical solutions of the Boltzmann equation with Long-Range interactions. Proc. Nat. Acad. Sci. U. S. A. March 30, 2010; 107 (13), 5744-5749.

[H] Hilton, F.: Collisional transport in plasma. Handbook of Plasma Physics. (1) Amsterdam: North-Holland, 1983.

[Ha] Hadzic, M.: Orthogonality conditions and asymptotic stability in the Stefan problem with surface tension. arXiv:1101.5177

[HY] Hsiao, L.; Yu, H.: On the Cauchy problem of the Boltzmann and Landau equations with soft potentials. Quart. Appl. Math. 65 (2007), no. 2, 281–315.

[L] Lions, P-L., On Boltzmann and Landau equations. Phil Trans. R. Soc. Lond. A 346, 191-204 (1994).

[SG1] Strain, R. M.; Guo, Y.: Almost exponential decay near Maxwellian. Comm. Partial Differential Equations 31 (2006), no. 1-3, 417–429.

[SG2] Strain, R. M.; Guo, Y.:Exponential decay for soft potentials near Maxwellian. Arch. Ration. Mech. Anal. 187 (2008), no. 2, 287–339.

[SG3] Strain, R. M.; Guo, Y.: Stability of the relativistic Maxwellian in a collisional plasma. Comm. Math. Phys. 251 (2004), no. 2, 263–320.

[V] Villani, C.: On the Landau equation: Weak stability, global existence. Adv. Diff. Eq. 1 (5), 793-816.

[Z1] Zhan, M.-Q.: Local existence of solutions to the Landau-Maxwell system. Math. Methods Appl. Sci. 17 (1994), no. 8, 613–641.

[Z2] Zhan, M.-Q.: Local existence of classical solutions to the Landau equations. Transport Theory Statist. Phys. 23 (1994), no. 4, 479–499.