Convergence of variational approximation schemes for three dimensional elastodynamics with polyconvex energy *

Alexey Miroshnikov† and Athanasios E. Tzavaras ‡

Abstract

We consider a variational scheme developed in [10] that approximates the equations describing the dynamics of three dimensional motions for isotropic elastic materials; these form a system of conservation laws. We establish the convergence of the time-continuous interpolates constructed in the scheme to a smooth solution of the elastodynamics system by adapting the relative entropy method to the subject of time-discretized approximations and employing the method in an environment with L^p-theory bounds.

keywords: nonlinear second-order hyperbolic equations, nonlinear elasticity, mechanics of deformable bodies, variational approximation scheme.

1 Introduction

The equations of nonlinear elasticity are the system

$$y_{tt} = \operatorname{div} S(\nabla y) \tag{1}$$

where $y : \Omega \times \mathbb{R}^+ \to \mathbb{R}^3$ stands for the displacement and S for the Piola-Kirchhoff stress tensor. (1) can be expressed as a system of conservation laws,

$$\begin{aligned} \partial_t v_i &= \partial_\alpha S_{i\alpha}(F) \\ \partial_t F_{i\alpha} &= \partial_\alpha v_i, \end{aligned} \tag{2}$$

for the velocity $v_i = \partial_t y$ and the deformation gradient $F = \nabla y$. The property of F being a gradient is equivalent to the differential constraints

$$\partial_\beta F_{i\alpha} - \partial_\alpha F_{i\beta} = 0. \tag{3}$$

*Research partially supported by the EU FP7-REGPOT project "Archimedes Center for Modeling, Analysis and Computation", the EU EST-project "Differential Equations and Applications in Science and Engineering", and the National Science Foundation.

†Department of Mathematics, University of Maryland, College Park, USA

‡Department of Applied Mathematics, University of Crete, Heraklion, Greece.

Such constraints (3) are involutions (see [7]) and propagate from holding at $t = 0$ to holding at later times.

The Piola-Kirchhoff stress S is expressed as the gradient of the stored energy

$$S(F) = \frac{\partial W}{\partial F}(F).$$

(the corresponding constitutive theory is often referred to as hyperelasticity) where the stored-energy function $W(F) : M^{3\times 3}_{+} \to [0,\infty)$ is assumed to be invariant under rotations, reflecting the requirement of frame-indifference. Convexity of the stored energy is too restrictive as an assumption and has been replaced by various weaker conditions familiar from the theory of elastostatics (see [1, 2] and [3] for a recent survey). A commonly employed assumption is that of polyconvexity, postulating that

$$W(F) = G \circ \Phi(F)$$

where $\Phi(F) := (F, \operatorname{cof} F, \det F)$ is the vector of null-Lagrangians and $G = G(F, Z, w) = G(\Xi)$ is a convex function of $\Xi \in \mathbb{R}^{19}$; this encompasses certain physically realistic models (e.g. [5, Sec 4.9, 4.10].

Substantial progress was achieved in handling the lack of convexity in elastostatics starting with the work of J.Ball [1]. The analysis is less developed in elastodynamics. We refer to [9] for local existence of smooth solutions, and to Dafermos [6, 8] for a discussion of uniqueness when convexity of the stored energy is lacking. The existence of global weak solutions is an open problem, except in one-space dimension, see DiPerna [12]. Construction of entropic measure valued solutions has been achieved in Demoulini-Stuart-Tzavaras [10] using a variational approximation method associated with a time-discretized scheme.

The variational approximation scheme in [10] is developed for the equations of polyconvex elastodynamics, it dissipates the mechanical energy and it establishes a certain link between the theory of elastostatics and the equations of elastodynamics. The analysis is based on the observation of T. Qin[14] that for three-dimensional elastodynamics null-Lagrangians $\Phi^A(F)$, $A = 1, \ldots, 19$ satisfy nonlinear transport identities

$$\partial_t \Phi^A(F) = \partial_t \left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) v_i \right)$$

which allow to view the equations of elasticity (2) as constrained evolution of an enlarged system

$$\begin{aligned} \partial_t\, v_i &= \partial_\alpha \Big(\frac{\partial G}{\partial \Xi^A}(\Xi)\, \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) \Big) \\ \partial_t\, \Xi^A &= \partial_\alpha \Big(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)\, v_i \Big). \end{aligned} \tag{4}$$

The extension has the following property: if $F(\cdot, 0)$ is a gradient and $\Xi(\cdot, 0) = \Phi(F(\cdot, 0))$, then $F(\cdot, t)$ remains a gradient and $\Xi(\cdot, t) = \Phi(F(\cdot, t)), \forall t$ and (v, F)

solves (2). The extension admits entropy pair

$$\partial_t\Big(\frac{|v|^2}{2}+G(\Xi)\Big)-\partial_\alpha\Big(v_i\,\frac{\partial G}{\partial \Xi_A}(\Xi)\,\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)\Big)=0 \tag{5}$$

and is thus symmetrizable.

In [10] the authors work with periodic solutions on the torus $\Omega:=\mathbb{T}^3$ and develop a variational approximation method based on the time-discretization of (4): given time-step $h>0$ and initial data (v^0,Ξ^0) the scheme provides the sequence of iterates (v^j,Ξ^j), $j\geqslant 1$ such that

$$\begin{aligned}\frac{v_i^j-v_i^{j-1}}{h}&=\partial_\alpha\Big(\frac{\partial G}{\partial \Xi^A}(\Xi^j)\;\frac{\partial \Phi^A}{\partial F_{i\alpha}}\;\big(F^{j-1}\big)\Big)\\ \frac{\big(\Xi^j-\Xi^{j-1}\big)_A}{h}&=\partial_\alpha\Big(\frac{\partial \Phi^A}{\partial F_{i\alpha}}\;\big(F^{j-1}\big)\;v_i^j\Big).\end{aligned}\qquad \text{in}\quad \mathcal{D}'(\Omega) \tag{6}$$

Moreover, the iterates (v^j,Ξ^j) induce time-continues approximate solutions $\Theta^{(h)}=(V^{(h)},\Xi^{(h)})$ which, in turn, generate a measure-valued solution of the equations of polyconvex elastodynamics.

The objective of this article is to prove that approximate solutions $\Theta^{(h)}$ constructed in [10] converge to a solution of (4) with a certain rate as long as the limit solution $\bar\Theta=(\bar V,\bar\Xi)$ defined on $[0,T]\times\Omega$ is smooth. This is effected by controlling

$$\Psi_d(t):=\int_\Omega\Big((1+|F^{(h)}|^{p-2}+|\bar F|^{p-2})|F^{(h)}-\bar F|^2+|\Theta^{(h)}-\bar\Theta|^2\Big)\,dx.$$

Specifically, we prove that there exists constant $C=C(T,\bar\Theta)>0$, which is independent of h, such that

$$\Psi_d(\tau)\leqslant C\Big(\Psi_d(0)+h\Big),\quad \tau\in[0,T].$$

Moreover, if initial data $\Psi_d^h(0)\to 0$ as $h\downarrow 0$, then $\sup_{t\in[0,T]}|\Psi_d^h(t)|\to 0$, as $h\downarrow 0$. In particular, if initial data for approximates and the limit solution coincide, then

$$\sup_{t\in[0,T]}\|\Theta^{(h)}-\bar\Theta\|_{L^2(\mathbb{T}^3)}=O(h^{1/2}).$$

For the convergence proof we will employ the relative entropy method introduced by Dafermos [6] and DiPerna [11], which allows to estimate the difference between two solutions by monitoring the evolution the relative entropy η^r defined by

$$\eta^r=\frac{1}{2}|V^{(h)}-\bar V|^2+(G(\Xi^{(h)})-G(\bar\Xi))+\nabla G(\bar\Xi)(\Xi^{(h)}-\bar\Xi)$$

with relative flux

$$q_\alpha^r=\big(v_i^{(h)}-\bar V_i\big)\Big(\frac{\partial G}{\partial \Xi^A}(\xi^{(h)})-\frac{\partial G}{\partial \Xi^A}(\bar\Xi)\Big)\frac{\partial \Phi^A}{\partial F_{i\alpha}}(\tilde f).$$

It turns out that η^r satisfies the identity (51) for the evolution of the relative energy. Analyzing this identity allows to estimate the time-growth of η^r and draw conclusions to prove convergence to classical solutions. There are two novelties in the present work: (a) In adapting the relative entropy method to the subject of time-discretized approximations. (b) In employing the method in an environment where L^p-theory is used for that bounds. This is accomplished by using an equivalent semi-metric for the relative entropy, see section 4.1.

Our approach is motivated by [13] where convergence of zero-viscosity limits,

$$\partial_t \hat{\Xi}^\varepsilon = \partial_\alpha \left(\frac{\partial \Phi}{\partial F_{i\alpha}}(F) v_i \right)$$
$$\partial_t v_i^\varepsilon = \partial_\alpha \left(\frac{\partial G(\Xi^\varepsilon)}{\partial \Xi^A} \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) \right) + \varepsilon \partial_\alpha \partial_\alpha v_i \, ,$$

to classical solutions of polyconvex elasticity is proved. There is, however, a significant difference between the analysis in [13] and the analysis required in the case of time-discrete approximations (6). The extended system in the case of viscosity approximations has the property (see [13, p. 475]) that if the constraint $\Xi^\varepsilon = \Phi(F^\varepsilon)$ is satisfied initially then it is satisfied for all times. By contrast, the time-discretized equations (6) do not achieve this property, namely,

$$\Xi^{(h)}(\cdot, 0) = \Phi(F^{(h)}(\cdot, 0)) \quad \not\Rightarrow \quad \Xi^{(h)}(\cdot, t) = \Phi(F^{(h)}(\cdot, t)), \ t > 0.$$

This presents various new technical difficulties. For instance, control of the term $|\Xi - \bar{\Xi}|$ does not necessarily imply the control of

$$|\nabla \Phi(F) - \nabla \Phi(\bar{F})| \sim (1 + |F|^2 + |\bar{F}|^2)|F - \bar{F}|,$$

(as is the case in [13]) a term that appears in the relative entropy identity and has to be estimated. To resolve this issue we decompose G into two parts

$$G(\Xi) = H(F) + R(\Xi)$$

where H, R are strictly convex functions satisfying hypothesis (H1)-(H6).

The paper is organized as follows. In Section 2 we state Lemmas 1 and 2 from [10] that establish the existence and properties of the iterates (v^j, Ξ^j), define the time-continuous and time-constant interpolates and state the main convergence Theorem. In Section 3 we discuss the null-Lagrangian properties, the product rule for divergence in general functional settings and derive the relative entropy identity. Finally, in Section 4, we discuss the equivalence between Ψ_d and η^r and carry out the detailed and cumbersome estimations in order to prove the main theorem.

2 Time discrete variational scheme and statement of Main Theorem

We assume that the number of dimensions is $n = 3$ and that the stored energy $W : M^{3\times3}_+ \to \mathbb{R}$ is ***polyconvex***, that is

$$W(F) = G \circ \Phi(F) \tag{7}$$

where $G = G(\Xi) : M^{3\times3} \times M^{3\times3} \times \mathbb{R} \to \mathbb{R}$ is some *convex* function of $\Xi = (F, Z, w) \in M^{3\times3} \times M^{3\times3} \times \mathbb{R} \simeq \mathbb{R}^{19}$ and

$$\Phi(F) = (F, \operatorname{cof} F, \det F). \tag{8}$$

We view Ξ as a vector in $\mathbb{R}^{19}$ with the convention that

$$\begin{aligned} &\Xi_A = F_{i\alpha}, \quad &&A = 3(i-1) + \alpha, \quad &&i, \alpha = 1, \dots, 3 \\ &\Xi_A = Z_{k\gamma}, \quad &&A = 3(k+2) + \gamma, \quad &&k, \gamma = 1, \dots, 3 \\ &\Xi_A = w, \quad &&A = 19. && \end{aligned} \tag{9}$$

We next define for $i, \alpha = 1, 2, 3$

$$\begin{aligned} g_{i\alpha}(\Xi, \hat{F}) &= \frac{\partial G}{\partial \Xi^A}(\Xi)\, \frac{\partial \Phi^A}{\partial F_{i\alpha}}(\hat{F}) \\ &= \frac{\partial G}{\partial F_{i\alpha}}(\Xi) + \frac{\partial G}{\partial Z_{k\gamma}}(\Xi)\, \varepsilon_{imk}\, \varepsilon_{\alpha\beta\gamma}\, \hat{F}_{m\beta} + \big(\operatorname{cof} \hat{F}\big)_{i\alpha} \frac{\partial G}{\partial w}(\Xi) \end{aligned} \tag{10}$$

and set the corresponding fields $g_i : \mathbb{R}^{19} \times \mathbb{R}^9 \to \mathbb{R}^3$ as follows

$$g_i(\Xi, \hat{F}) = (g_{i1}, g_{i2}, g_{i3})(\Xi, \hat{F}). \tag{11}$$

Assumptions

We will work with periodic boundary conditions, *i.e.* the spatial domain is taken to be the three dimensional torus $\Omega := \mathbb{T}^3$. The indices $i, \alpha, \dots$ generally run over $1, \dots, 3$ while $A, B, \dots$ run over $1, \dots, 19$. Also, we use the notation $L^p = L^p(\Omega)$ and $W^{1,p} = W^{1,p}(\Omega)$. Finally, we impose the following convexity and growth assumptions on G:

(H1) $G \in C^3(M^{3\times3} \times M^{3\times3} \times \mathbb{R}; [0, \infty))$ has the following form

$$G(\Xi) = H(F) + R(\Xi) \tag{12}$$

with $H \in C^3(M^{3\times3}; [0, \infty))$ and $R \in C^3(M^{3\times3} \times M^{3\times3} \times \mathbb{R}; [0, \infty))$ strictly convex that satisfy

$$\kappa |F|^{p-2} |z|^2 \leqslant z^T \nabla^2 H(F) z \leqslant \kappa' |F|^{p-2} |z|^2, \ \ \forall z \in \mathbb{R}^9$$

and $\gamma I \leqslant \nabla^2 R \leqslant \gamma' I$ for some fixed $\gamma, \gamma', \kappa, \kappa' > 0$ and $p \in (6, \infty)$.

(H2) $G(\Xi) \geqslant c_1 |F|^p + c_2 |Z|^2 + c_3 |w|^2 - c_4$.

(H3) $G(\Xi) \leqslant c_5(|F|^p + |Z|^2 + |w|^2 + 1)$.

(H4) $|G_F|^{\frac{p}{p-1}} + |G_Z|^{\frac{p}{p-2}} + |G_w|^{\frac{p}{p-3}} \leqslant c_6 \left(|F|^p + |Z|^2 + |w|^2 + 1\right)$.

(H5) $\left|\frac{\partial^2 H}{\partial F_{i\alpha}\partial F_{ml}}\right| \leqslant c_7|F|^{p-2}$ and $\left|\frac{\partial^3 H}{\partial F_{i\alpha}\partial F_{ml}\partial F_{rs}}\right| \leqslant c_8|F|^{p-3}$.

(H6) $\left|\frac{\partial^3 R}{\partial \Xi_A \partial \Xi_B \partial \Xi_D}\right| \leqslant c_9$.

In addition, to simplify notation, we write

$$\begin{aligned} G_{,A}(\Xi) &= \frac{\partial G}{\partial \Xi^A}(\Xi), & R_{,A}(\Xi) &= \frac{\partial R}{\partial \Xi^A}(\Xi), \\ H_{,i\alpha}(\Xi) &= \frac{\partial H}{\partial F_{i\alpha}}(\Xi), & \Phi^A_{,i\alpha}(F) &= \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F). \end{aligned}$$

2.1 Time-discrete variational scheme

The equations of elastodynamics (1) for the case of polyconvex stored-energy (7) can be expressed as a system of conservation laws,

$$\begin{aligned} \partial_t v_i &= \partial_\alpha\Big(\frac{\partial G}{\partial \Xi^A}(\Phi(F))\ \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)\Big) \\ \partial_t F_{i\alpha} &= \partial_\alpha v_i \end{aligned} \tag{13}$$

which is equivalent to (1) subject to differential constrains

$$\partial_\beta F_{i\alpha} - \partial_\alpha F_{i\beta} = 0 \tag{14}$$

that are involutions [7]: if they are satisfied for $t = 0$ then $(13)_2$ propagates (14) to satisfy for all times. Therefore systems (1) and (13) are equivalent when (14) is satisfied for initial data.

The components of Φ^A in (8), for $A = 1, \dots, 19$, are null-Lagrangians and satisfy

$$\partial_\alpha\Big(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(\nabla u)\Big) = 0, \quad x \in \mathbb{R}^3 \tag{15}$$

for any smooth function $u(x) : \mathbb{R}^3 \to \mathbb{R}^3$. Thus, if smooth (v, F) satisfies compatibility condition $(13)_2$, then [14]

$$\partial_t \Phi^A(F) = \partial_\alpha\left(\frac{\partial \Phi}{\partial F_{i\alpha}}(F) v_i\right), \quad \forall F \text{ with } \partial_\beta F_{i\alpha} = \partial_\alpha F_{i\beta}. \tag{16}$$

The transport identities (16) allow to embed (13) into the system of conservation laws [10]

$$\begin{aligned} \partial_t v_i &= \partial_\alpha\Big(\frac{\partial G}{\partial \Xi^A}(\Xi)\ \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)\Big), & i &= 1, 2, 3 \\ \partial_t \Xi^A &= \partial_\alpha\Big(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)\, v_i\Big), & A &= 1 \dots 19. \end{aligned} \tag{17}$$

The extension has the following properties:

(i) If $F(\cdot,0)$ is a gradient then $F(\cdot,t)$ remains a gradient $\forall t$.

(ii) If $F(\cdot,0)$ is a gradient and $\Xi(\cdot,0)=\Phi(F(\cdot,0))$, then $F(\cdot,t)$ remains a gradient and $\Xi(\cdot,t)=\Phi(F(\cdot,t))$, $\forall t$. In other words, the system of polyconvex elastodynamics can be viewed as a constrained evolution of (17).

(iii) The enlarged system is equipped with entropy-entropy flux pair

$$\partial_t\left(\frac{|v|^2}{2}+G(\Xi)\right)-\partial_\alpha\left(v_i\,\frac{\partial G}{\partial \Xi_A}(\Xi)\,\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)\right)=0 \tag{18}$$

and thus is symmetrizable along the solutions that are gradients.

The variational scheme, developed by S. Demoulini, D. Stuart, and A. Tzavaras [10], is based upon time-discretization of the extended system (17): given initial data

$$\Theta^0:=(v^0,\Xi^0)=(v^0,F^0,Z^0,w^0)\in L^2\times L^p\times L^2\times L^2 \tag{19}$$

and fixed $h>0$ the scheme provides the sequence of iterates

$$\Theta^j:=(v^j,\Xi^j)=(v^j,F^j,Z^j,w^j)\in L^2\times L^p\times L^2\times L^2,\;\; j\geqslant 1 \tag{20}$$

that satisfy

$$\left.\begin{aligned}\frac{v_i^j-v_i^{j-1}}{h}&=\partial_\alpha\Big(\frac{\partial G}{\partial \Xi^A}(\Xi^j)\;\frac{\partial \Phi^A}{\partial F_{i\alpha}}\;\big(F^{j-1}\big)\Big)\\ \frac{\big(\Xi^j-\Xi^{j-1}\big)_A}{h}&=\partial_\alpha\Big(\frac{\partial \Phi^A}{\partial F_{i\alpha}}\;\big(F^{j-1}\big)\;v_i^j\Big)\end{aligned}\right\}\quad\text{in}\quad \mathcal{D}'(\Omega). \tag{21}$$

Concerning the existence and properties of the iterates (v^j,Ξ^j), it was proved :

Lemma 1 ([10], p. 333)**.** *Given $(v^0,F^0,Z^0,w^0)\in L^2\times L^p\times L^2\times L^2$ there exists*

$$(v,F,Z,w)\in L^2\times L^p\times L^2\times L^2$$

which minimizes the functional

$$\mathcal{J}(v,F,Z,w)=\int_\Omega \frac{1}{2}|v-v^0|^2+G(F,Z,w)\,dx$$

on the weakly closed affine subspace $\mathcal{C}$ defined by the weak form of equations $(21)_2$, i.e. the set $\mathcal{C}\subset L^2\times L^p\times L^2\times L^2$ of (v,F,Z,w) such that for all $\varphi\in C^\infty(\mathbb{T}^3)$:

$$\begin{aligned}\int\varphi\frac{1}{h}(F_{i\alpha}-F^0_{i\alpha})dx&=-\int v_i\partial_\alpha\varphi\;dx\\ \int\varphi\frac{1}{h}(Z_{k\gamma}-Z^0_{k\gamma})dx&=-\int \varepsilon_{imk}\varepsilon_{\alpha\beta\gamma}F^0_{m\beta}v_i\,\partial_\alpha\varphi\;dx\\ \int\varphi\frac{1}{h}(w-w^0)dx&=-\int(\operatorname{cof}F^0)_{i\alpha}v_i\,\partial_\alpha\varphi\;dx\end{aligned}$$

The minimizer satisfies the Euler-Lagrange equation $(21)_1$ *in the sense of distributions, i.e.*

$$\int \varphi \frac{1}{h}(v_i - v_i^0) dx = -\int g_{i\alpha}(\Xi, F^0)\partial_\alpha \varphi \, dx$$

for all smooth φ*. Furthermore the constraints*

$$\begin{aligned} \partial_{\alpha i} Z &= 0 \\ \partial_\beta F_{i\alpha} - \partial_\alpha F_{i\beta} &= 0 \end{aligned}$$

are preserved by the map

$$S_h : (v^0, F^0, Z^0, w^0) \to (v, F, Z, w),$$

the solution operator induced by the lemma. In fact if F^0 *is a gradient then so is* F*, and thus we can assert the existence of a* $W^{1,p}$ *function* $y : \mathbb{T}^3 \to \mathbb{R}^3$ *such that* $\partial_\alpha y_i = F_{i\alpha}$*.*

Corollary 1 (Regularity)**.** *The iterates* $\Theta^j = (v^j, \Xi^j)$*,* $j \geqslant 1$*, in Lemma 1 satisfy the additional smoothness* $v^j \in W^{1,p}$*.*

Next, define $\eta : \mathbb{R}^{22} \to \mathbb{R}$ by

$$\eta(v, \Xi) = \frac{|v|^2}{2} + G(\Xi), \quad (v, \Xi) \in \mathbb{R}^{22} \tag{22}$$

The iterates satisfy the following uniform estimates:

Lemma 2 ([10], p. 335)**.** *Let* $\Theta^0 = (v^0, F^0, Z^0, w^0)$ *and* $\Theta = (v, F, Z, w)$ *be as in Lemma 1. Then if* G *is strictly convex function, i.e.* $\exists \gamma > 0$ *such that* $\nabla^2 G \geqslant \gamma$*, there exists* $c > 0$ *such that*

$$\int_\Omega \Big(\eta(\Theta) + c|\Theta - \Theta^0|^2 \Big) dx \leqslant \int_\Omega \eta(\Theta^0) \, dx.$$

Corollary 2 ([10], p. 335)**.** *The iterates* $\Theta^j = (v^j, F^j, Z^j, w^j)$*, satisfy the energy dissipation inequality, for* $j \geqslant 1$

$$\frac{1}{h}\Big(\eta(\Theta^j) - \eta(\Theta^{j-1}) \Big) - \partial_\alpha \big(g_{i\alpha}(\Xi^j, F^{j-1}) v_i^j \big) \leqslant 0$$

in the sense of distributions. There exists a number E_0*, determined by the initial data, such that*

$$\begin{aligned} &\sup_{j \geqslant 0} \Big(\|v^j\|^2_{L^2_{dx}} + \int_\Omega G(\Xi^j) dx \Big) \\ &\qquad + \sum_{j=1}^{\infty} \big(\|v^j - v^{j-1}\|^2_{L^2_{dx}} + \|\Xi^j - \Xi^{j-1}\|^2_{L^2_{dx}} \big) \leqslant E_0. \end{aligned} \tag{23}$$

Following [10], we construct the time-space version of the approximates: the time-continuous, piecewise linear interpolates $\Theta^{(h)} := (V^{(h)}, \Xi^{(h)})$ defined by

$$\begin{aligned} V^{(h)}(x,t) &= \sum_{j=1}^{\infty} \mathcal{X}^j(t)\Big(v^{j-1} + \frac{t-h(j-1)}{h}(v^j - v^{j-1})\Big) \\ \Xi^{(h)}(x,t) &= \big(F^{(h)}, Z^{(h)}, w^{(h)}\big)(t) \\ &= \sum_{j=1}^{\infty} \mathcal{X}^j(t)\Big(\Xi^{j-1} + \frac{t-h(j-1)}{h}(\Xi^j - \Xi^{j-1})\Big), \end{aligned} \tag{24}$$

and the piecewise constant interpolates $\theta^{(h)} := (v^{(h)}, \xi^{(h)})$ and $\tilde{f}^{(h)}$ given by

$$\begin{aligned} v^{(h)}(x,t) &= \sum_{j=1}^{\infty} \mathcal{X}^j(t) v^j \\ \xi^{(h)}(x,t) &= (f^{(h)}, z^{(h)}, \omega^{(h)})(t) = \sum_{j=1}^{\infty} \mathcal{X}^j(t)\Xi^j \\ \tilde{f}^{(h)}(x,t) &= \sum_{j=1}^{\infty} \mathcal{X}^j(t) F^{j-1} \end{aligned} \tag{25}$$

where $\mathcal{X}^j$ is the characteristic function of the interval $I_j := [(j-1)h, jh)$. Also, we denote by $I_j^\circ := ((j-1)h, jh)$ and $\bar{I}_j := [(j-1)h, jh]$, the interior and closure of I_j. In addition, we note that $\tilde{f}^{(h)}$ is the time-shifted version of $f^{(h)}$ and it will be used later on in defining a relative entropy flux as well as in the time-continuous equations (28).

The main objective of this article is to prove convergence of the time-continuous approximates $\Theta^{(h)} = (V^{(h)}, \Xi^{(h)})$ to a solution of the enlarged system (17) as long as the limit solution $\bar{\Theta} = (\bar{V}, \bar{\Xi})$ is smooth, by controlling

$$\Psi_d(t) := \int_\Omega \Big((1 + |F^{(h)}|^{p-2} + |\bar{F}|^{p-2}) |F^{(h)} - \bar{F}|^2 + |\Theta^{(h)} - \bar{\Theta}|^2 \Big)\, dx.$$

We now state:

Main Theorem. *Let W be given by* (7) *where G satisfies (H1)-(H6). Let $\Theta^{(h)} = (V^{(h)}, \Xi^{(h)})$, $\theta^{(h)} = (v^{(h)}, \xi^{(h)})$ and $\tilde{f}^{(h)}$ be the time interpolates defined in* (24)-(25) *and induced by spatial iterates*

$$\Theta^j = (v^j, \Xi^j) = (v^j, F^j, Z^j, w^j) \in L^2 \times L^p \times L^2 \times L^2, \;\; j \geqslant 0\,, \tag{26}$$

where Θ^0 is the given initial data and Θ^j, $j \geqslant 1$, generated by Lemma 1 satisfy (21). *Let $\bar{\Theta} = (\bar{V}, \bar{\Xi}) = (\bar{V}, \bar{F}, \bar{Z}, \bar{w})$ be a smooth solution of* (17) *defined on $\Omega \times [0,T]$ and emanating from smooth data $\bar{\Theta}^0 = (\bar{V}^0, \bar{F}^0, \bar{Z}^0, \bar{w}^0)$; we assume that $F^0, \bar{F}^0$ are initially gradients. Then:*

(a) *The relative entropy* $\eta^r = \eta^r(\Theta, \bar{\Theta})$ *defined in* (43) *satisfies the identity* (51) *and there exist constants* $\mu, \mu' > 0$ *such that*

$$\mu\,\Psi_d(t) \leqslant \int_\Omega \eta^r(x,t)\,dx \leqslant \mu'\Psi_d(t), \quad t \in [0,T]$$

(b) *There exists* $\varepsilon > 0$ *and* $C = C(T, E_0, \gamma, \gamma', \kappa, \kappa', \mu, \mu', \lambda, \varepsilon, \bar{\Theta},) > 0$ *such that for all* $h \in (0, \varepsilon)$

$$\Psi_d(\tau) \leqslant C\Big(\Psi_d(0) + h\Big), \quad \tau \in [0,T]\,.$$

Moreover, if the data satisfy $\Psi^h_d(0) \to 0$ *as* $h \downarrow 0$*, then*

$$\sup_{t\in[0,T]} \int_\Omega \Big(|\Theta^{(h)} - \bar{\Theta}|^2 + |F^{(h)} - \bar{F}|^2(1 + |F^{(h)}|^{p-2} + |\bar{F}|^{p-2})\Big)\,dx \to 0$$

as $h \downarrow 0$.

Remark. If the initial data for the approximates $\Theta^{(h)}$ and the limit solution $\bar{\Theta}$ coincide, then

$$\sup_{t\in[0,T]} \|\Theta^{(h)} - \bar{\Theta}\|_{L^2(\mathbb{T}^3)} = O(h^{1/2})$$

which provides a rate of convergence in $L^\infty([0,T], L^2)$.

For the remainder of the paper, we drop the dependence on h in order to simplify the notations.

3 Relative entropy identity

We now take $h > 0$ and fix $T > 0$. For the rest of the article, *c.f.* Main Theorem, we assume the following:

(a) $\Theta = (V, \Xi)$, $\theta = (v, \xi)$, $\tilde{f}$ denote the time approximates defined in (24)-(25) with the properties as described in the statement of Main Theorem.

(b) $\bar{\Theta} = (\bar{V}, \bar{\Xi}) = (\bar{V}, \bar{F}, \bar{Z}, \bar{w})$ is a smooth solution of (17) defined in $\Omega \times [0,T]$ with the property that $\bar{F}^0 := \bar{F}(\cdot, 0)$ is a gradient.

We next observe that for each $t \in I^\circ_j$, for $j \geqslant 1$,

$$\begin{aligned} \partial_t V(\cdot,t) &= \frac{v^j - v^{j-1}}{h} =: \frac{1}{h}\delta v^j \\ \partial_t \Xi(\cdot,t) &= \frac{\Xi^j - \Xi^{j-1}}{h} =: \frac{1}{h}\delta \Xi^j. \end{aligned} \tag{27}$$

Hence, formula (21) together with (27) implies for a.e. $t \in [0, \infty)$

$$\left.\begin{aligned} \partial_t V_i(\cdot,t) &= \partial_\alpha\Big(\frac{\partial G}{\partial \Xi^A}(\xi)\,\frac{\partial \Phi^A}{\partial F_{i\alpha}}(\tilde{f})\Big) \\ \partial_t\,\Xi_A(\cdot,t) &= \partial_\alpha\Big(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(\tilde{f})\,v_i\Big) \end{aligned}\right. \quad \text{in} \quad \mathcal{D}'(\Omega). \tag{28}$$

Next, for the smooth $\bar{\Theta}$, we rewrite extended system (17) and entropy-entropy flux identity (18) in the short notation by using (10):

$$\begin{aligned} \partial_t \bar{V}_i &= \operatorname{div}\big(g_i(\bar{\Xi}, \bar{F})\big) \\ \partial_t \bar{\Xi}_A &= \partial_\alpha \Big(\Phi^A_{,i\alpha}(\bar{F})\, \bar{V}_i\Big) \end{aligned} \tag{29}$$

and

$$\partial_t \left(\eta(\bar{\Theta})\right) - \operatorname{div}\left(v_i g_i(\bar{\Xi}, \bar{F})\right) = 0. \tag{30}$$

3.1 Null-Lagrangians

Let us investigate the properties of Φ defined in (8). First, we note that $\forall F_1, F_2 \in M^{3\times 3}$

$$\left|\Phi^A_{,i\alpha}(F_1) - \Phi^A_{,i\alpha}(F_2)\right| \leqslant \begin{cases} 0, & A = 1, \dots, 9 \\ |F_1 - F_2|, & A = 10, \dots, 18 \\ 3\big(|F_1| + |F_2|\big)|F_1 - F_2|, & A = 19, \end{cases} \tag{31}$$

and

$$\left|\Phi^A_{,i\alpha}(F_1)\right| \leqslant 1 + |F_1| + |F_1|^2, \quad A = 1 \dots 19. \tag{32}$$

We now recall that components of Φ are null-Lagrangians. Then extending property (15) we claim: if $u \in W^{1,q}_{loc}(\Omega; \mathbb{R}^3)$ with $q \in [2, \infty)$, then

$$\partial_\alpha \Big(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(\nabla u)\Big) = 0 \quad \text{in} \quad \mathcal{D}'(\Omega). \tag{33}$$

The proof of (33) follows from (32) and the density argument.

Lemma 3. *Let $u \in W^{1,q}_{loc}(\Omega; \mathbb{R}^3)$ with $q \in (2, \infty)$ and $z \in W^{1,r}_{loc}(\Omega)$ with $r \in [q^*, \infty)$ where $q^* = q/(q-2)$. Then*

$$\partial_\alpha \Big(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(\nabla u) z\Big) = \frac{\partial \Phi^A}{\partial F_{i\alpha}}(\nabla u)\, \partial_\alpha z \quad \textit{in} \quad \mathcal{D}'(\Omega). \tag{34}$$

Proof. Observe that (32) implies $\frac{\partial \Phi^A}{\partial F_{i\alpha}}(\nabla u) \in L^{q/2}_{loc}$. Hence we must have both $\frac{\partial \Phi^A}{\partial F_{i\alpha}}(\nabla u) z$ and $\frac{\partial \Phi^A}{\partial F_{i\alpha}}(\nabla u)\, \partial_\alpha z$ in L^1_{loc}. Then for $\varphi \in C^\infty_0(\Omega)$ we have

$$\begin{aligned} &\int_\Omega \Big(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(\nabla u)\, z\Big) \partial_\alpha \varphi \, dx \\ &\qquad = \int_\Omega \Big(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(\nabla u)\Big) \partial_\alpha (z\, \varphi)\, dx - \int_\Omega \Big(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(\nabla u)\, \partial_\alpha z\Big) \varphi \, dx := I_1 - I_2. \end{aligned}$$

Since $z\varphi \in W^{1,r}_0 \bigcap W^{1,q^*}_{loc}$, by (33) together with the density argument we obtain that $I_1 = 0$ and hence

$$\int_\Omega \Big(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(\nabla u)\, z\Big) \partial_\alpha \varphi \, dx = -I_2 = \int_\Omega \Big(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(\nabla u)\, \partial_\alpha z\Big) \varphi \, dx.$$

□

Lemma 4. *Let $q \in (1,\infty)$ and $q' = \frac{q}{q-1}$. Assume*

$$f \in W^{1,q}(\Omega),\ h \in L^{q'}(\Omega;\mathbb{R}^3) \text{ and } z = \operatorname{div} h \in L^{q'}(\Omega).$$

Then $fh \in L^1(\Omega;\mathbb{R}^3)$, $\operatorname{div}(fh) \in L^1(\Omega)$ and

$$\operatorname{div}(fh) = f \operatorname{div} h + \nabla f h. \tag{35}$$

Proof. Clearly $h \in L^{q'}(\Omega;\mathbb{R}^3)$, $f \in L^q(\Omega) \Rightarrow fh \in L^1(\Omega;\mathbb{R}^3)$. Now, take arbitrary $\varphi \in C_0^\infty(\Omega)$. Then, since $f \in W^{1,q}(\Omega)$, we get

$$-\int_\Omega f h_\alpha \partial_\alpha \varphi\, dx = -\int_\Omega h_\alpha \partial_\alpha (f\varphi)\, dx + \int_\Omega (h_\alpha \partial_\alpha f)\, \varphi\, dx.$$

Further, we notice that $f\varphi \in W_0^{1,q}(\Omega)$ and we obtain

$$-\int_\Omega h_\alpha \partial_\alpha (f\varphi)\, dx = \int_\Omega z(f\varphi)\, dx$$

where we used the density argument and the assumption $z = \operatorname{div} h \in L^{q'}(\Omega)$. Hence

$$-\int_\Omega f h_\alpha\, \partial_\alpha \varphi\, dx = \int_\Omega (zf + h_\alpha \partial_\alpha f)\, \varphi\, dx$$

and this proves (35). Finally, since zf, $h_\alpha \partial_\alpha f \in L^1(\Omega)$, we clearly have $\operatorname{div}(fh) \in L^1(\Omega)$. □

3.2 Derivation of the relative entropy identity

At this point we are ready to establish identities for entropy-entropy flux pair similar to formula (30) corresponding to smooth solutions of (29). In this chapter, we fix an arbitrary $j \geqslant 1$ and derive identities for the interval $t \in I_j$.

In the sequel, we will perform a series of calculations that hold for smooth functions. A technical difficulty arises, since the iterates (v^j, Ξ^j) obtained in Lemma 1 are, in general, not smooth. To bypass this we first recall that by assumption initial $\bar{F}^0$ and F^0 are gradients. Therefore by (24)-(25), Lemma 1 and the properties of extension (17) we must have

$$F, f, \tilde{f}, \bar{F} \text{ are gradients } \forall t \geqslant 0. \tag{36}$$

Further, we observe that (10) and (H4) imply for $p' = \frac{p}{p-1}$ with $p \in (6,\infty)$

$$\begin{aligned}
&\Big| g_{i\alpha}(\Xi^\circ, \widehat{F}) \Big|^{p'} \\
&\quad \leqslant C_g \bigg(\Big| \frac{\partial G}{\partial F_{i\alpha}} \Big|^{\frac{p}{p-1}} + \big|\widehat{F}\big|^{\frac{p}{p-1}} \Big| \frac{\partial G}{\partial Z_{k\gamma}} \Big|^{\frac{p}{p-1}} + \big|\widehat{F}\big|^{\frac{2p}{p-1}} \Big| \frac{\partial G}{\partial w} \Big|^{\frac{p}{p-1}} \bigg) \\
&\quad \leqslant C_g' \bigg(|\widehat{F}|^p + \Big| \frac{\partial G}{\partial F_{i\alpha}} \Big|^{\frac{p}{p-1}} + \Big| \frac{\partial G}{\partial Z_{k\gamma}} \Big|^{\frac{p}{p-2}} + \Big| \frac{\partial G}{\partial w} \Big|^{\frac{p}{p-3}} \bigg) \\
&\quad \leqslant C_g'' \big(|\widehat{F}|^p + |F^\circ|^p + |Z^\circ|^2 + |w^\circ|^2 + 1 \big), \quad \forall \Xi^\circ \in \mathbb{R}^{19},\ \widehat{F} \in \mathbb{R}^9.
\end{aligned} \tag{37}$$

Finally, with the help of (36), (37) and Corollary 1, we use Lemmas 3 and 4 which provide the null-Lagrangian property and the product rule in the smoothness class appropriate for the time approximates $\tilde{f}$, θ and Θ induced, via (24)-(25), by solutions of the variational approximation scheme (v^j, Ξ^j) which, by assumption of Main Theorem, satisfy (19)-(21), (see also Lemma 1).

Thus, using (10), (36) and Lemma 3, we rewrite (28) as follows

$$\begin{aligned} \partial_t V_i(t) &= \operatorname{div}\big(g_i(\xi, \tilde{f})\big) \\ \partial_t \Xi_A(t) &= \Phi^A_{,i\alpha}(\tilde{f})\, \partial_\alpha v_i. \end{aligned} \tag{38}$$

Next, by Corollary 1, Lemmas 3 and 4, (29) and (36)-(38), we obtain

$$\begin{aligned} \operatorname{div}\big(v_i g_i(\xi, \tilde{f})\big) &= v_i \partial_t V_i + \nabla v_i g_i(\xi, \tilde{f}) \\ \operatorname{div}\big(\bar{V}_i g_i(\xi, \tilde{f})\big) &= \bar{V}_i \partial_t V_i + \nabla \bar{V}_i g_i\,(\xi, \tilde{f}) \\ \operatorname{div}\big(v_i g_i(\bar{\Xi}, \tilde{f})\big) &= v_i \Phi^A_{,i\alpha}(\tilde{f})\, \partial_\alpha (G_{,A}(\bar{\Xi})) + \nabla v_i g_i(\bar{\Xi}, \tilde{f}) \\ \operatorname{div}\big(\bar{V}_i g_i(\bar{\Xi}, \tilde{f})\big) &= \bar{V}_i \Phi^A_{,i\alpha}(\tilde{f})\, \partial_\alpha (G_{,A}(\bar{\Xi})) + \nabla \bar{V}_i g_i(\bar{\Xi}, \tilde{f}). \end{aligned} \tag{39}$$

Hence (38)-$(39)_1$ imply the following important identity

$$\begin{aligned} \partial_t\big(\eta(\Theta)\big) &= V_i \partial_t V_i + G_{,A}(\Xi)\partial_t \Xi_A \\ &= (V_i - v_i)\partial_t V_i + (G_{,A}(\Xi) - G_{,A}(\xi))\partial_t \Xi_A \\ &\quad + \big(v_i \partial_t V_i + \nabla v_i\, g_i(\xi, \tilde{f})\big) \\ &= \big(\nabla\eta(\Theta) - \nabla\eta(\theta)\big)\frac{\delta\Theta^j}{h} + \operatorname{div}\big(v_i g_i(\xi, \tilde{f})\big). \end{aligned} \tag{40}$$

Similarly, by (29), $(39)_2$ and (41) we obtain

$$\begin{aligned} \partial_t\big(\bar{V}_i(V_i - \bar{V}_i)\big) &= \big(\bar{V}_i \partial_t V_i + \nabla \bar{V}_i g_i(\xi, \tilde{f})\big) \\ &\quad - \big(\bar{V}_i \partial_t \bar{V}_i + \nabla \bar{V}_i g_i(\bar{\Xi}, \bar{F})\big) \\ &\quad - \nabla \bar{V}_i\big(g_i(\xi, \tilde{f}) - g_i(\bar{\Xi}, \bar{F})\big) + \partial_t \bar{V}_i (V_i - \bar{V}_i) \\ &= \operatorname{div}\Big(\bar{V}_i g_i(\xi, \tilde{f}) - \bar{V}_i g_i(\bar{\Xi}, \bar{F})\Big) \\ &\quad - \nabla \bar{V}_i\,\Big(g_i(\xi, \tilde{f}) - g_i(\bar{\Xi}, \bar{F})\Big) + \ \partial_t \bar{V}_i (V_i - \bar{V}_i). \end{aligned} \tag{41}$$

Next, by $(29)_2$, $(38)_2$ and $(39)_3$ we have

$$\begin{aligned} G_{,A}(\bar{\Xi})\partial_t \bar{\Xi}_A &= \operatorname{div}\big(\bar{V}_i g_i(\bar{\Xi}, \bar{F})\big) - \bar{V}_i \Phi^A_{,i\alpha}(\bar{F})\partial_\alpha\big(G_{,A}(\bar{\Xi})\big) \\ G_{,A}(\bar{\Xi})\partial_t \Xi_A &= \operatorname{div}\big(v_i g_i(\bar{\Xi}, \tilde{f})\big) - v_i \Phi^A_{,i\alpha}(\tilde{f})\partial_\alpha\big(G_{,A}(\bar{\Xi})\big) \end{aligned}$$

and hence

$$\begin{aligned} \partial_t\big(G_{,A}(\bar{\Xi})(\Xi - \bar{\Xi})_A\big) &= \operatorname{div}\big(v_i g_i(\bar{\Xi}, \tilde{f}) - \bar{V}_i g_i(\bar{\Xi}, \bar{F})\big) \\ &\quad - \partial_\alpha(G_{,A}(\bar{\Xi}))\Big(v_i \Phi^A_{,i\alpha}(\tilde{f}) - \bar{V}_i \Phi^A_{,i\alpha}(\bar{F})\Big) \\ &\quad + \partial_t(G_{,A}(\bar{\Xi}))(\Xi - \bar{\Xi})_A. \end{aligned} \tag{42}$$

Then, denoting the *relative entropy* by

$$\eta^r = \eta^r(\Theta, \bar{\Theta}) := \eta(\Theta) - \eta(\bar{\Theta}) - \nabla\eta(\bar{\Theta})(\Theta - \bar{\Theta}) \tag{43}$$

and using identities (30), (40)-(42), we establish

$$\begin{aligned} \partial_t\big(\eta^r(\Theta, \bar{\Theta})\big) &= \operatorname{div}\big(v_i g_i(\xi, \tilde{f}) - \bar{V} g_i(\xi, \tilde{f}) - v_i g_i(\bar{\Xi}, \tilde{f}) + \bar{V} g_i(\bar{\Xi}, \bar{F})\big) \\ &\quad + \big(\nabla\eta(\Theta) - \nabla\eta(\theta)\big)\frac{\delta\Theta^j}{h} + J \end{aligned} \tag{44}$$

where

$$\begin{aligned} J :=\ & \nabla\bar{V}_i\big(g_i(\xi, \tilde{f}) - g_i(\bar{\Xi}, \bar{F})\big) \\ &+ \partial_\alpha(G_{,A}(\bar{\Xi}))\Big(v_i\Phi^A_{,i\alpha}(\tilde{f}) - \bar{V}_i\Phi^A_{,i\alpha}(\bar{F})\Big) \\ &- \partial_t\bar{V}_i(V_i - \bar{V}_i) - \partial_t(G_{,A}(\bar{\Xi}))(\Xi - \bar{\Xi})_A. \end{aligned}$$

Now we rearrange the term J. The aim is to decompose its parts into several separate groups. One that contains the error of interpolation, that is terms Θ and θ, and the other that contains the difference between Θ and $\bar{\Theta}$. This rearrangement is exploited later on to make use of Gronwall's inequality.

First, by (29) and null Lagrangian properties (33) we get

$$\begin{aligned} J &= \partial_\alpha\bar{V}_i\big(g_{i\alpha}(\xi, \tilde{f}) - g_{i\alpha}(\bar{\Xi}, \bar{F})\big) \\ &\quad + \partial_\alpha(G_{,A}(\bar{\Xi}))\Big(v_i\Phi^A_{,i\alpha}(\tilde{f}) - \Phi^A_{,i\alpha}(\bar{F})\bar{V}_i - \Phi^A_{,i\alpha}(\bar{F})(V_i - \bar{V}_i)\Big) \\ &\quad - G_{,AB}(\bar{\Xi})(\Xi - \bar{\Xi})_A\Phi^B_{,i\alpha}(\bar{F})\,\partial_\alpha\bar{V}_i \\ &= \partial_\alpha(G_{,i\alpha}(\bar{\Xi}))\Big(v_i\Phi^A_{,i\alpha}(\tilde{f}) - \Phi^A_{,i\alpha}(\bar{F})V_i\Big) \\ &\quad + \partial_\alpha\bar{V}_i\Big(g_{i\alpha}(\xi, \tilde{f}) - g_{i\alpha}(\bar{\Xi}, \bar{F}) - G_{,AB}(\bar{\Xi})(\Xi - \bar{\Xi})_B\Phi^A_{,i\alpha}(\bar{F})\Big) \\ &=: J_1 + J_2 \end{aligned} \tag{45}$$

where we used $G_{,AB} = G_{,BA}$. We next rearrange J_1 as follows

$$\begin{aligned} J_1 &= \partial_\alpha(G_{,i\alpha}(\bar{\Xi}))\Big(v_i\Phi^A_{,i\alpha}(\tilde{f}) - \Phi^A_{,i\alpha}(\bar{F})V_i\Big) \\ &= \partial_\alpha(G_{,A}(\bar{\Xi}))\Big(\Phi^A_{,i\alpha}(\tilde{f}) - \Phi^A_{,i\alpha}(\bar{F})\Big)(v_i - \bar{V}_i) \\ &\quad + \partial_\alpha(G_{,A}(\bar{\Xi}))\Big(\Phi^A_{,i\alpha}(\tilde{f}) - \Phi^A_{,i\alpha}(\bar{F})\Big)\bar{V}_i \\ &\quad + \partial_\alpha(G_{,A}(\bar{\Xi}))\,\Phi^A_{,i\alpha}(\bar{F})\,(v_i - V_i). \end{aligned} \tag{46}$$

By $(39)_{3,4}$ we get

$$\begin{aligned} \partial_\alpha\bar{V}_i\Big(g_{i\alpha}\big(\bar{\Xi}, \tilde{f}\big) - g_{i\alpha}\big(\bar{\Xi}, \bar{F}\big)\Big) &= \operatorname{div}\big(\bar{V}_i g_i(\bar{\Xi}, \tilde{f}) - \bar{V}_i g_i(\bar{\Xi}, \bar{F})\big) \\ &\quad - \bar{V}_i\Big(\Phi^A_{,i\alpha}(\tilde{f}) - \Phi^A_{,i\alpha}(\bar{F})\Big)\partial_\alpha(G_{,A}(\bar{\Xi})) \end{aligned}$$

and this allows us to write J_2 as follows

$$\begin{aligned}
J_2 &= \partial_\alpha \bar{V}_i\Big(g_{i\alpha}(\xi,\tilde{f}) - g_{i\alpha}(\bar{\Xi},\bar{F}) - G_{,AB}(\bar{\Xi})(\Xi-\bar{\Xi})_B\Phi^A_{,i\alpha}(\bar{F})\Big)\\
&= \partial_\alpha \bar{V}_i\big(G_{,A}(\xi) - G_{,A}(\bar{\Xi})\big)\Big(\Phi^A_{,i\alpha}(\tilde{f}) - \Phi^A_{,i\alpha}(\bar{F})\Big)\\
&\quad + \operatorname{div}\big(\bar{V}_i g_i(\bar{\Xi},\tilde{f}) - \bar{V}_i g_i(\bar{\Xi},\bar{F})\big)\\
&\quad - \bar{V}_i\Big(\Phi^A_{i\alpha}(\tilde{f}) - \Phi^A_{i\alpha}(\bar{F})\Big)\partial_\alpha(G_{,A}(\bar{\Xi}))\\
&\quad + \partial_\alpha \bar{V}_i\big(G_{,A}(\xi) - G_{,A}(\Xi)\big)\Phi^A_{,i\alpha}(\bar{F})\\
&\quad + \partial_\alpha \bar{V}_i\Big(G_{,A}(\Xi) - G_{,A}(\bar{\Xi}) - G_{,AB}(\bar{\Xi})(\Xi-\bar{\Xi})_B\Big)\Phi^A_{,i\alpha}(\bar{F}).
\end{aligned} \tag{47}$$

Combining (46) and (47) we obtain

$$\begin{aligned}
J &= J_1 + J_2\\
&= \partial_\alpha(G_{,A}(\bar{\Xi}))\left(\Phi^A_{,i\alpha}(\tilde{f}) - \Phi^A_{,i\alpha}(\bar{F})\right)(v_i - \bar{V}_i)\\
&\quad + \partial_\alpha(G_{,A}(\bar{\Xi}))\,\Phi^A_{,i\alpha}(\bar{F})(v_i - \bar{V}_i)\\
&\quad + \partial_\alpha \bar{V}_i\left(G_{,A}(\xi) - G_{,A}(\bar{\Xi})\right)\left(\Phi^A_{,i\alpha}(\tilde{f}) - \Phi^A_{,i\alpha}(\bar{F})\right)\\
&\quad + \partial_\alpha \bar{V}_i\left(G_{,A}(\xi) - G_{,A}(\Xi)\right)\Phi^A_{,i\alpha}(\bar{F})\\
&\quad + \partial_\alpha \bar{V}_i\left(G_{,A}(\Xi) - G_{,A}(\bar{\Xi}) - G_{,AB}(\bar{\Xi})(\Xi-\bar{\Xi})_B\right)\Phi^A_{,i\alpha}(\bar{F}).
\end{aligned} \tag{48}$$

Finally, we decompose

$$\begin{aligned}
&\Big(\Phi^A_{,i\alpha}(\tilde{f}) - \Phi^A_{,i\alpha}(\bar{F})\Big)(v_i - \bar{V}_i)\\
&= \Big(\Phi^A_{,i\alpha}(\tilde{f}) - \Phi^A_{,i\alpha}(F)\Big)\big(v_i - V_i\big) + \Big(\Phi^A_{,i\alpha}(\tilde{f}) - \Phi^A_{,i\alpha}(F)\Big)\big(V_i - \bar{V}_i\big)\\
&\quad + \Big(\Phi^A_{,i\alpha}(F) - \Phi^A_{,i\alpha}(\bar{F})\Big)\big(v_i - V_i\big) + \Big(\Phi^A_{,i\alpha}(F) - \Phi^A_{,i\alpha}(\bar{F})\Big)\big(V_i - \bar{V}_i\big).
\end{aligned} \tag{49}$$

and

$$\begin{aligned}
\big(G_{,A}(\xi) - G_{,A}(\bar{\Xi})\big)\Big(\Phi^A_{,i\alpha}(\tilde{f}) - \Phi^A_{,i\alpha}(\bar{F})\Big) &=\\
&\big(G_{,A}(\xi) - G_{,A}(\Xi)\big)\Big(\Phi^A_{,i\alpha}(\tilde{f}) - \Phi^A_{,i\alpha}(F)\Big)\\
&+\big(G_{,A}(\xi) - G_{,A}(\Xi)\big)\Big(\Phi^A_{,i\alpha}(F) - \Phi^A_{,i\alpha}(\bar{F})\Big)\\
&+\big(G_{,A}(\Xi) - G_{,A}(\bar{\Xi})\big)\Big(\Phi^A_{,i\alpha}(\tilde{f}) - \Phi^A_{,i\alpha}(F)\Big)\\
&+\big(G_{,A}(\Xi) - G_{,A}(\bar{\Xi})\big)\Big(\Phi^A_{,i\alpha}(F) - \Phi^A_{,i\alpha}(\bar{F})\Big).
\end{aligned} \tag{50}$$

In summary, by (44) together with (48)-(50), we obtain the relative entropy identity

$$\partial_t \eta^r - \operatorname{div} q^r = -\frac{1}{h}D^j + S + Q, \ \ \forall t \in I_j^\circ \tag{51}$$

where q^r is the relative entropy-flux defined by

$$q^r_\alpha(\theta, \bar{\Theta}, \tilde{f}) := (v_i - \bar{V}_i)\big(G_{,A}(\xi) - G_{,A}(\bar{\Xi})\big)\, \Phi^A_{,i\alpha}(\tilde{f}), \tag{52}$$

the term D^j is given by

$$D^j := \big(\nabla\eta(\theta) - \nabla\eta(\Theta)\big)\delta\Theta^j \tag{53}$$

and will turn out to be dissipative, the term S, given by

$$\begin{aligned} S := {} & \partial_\alpha(G_{,A}(\bar{\Xi}))\, \Phi^A_{,i\alpha}(\bar{F})\,(v_i - V_i) \\ & + \partial_\alpha(G_{,A}(\bar{\Xi}))\Big(\Phi^A_{,i\alpha}(\tilde{f}) - \Phi^A_{,i\alpha}(F)\Big)\big(v_i - V_i\big) \\ & + \partial_\alpha(G_{,A}(\bar{\Xi}))\Big(\Phi^A_{,i\alpha}(\tilde{f}) - \Phi^A_{,i\alpha}(F)\Big)\big(V_i - \bar{V}_i\big) \\ & + \partial_\alpha(G_{,A}(\bar{\Xi}))\Big(\Phi^A_{,i\alpha}(F) - \Phi^A_{,i\alpha}(\bar{F})\Big)\big(v_i - V_i\big) \\ & + \partial_\alpha\bar{V}_i\, \Phi^A_{,i\alpha}(\bar{F})\Big(G_{,A}(\xi) - G_{,A}(\Xi)\Big) \\ & + \partial_\alpha\bar{V}_i\,(G_{,A}(\xi) - G_{,A}(\Xi))\,\Big(\Phi^A_{,i\alpha}(\tilde{f}) - \Phi^A_{,i\alpha}(F)\Big) \\ & + \partial_\alpha\bar{V}_i\,(G_{,A}(\xi) - G_{,A}(\Xi))\,\Big(\Phi^A_{,i\alpha}(F) - \Phi^A_{,i\alpha}(\bar{F})\Big) \\ & + \partial_\alpha\bar{V}_i\,\big(G_{,A}(\Xi) - G_{,A}(\bar{\Xi})\big)\,\Big(\Phi^A_{,i\alpha}(\tilde{f}) - \Phi^A_{,i\alpha}(F)\Big), \end{aligned} \tag{54}$$

contains mostly error terms, and finally the term Q,

$$\begin{aligned} Q := {} & \partial_\alpha(G_{,A}(\bar{\Xi}))\Big(\Phi^A_{,i\alpha}(F) - \Phi^A_{,i\alpha}(\bar{F})\Big)(V_i - \bar{V}_i) \\ & + \partial_\alpha\bar{V}_i\Big(G_{,A}(\Xi) - G_{,A}(\bar{\Xi})\big)\,\big(\Phi^A_{,i\alpha}(F) - \Phi^A_{,i\alpha}(\bar{F})\big) \\ & + \partial_\alpha\bar{V}_i\Big(G_{,A}(\Xi) - G_{,A}(\bar{\Xi}) - G_{,AB}(\bar{\Xi})(\Xi - \bar{\Xi})_B\Big)\Phi^A_{,i\alpha}(\bar{F}), \end{aligned} \tag{55}$$

holds the information regarding the difference between Θ and $\bar{\Theta}$ and can be estimated from above by the relative entropy $\eta^r(\Theta, \bar{\Theta})$.

4 Proof of Main Theorem

The identity (51) is central to our paper. In this section, we will estimate each of the terms described above and then via Gronwall's inequality complete the proof.

4.1 An equivalent semi-metric $d(\cdot, \cdot)$ for the relative entropy

The goal of this section is to show that the relative entropy η^r can be equivalently represented via a specific function $d(\cdot, \cdot)$ defined via power functions.

First, let us prove two lemmas that we use in our further computations.

Lemma 5. *For every $u, v \in \mathbb{R}^n$*

$$\int_0^1 |u + \tau(v-u)|\, d\tau \;\geqslant\; \frac{1}{4\sqrt{n}}\,(|u| + |v|)\,. \tag{56}$$

Proof. First, we consider the scalar case $u, v \in \mathbb{R}$. Then, for $u, v \geqslant 0$, $u + \tau(v - u) = u(1-\tau) + v\tau \geqslant 0$ and

$$\int_0^1 |u + \tau(v-u)| d\tau = u + \frac{1}{2}(v-u) = \frac{1}{2}(|u| + |v|).$$

The same is true when $u, v \leqslant 0$. Now consider the case when $uv < 0$ and assume that $u > 0$ and $v < 0$. Setting $\gamma = \frac{u}{u-v} = \frac{|u|}{|u|+|v|} \in (0,1)$ we obtain

$$\begin{aligned}
&\int_0^1 |u + \tau(v-u)| d\tau \\
&\quad = \int_0^\gamma (u + \tau(v-u)) d\tau - \int_\gamma^1 (u + \tau(v-u)) d\tau \\
&\quad = (u - v)\left(\gamma^2 - \gamma + \frac{1}{2}\right) = \frac{1}{2}\big(|u| + |v|\big)\big(\gamma^2 + (1-\gamma)^2\big) \\
&\quad \geqslant \frac{1}{4}\big(|u| + |v|\big).
\end{aligned} \tag{57}$$

Consider next the case $u, v \in \mathbb{R}^n$, $n > 1$. Clearly, $\frac{1}{\sqrt{n}}\left(\sum_{i=1}^n |z_i|\right) \leqslant |z| \leqslant \left(\sum_{i=1}^n |z_i|\right)$, $\forall z \in \mathbb{R}^n$ where $|z|$ is the Euclidean norm of a vector z. Then by (57)

$$\begin{aligned}
\int_0^1 |u + \tau(v-u)|\, d\tau \;&\geqslant\; \frac{1}{\sqrt{n}} \int_0^1 \left(\sum_{i=1}^n |u_i + \tau(v_i - u_i)|\right) d\tau \\
&\geqslant\; \frac{1}{4\sqrt{n}} \sum_{i=1}^n (|u_i| + |v_i|) \;\geqslant\; \frac{1}{4\sqrt{n}}\,(|u| + |v|)\,.
\end{aligned}$$

□

Lemma 6. *Let $q \in [1, \infty)$. Then $\forall u, v \in \mathbb{R}^n$ and each $\bar{\beta} \in [0,1]$*

$$\int_0^{\bar{\beta}} \int_0^1 (1-\beta)\, |u + \tau(1-\beta)(v-u)|^q\, d\tau\, d\beta \;\geqslant\; c'\bar{\beta}\,(|u|^q + |v|^q)\,, \tag{58}$$

where

$$c' = c'(n, q) = \frac{1}{2^{q+2}(q+2)(4\sqrt{n})^q}. \tag{59}$$

Proof. Let $\bar{\beta} \in [0,1]$ and set $\bar{c} = \frac{1}{4\sqrt{n}}$. By applying Jensen's inequality and then using estimate (56) we get

$$
\begin{aligned}
&\int_0^{\bar{\beta}} \int_0^1 (1-\beta)\big|u + \tau(1-\beta)(v-u)\big|^q d\tau\, d\beta \\
&\qquad \geqslant \int_0^{\bar{\beta}} (1-\beta)\bigg(\int_0^1 \big|u + \tau\big((1-\beta)v + \beta u - u\big)\big| d\tau\bigg)^q d\beta \\
&\qquad \geqslant \bar{c}^q \int_0^{\bar{\beta}} (1-\beta)\big(|u| + |(1-\beta)v + \beta u|\big)^q d\beta \\
&\qquad \geqslant \frac{\bar{c}^q}{2}\big(|u|^q + |v|^q\big) \int_0^{\bar{\beta}} (1-\beta)^{q+1}\, d\beta.
\end{aligned}
$$

Notice, if $0 \leqslant \bar{\beta} \leqslant \frac{1}{2}$, we have

$$
\int_0^{\bar{\beta}} (1-\beta)^{q+1} d\beta \;\geqslant\; \int_0^{\bar{\beta}} (1/2)^{q+1}\, d\beta = \frac{\bar{\beta}}{2^{q+1}}.
$$

On the other hand, if $\frac{1}{2} < \bar{\beta} \leqslant 1$, then

$$
\int_0^{\bar{\beta}} (1-\beta)^{q+1} d\beta \;\geqslant\; \int_0^{1/2} (1-\beta)^{q+1} d\beta \;\geqslant\; \frac{\bar{\beta}}{2(q+2)}.
$$

Combining the last three inequalities we obtain (58). □

Definition. *Let* $\Theta_1 = (V_1, \Xi_1), \Theta_2 = (V_2, \Xi_2) \in \mathbb{R}^{22}$. *We set*

$$
d(\Theta_1, \Theta_2) = \big(1 + |F_1|^{p-2} + |F_2|^{p-2}\big)\,|F_1 - F_2|^2 + |\Theta_1 - \Theta_2|^2 \tag{60}
$$

where $(F_1, Z_1, w_1) = \Xi_1, (F_2, Z_2, w_2) = \Xi_2 \in \mathbb{R}^{19}$.

Before we proceed, we notice that hypotheses (H1) and (H5), imply

$$
\big|R_{,A}(\Xi_1) - R_{,A}(\Xi_2)\big| \;\leqslant C|\Xi_1 - \Xi_2|, \quad \forall\, \Xi_1, \Xi_2 \in \mathbb{R}^{19} \tag{61}
$$

and

$$
\begin{aligned}
&\big|H_{,i\alpha}(F_1) - H_{,i\alpha}(F_2)\big| \\
&\quad = \bigg|\int_0^1 \sum_{l,m=1}^{3} \frac{\partial^2 H}{\partial F_{i\alpha}\partial F_{lm}}\big(sF_1 + (1-s)F_2\big)(F_1 - F_2)_{lm}\, ds\bigg| \\
&\quad \leqslant C|F_1 - F_2| \int_0^1 |sF_1 + (1-s)F_2|^{p-2}\, ds, \quad \forall F_1, F_2 \in M^{3\times 3}.
\end{aligned} \tag{62}
$$

This together with Lemma 6 will help us to establish the relation between the relative entropy function η^r and the semi-metric $d(\cdot,\cdot)$.

Lemma 7. *There exist constants $\mu, \mu' > 0$ such that*

$$\mu\, d(\Theta_1, \Theta_2) \;\leqslant\; \eta^r(\Theta_1, \Theta_2) \;\leqslant\; \mu' d(\Theta_1, \Theta_2) \tag{63}$$

for all $\Theta_l = (V_l, \Xi_l) \in \mathbb{R}^{22}$ with $\Xi_l = (F_l, Z_l, w_l) \in \mathbb{R}^{19}$, $l = 1, 2$.

Proof. Consider

$$\begin{aligned}
&\eta^r(\Theta_1, \Theta_2)\\
&\quad = \eta(\Theta_1) - \eta(\Theta_2) - \nabla\eta(\Theta_2)(\Theta_1 - \Theta_2)\\
&\quad = \int_0^1 \frac{d}{ds}\left(\eta(s\Theta_1 + (1-s)\Theta_2)\right) ds - \nabla\eta(\Theta_2)(\Theta_1 - \Theta_2)\\
&\quad = \int_0^1 \big(\nabla\eta(s\Theta_1 + (1-s)\Theta_2) - \nabla\eta(\Theta_2)\big)(\Theta_1 - \Theta_2)\, ds \qquad (64)\\
&\quad = \int_0^1 \left(\int_0^1 \frac{d}{d\tau}\nabla\eta\left(\Theta_2 + \tau s(\Theta_1 - \Theta_2)\right) d\tau\right)(\Theta_1 - \Theta_2)\, ds\\
&\quad = \int_0^1 \int_0^1 s(\Theta_1 - \Theta_2)^T \big(\nabla^2\eta(\hat\Theta)\big)(\Theta_1 - \Theta_2)\, ds d\tau.
\end{aligned}$$

where

$$\hat\Theta(\tau, s) = \big(\hat V(\tau, s), \hat\Xi(\tau, s)\big) := \Theta_2 + \tau s(\Theta_1 - \Theta_2), \quad \tau, s \in [0, 1].$$

Observe that (12) implies

$$\nabla_\Xi G = \begin{bmatrix}\nabla_F H & 0 & 0\end{bmatrix}^T + \nabla_\Xi R \tag{65}$$

and therefore by (22) we have

$$\begin{aligned}
&(\Theta_1 - \Theta_2)^T \big(\nabla^2\eta(\hat\Theta)\big)(\Theta_1 - \Theta_2)\\
&\qquad = |V_1 - V_2|^2 + (\Xi_1 - \Xi_2)^T \nabla^2 R(\hat\Xi)(\Xi_1 - \Xi_2) \qquad (66)\\
&\qquad\qquad + (F_1 - F_2)^T \nabla^2 H(\hat F)(F_1 - F_2).
\end{aligned}$$

Thus (H1), (64) and (66) imply

$$\begin{aligned}
&\frac{1}{2}|V_1 - V_2|^2 + \frac{\gamma}{2}|\Xi_1 - \Xi_2|^2 + \kappa\,|F_1 - F_2|^2 \int_0^1\int_0^1 s|\hat F|^{p-2} ds\, d\tau\\
&\qquad \leqslant \eta^r(\Theta_1, \Theta_2) \leqslant \qquad (67)\\
&\frac{1}{2}|V_1 - V_2|^2 + \frac{\gamma'}{2}|\Xi_1 - \Xi_2|^2 + \kappa'\,|F_1 - F_2|^2 \int_0^1\int_0^1 s|\hat F|^{p-2} ds\, d\tau.
\end{aligned}$$

At this point we estimate those terms of (67) which contain integrals. First, we consider

$$\begin{aligned}
&\int_0^1\int_0^1 s|\hat F|^{p-2} ds\, d\tau = \int_0^1\int_0^1 s\,|\tau s F_1 + (1-\tau s)F_2|^{p-2}\, ds\, d\tau\\
&\leqslant \int_0^1\int_0^1 s\,\left(|F_1|^{p-2} + |F_2|^{p-2}\right) 2^{p-2} ds\, d\tau \;\leqslant\; 2^{p-3}\left(|F_1|^{p-2} + |F_2|^{p-2}\right).
\end{aligned}$$

Next, by Lemma 6, after the appropriate substitution ($s = 1-\beta$, $\bar{\beta} = 1$), we obtain

$$\int_0^1 \int_0^1 s|\hat{F}|^{p-2} ds\, d\tau = \int_0^1 \int_0^1 s\,|F_2 + \tau s(F_1 - F_2)|^{p-2}\, ds\, d\tau \geqslant c'\left(|F_1|^{p-2} + |F_2|^{p-2}\right).$$

Then combining the two last inequalities with the estimate (67) and the fact that $|\Xi_1 - \Xi_2|^2 + |F_1 - F_2|^2 \leqslant 2|\Xi_1 - \Xi_2|^2$ we conclude

$$\mu\, d(\Theta_1, \Theta_2) \leqslant \eta^r(\Theta_1, \Theta_2) \leqslant \mu' d(\Theta_1, \Theta_2)$$

where $\mu = \min\left(\frac{1}{2}, \frac{\gamma}{4}, \kappa 2^{p-3}\right)$ and $\mu' = \max\left(\frac{1}{2}, \frac{\gamma'}{2}, \kappa' 2^{p-3}\right)$. □

Observe now that smoothness of $\bar{\Theta}$ implies that $\exists M = M(T) > 0$ such that

$$M \geqslant |\bar{\Theta}| + |\nabla_x \bar{\Theta}| + |\partial_t \bar{\Theta}|, \quad (x,t) \in \Omega \times [0,T]. \tag{68}$$

We next prove:

Lemma 8. *$d(\Theta, \bar{\Theta})$, $\eta^r(\Theta, \bar{\Theta}) \in L^\infty\big([0,T]; L^1\big)$ and*

$$\mu \Psi_d(t) \leqslant \int_\Omega \eta^r\big(\Theta(x,t), \bar{\Theta}(x,t)\big)\, dx \leqslant \mu' \Psi_d(t), \quad \forall t \in [0,T] \tag{69}$$

where

$$\Psi_d(t) := \int_\Omega d\big(\Theta(x,t), \bar{\Theta}(x,t)\big)\, dx, \quad t \in [0,T]. \tag{70}$$

Proof. Take $t \in [0,T]$. Then $t \in I_j$ for some $j \geqslant 1$. Hence (24), (60), (68) and (H2) imply for $p \in (6, \infty)$

$$\begin{aligned} d(\Theta(\cdot,t), \bar{\Theta}(\cdot,t)) &\leqslant C\Big((1+|F|^{p-2})(1+|F|^2) + (1+|\Theta|^2)\Big) \\ &\leqslant C\Big(1 + |F|^p + |Z|^2 + |w|^2 + |V|^2\Big) \\ &\leqslant C\Big(1 + G(\Xi^{j-1}) + G(\Xi^j) + |v^{j-1}|^2 + |v^j|^2\Big) \end{aligned} \tag{71}$$

where $C = C(M) > 0$ is a constant independent of h, j and t. Since t is arbitrary, by (23) and (71), we conclude that $\exists C' = C'(M) > 0$ such that

$$\int_\Omega d(\Theta(\cdot,t), \bar{\Theta}(\cdot,t))\, dx \leqslant C'(1 + E_0), \quad \forall t \in [0,T]. \tag{72}$$

Then (63) and (72) imply the lemma. □

4.2 Estimate for term Q on $t \in [0,T]$

We next prove:

Lemma 9. *Let Q be defined in* (55)*. Then* $\exists\, \lambda = \lambda(M) > 0$ *such that*

$$|Q(x,t)| \leqslant \lambda\, d(\Theta, \bar{\Theta}), \quad (x,t) \in \Omega \times [0,T]. \tag{73}$$

Proof. Let $C = C(M) > 0$ be a generic constant. Then $(31)_3$ and (68) imply

$$|\Phi^A_{,i\alpha}(F) - \Phi^A_{,i\alpha}(\bar{F})| \leqslant C\,(1+|F|)\,\big|F-\bar{F}\big| \qquad A = 1\ldots 19. \tag{74}$$

Since $G_{,A}$ is smooth, (68) implies $|G_{,A}(\bar{\Xi})| \leqslant C$ and hence by (74) we get

$$\begin{aligned}&\big|\partial_\alpha(G_{,A}(\bar{\Xi}))\left(\Phi^A_{,i\alpha}(F) - \Phi^A_{,i\alpha}(\bar{F})\right)(V_i - \bar{V}_i)\big| \\ &\qquad \leqslant C\left((1+|F|^2)|F-\bar{F}|^2 + |V-\bar{V}|^2\right).\end{aligned} \tag{75}$$

We next set for each $A = 1,\ldots,19$

$$I_A := \sum_{i,\alpha=1}^{3} \partial_\alpha \bar{V}_i \left(G_{,A}(\Xi) - G_{,A}(\bar{\Xi})\right)\left(\Phi^A_{,i\alpha}(F) - \Phi^A_{,i\alpha}(\bar{F})\right). \tag{76}$$

By $(31)_1$ we have $I_A = 0$, $A = 1,\ldots,9$. Also, by (61) and (65) we get

$$|G_{,A}(\Xi) - G_{,A}(\bar{\Xi})| = |R_{,A}(\Xi) - R_{,A}(\bar{\Xi})| \leqslant C|\Xi - \bar{\Xi}|, \quad A = 10\ldots 19. \tag{77}$$

Thus, by (68), (74) and (76)-(77), we conclude

$$\sum_{A=1}^{19} |I_A| \leqslant C\Big(|\Xi-\bar{\Xi}|^2 + (1+|F|^2)|F-\bar{F}|^2\Big). \tag{78}$$

Next, following computations in (64), for each $A = 1,\ldots,19$ we have

$$\begin{aligned} J_A &:= G_{,A}(\Xi) - G_{,A}(\bar{\Xi}) - G_{,AB}(\bar{\Xi})\left(\Xi - \bar{\Xi}\right)_B \\ &= \int_0^1\!\!\int_0^1 s(\Xi-\bar{\Xi})^T\left(\nabla^2 G_{,A}(\hat{\Xi})\right)(\Xi-\bar{\Xi})\,ds d\tau \end{aligned} \tag{79}$$

where

$$\hat{\Xi}(s,\tau) := (\hat{F}, \hat{Z}, \hat{w})(\tau,s) = \bar{\Xi} + \tau s(\Xi - \bar{\Xi}), \quad \tau, s \in [0,1].$$

First, take $A \in \{1,\ldots,9\}$. Then (65) implies $G_{,A}(\hat{\Xi}) = H_{,i\alpha}(\hat{F}) + R_{,A}(\hat{\Xi})$ where, according to $(9)_1$, $A = 3(i-1)+\alpha$ for some unique $i, \alpha \in \{1,2,3\}$ and therefore

$$\nabla^2_{\Xi}(G_{,A}) = \begin{bmatrix} \nabla^2_F(H_{,i\alpha}) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \nabla^2_{\Xi}(R_{,A}).$$

Hence

$$(\Xi-\bar\Xi)^T\big(\nabla^2 G_{,A}(\hat\Xi)\big)(\Xi-\bar\Xi) = (F-\bar F)^T\big(\nabla^2 H_{,i\alpha}(\hat F)\big)(F-\bar F) + (\Xi-\bar\Xi)^T\big(\nabla^2 R_{,A}(\hat\Xi)\big)(\Xi-\bar\Xi)$$

and therefore by (H1) we get for $A=1,\dots,9$

$$\big|(\Xi-\bar\Xi)^T(\nabla^2 G_{,A}(\hat\Xi))(\Xi-\bar\Xi)\big| \leqslant C\big(|F-\bar F|^2|\hat F|^{p-3}+|\Xi-\bar\Xi|^2\big). \tag{80}$$

Take next $A\in\{10,\dots,19\}$. Then (65) implies $G_{,A}(\hat\Xi)=R_{,A}(\hat\Xi)$ and hence by (H1)

$$|(\Xi-\bar\Xi)^T\big(\nabla^2 G_{,A}(\hat\Xi)\big)(\Xi-\bar\Xi)| \leqslant C\,|\Xi-\bar\Xi|^2, \quad A=10,\dots,19. \tag{81}$$

Thus (68) and (79)-(81) imply

$$\begin{aligned} &|\partial_\alpha \bar V_i\, \Phi^A_{,i\alpha}(\bar F)\, J_A| \\ &\leqslant C\Big(|\Xi-\bar\Xi|^2+|F-\bar F|^2\int_0^1\int_0^1 |\bar F+\tau s(F-\bar F)|^{p-3}ds\,d\tau\Big) \\ &\leqslant C\Big(|\Xi-\bar\Xi|^2+|F-\bar F|^2(1+|F|^{p-3})\Big). \end{aligned} \tag{82}$$

Then by (60), (75), (78) and (82) we conclude for $p\in(6,\infty)$

$$\begin{aligned} |Q(x,t)| &\leqslant C\Big(|V-\bar V|^2+|\Xi-\bar\Xi|^2+(1+|F|^2+|F|^{p-3})|F-\bar F|^2\Big) \\ &\leqslant C\big(|\Theta-\bar\Theta|^2+(1+|F|^{p-2})|F-\bar F|^2\big) \leqslant C\, d(\Theta,\bar\Theta). \end{aligned}$$

□

4.3 Estimates for terms D^j and S on $t\in I'_j\subset[0,T]$

For the rest of the section, we fix $j\geqslant 1$ such that

$$I'_j := I_j \bigcap [0,T] = [(j-1)h, jh)\bigcap[0,T]$$

is not empty and consider all estimations only for $t\in I'_j$.

Observe that the definitions of the time approximates (24)-(25) imply

$$\begin{aligned} V(\cdot,t)-v(\cdot,t) &= \Big(\frac{t-hj}{h}\Big)\delta v^j \\ \xi(\cdot,t)-\xi(\cdot,t) &= \Big(\frac{t-hj}{h}\Big)\delta\Xi^j \\ F(\cdot,t)-\tilde f(\cdot,t) &= \Big(\frac{t-h(j-1)}{h}\Big)\delta F^j. \end{aligned} \tag{83}$$

Then, using (83), we get

$$\big(v-V\big)\delta v^j = \Big(\frac{hj-t}{h}\Big)\,|\delta v^j|^2 \tag{84}$$

and

$$\begin{aligned}\big(\nabla R(\xi)-\nabla R(\Xi)\big)\delta\Xi^j &= \int_0^1 \frac{d}{ds}\Big(\nabla R\big(s\xi+(1-s)\Xi\big)\delta\Xi^j\Big)ds\\ &= \int_0^1 (\xi-\Xi)^T\nabla^2 R(\hat{\Xi})\,\delta\Xi^j\,ds \qquad (85)\\ &= \left(\frac{hj-t}{h}\right)\int_0^1 (\delta\Xi^j)^T\nabla^2 R(\hat{\Xi})\,(\delta\Xi^j)\,ds\end{aligned}$$

$$\begin{aligned}\big(\nabla H(f)-\nabla H(F)\big)\delta F^j &= \int_0^1 \frac{d}{ds}\Big(\nabla H\big(sf+(1-s)F\big)\,\delta F^j\Big)\,ds\\ &= \left(\frac{hj-t}{h}\right)\int_0^1 (\delta F^j)^T\nabla^2 H(\hat{F})\,(\delta F^j)\,ds\end{aligned} \qquad (86)$$

where

$$\hat{\Xi}(s,t)=(\hat{F},\hat{Z},\hat{w})(s,t):=s\xi(\cdot,t)+(1-s)\Xi(\cdot,t),\quad t\in\bar{I}_j,\ s\in[0,1].$$

We next observe that (22), (53) and (65) imply

$$\begin{aligned}D^j &= \big(\nabla\eta(\theta)-\nabla\eta(\Theta)\big)\delta\Theta^j\\ &= (v-V)\,\delta v^j+\big(\nabla G(\xi)-\nabla G(\Xi)\big)\delta\Xi^j\\ &= (v-V)\,\delta v^j+\big(\nabla H(f)-\nabla H(F)\big)\delta F^j\\ &\qquad\qquad +\big(\nabla R(\xi)-\nabla G(\Xi)\big)\delta\Xi^j.\end{aligned} \qquad (87)$$

Hence (H1), (84)-(87), and the fact that $\frac{hj-t}{h}\in[0,1]$ imply

$$|D^j(\cdot,t)|\leqslant\Big(|\delta v^j|^2+\gamma'|\delta\Xi^j|^2+\kappa'|\delta F^j|^2\int_0^1|\hat{F}(s,t)|^{p-2}ds\Big). \qquad (88)$$

By (H2) and (24)-(25) we estimate for $p\in(6,\infty)$

$$\begin{aligned}|\delta F^j|^2\int_0^1|\hat{F}(s,t)|^{p-2}ds &= |\delta F^j|^2\int_0^1|sf-(1-s)F|^{p-2}ds\\ &\leqslant|\delta F^j|^2\big(|f|+|F|\big)^{p-2}\\ &\leqslant C\left(|F^{j-1}|^p+|F^j|^p\right)\\ &\leqslant C\left(1+G(\Xi^{j-1})+G(\Xi^j)\right)\end{aligned} \qquad (89)$$

and, similarly,

$$\begin{aligned}|\delta\Xi^j|^2 &= |\delta F^j|^2+|\delta Z^j|^2+|\delta w^j|^2\\ &\leqslant 2\big(2+|F^{j-1}|^p+|F^j|^p+|Z^{j-1}|^2+|Z^j|^2+|w^{j-1}|^2+|w^j|\big)\\ &\leqslant C\big(1+G(\Xi^{j-1})+G(\Xi^j)\big)\end{aligned} \qquad (90)$$

where $C>0$ is a generic constant independent of h, j and t. Then (88)-(90) imply that there exists constant $\nu'>0$ independent of h, j, t such that

$$|D^j(\cdot,t)|\leqslant\nu'\big(1+|v^{j-1}|^2+|v^j|^2+G(\Xi^{j-1})+G(\Xi^j)\big),\quad t\in I'_j. \qquad (91)$$

Then, by (23) and (91), we obtain

$$\int_\Omega |D^j(x,t)|\,dx \leqslant 2\,\nu'\big(1+E_0\big), \quad \forall t \in I_j'. \tag{92}$$

and therefore

$$D^j \in L^\infty\big(I_j'; L^1(\Omega)\big) \subset L^1\big(I_j' \times \Omega\big). \tag{93}$$

We next show that $D^j \geqslant 0$ estimating it from below. Once again, using (H1), (84)-(87) and the fact that $\frac{hj-t}{h} \in [0,1]$, we obtain

$$\begin{aligned} D^j(\cdot,t) &\geqslant \left(\frac{hj-t}{h}\right)\left(|\delta v^j|^2 + \gamma|\delta\Xi^j|^2 + \kappa|\delta F^j|^2 \int_0^1 |\hat{F}(s,t)|^{p-2} ds\right) \\ &\geqslant \nu\left(\frac{hj-t}{h}\right)\left(|\delta\Theta^j|^2 + |\delta F^j|^2 \int_0^1 |\hat{F}(s,t)|^{p-2} ds\right) \geqslant 0 \end{aligned} \tag{94}$$

where $\nu = \min(1,\gamma,\kappa) > 0$.

We now fix arbitrary $\tau \in \bar{I}_j' := \bar{I}_j \bigcap [0,T]$ and set

$$\bar{a} := a(\tau) \quad \text{where} \quad a(t) := \frac{t-h(j-1)}{h} \in [0,1], \quad t \in \bar{I}_j'. \tag{95}$$

Observe next that

$$\hat{F}(s,t) = sf(t) + (1-s)F(t) = F^j + (1-s)(1-a(t))(F^{j-1} - F^j).$$

Then (95) and Lemma 6, after the appropriate substitution ($\beta = a$, $\bar{\beta} = \bar{a}$, $s = 1-\tau$), imply

$$\begin{aligned} &\int_{(j-1)h}^{\tau} \left(\left(\frac{t-hj}{h}\right)|\delta F^j|^2 \int_0^1 |\hat{F}(s,t)|^{p-2} ds\right) dt \\ &\quad = h|\delta F^j|^2 \int_0^{\bar{a}} \int_0^1 (1-a)|F^j + (1-s)(1-a)(F^{j-1} - F^j)|^{p-2} ds\, da \\ &\quad \geqslant h\bar{a}c'\big(|F^{j-1}|^{p-2} + |F^j|^{p-2}\big)|\delta F^j|^2. \end{aligned}$$

Similarly, using (95), we get

$$\int_{(j-1)h}^{\tau} \left(\left(\frac{hj-t}{h}\right)|\delta\Theta^j|^2\right) dt = \left(h\int_0^{\bar{a}} (1-a)\, da\right)|\delta\Theta^j|^2 \geqslant \frac{h\bar{a}}{2}|\delta\Theta^j|^2.$$

Therefore, using the two last estimates together with (93)-(94) and the fact that $2|\delta\Theta^j|^2 \geqslant |\delta F^j|^2 + |\delta\Theta^j|^2$, by Fubini's theorem we conclude

$$\begin{aligned} &\int_{(j-1)h}^{\tau} \int_\Omega \left(\frac{1}{h}D^j\right) dx\, dt \\ &\qquad \geqslant C_D\, \bar{a} \int_\Omega |\delta\Theta^j|^2 + \big(1 + |F^{j-1}|^{p-2} + |F^j|^{p-2}\big)|\delta F^j|^2\, dx \end{aligned} \tag{96}$$

where $C_D = \min(\nu c', \nu/2) > 0$ is independent of h, j and τ.

We next consider term S defined in (54). As before, we let $C > 0$ be a generic constant that, in general, will depend on M defined in (68). We now observe that

$$\frac{hj-t}{h}, \ \frac{t-h(j-1)}{h} \in [0,1], \quad t \in I_j'. \tag{97}$$

Then, by (24)-(25), (31), $(83)_3$ and (97), we find

$$\begin{aligned} \big|\Phi^A_{,i\alpha}(\tilde f) - \Phi^A_{,i\alpha}(F)\big| &\leqslant C\left(1+|\tilde f|+|F|\right)|F-\tilde f| \\ &\leqslant C\left(1+|F^{j-1}|+|F^j|\right)|\delta F^j|. \end{aligned} \tag{98}$$

Hence $(83)_1$, (97) and (98) imply

$$\begin{aligned} &\big|\partial_\alpha(G_{,A}(\bar\Xi))\big(\Phi^A_{,i\alpha}(\tilde f) - \Phi^A_{,i\alpha}(F)\big)(v_i - V_i)\big| \\ &\qquad \leqslant C\left(\left(1+|F^{j-1}|^2+|F^j|^2\right)|\delta F^j|^2 + |\delta v^j|^2\right). \end{aligned} \tag{99}$$

Similarly, by (98), we get

$$\begin{aligned} &\big|\partial_\alpha(G_{,A}(\bar\Xi))\big(\Phi^A_{,i\alpha}(\tilde f) - \Phi^A_{,i\alpha}(F)\big)(V_i - \bar V_i)\big| \\ &\qquad \leqslant C\Big(\left(1+|F^{j-1}|^2+|F^j|^2\right)|\delta F^j|^2 + |V-\bar V|^2\Big). \end{aligned} \tag{100}$$

By (74), $(83)_1$ and (97), we get

$$\begin{aligned} &\big|\partial_\alpha(G_{,A}(\bar\Xi))\big(\Phi^A_{,i\alpha}(F) - \Phi^A_{,i\alpha}(\bar F)\big)(v_i - V_i)\big| \\ &\qquad \leqslant C\Big(\left(1+|F|^2\right)|F-\bar F|^2 + |\delta v^j|^2\Big). \end{aligned} \tag{101}$$

Next, for each $A = 1,\dots,19$ we set

$$K_A := \sum_{i,\alpha=1}^{3} \partial_\alpha \bar V_i\big(G_{,A}(\xi) - G_{,A}(\Xi)\big)\big(\Phi^A_{,i\alpha}(\tilde f) - \Phi^A_{,i\alpha}(F)\big). \tag{102}$$

By $(31)_1$ we have $I_A = 0$, $A = 1,\dots,9$. By (61), (65), $(83)_2$ and (97) we get

$$|G_{,A}(\xi) - G_{,A}(\Xi)| = |R_{,A}(\xi) - R_{,A}(\Xi)| \leqslant C|\delta\Xi^j|, \quad A = 10,\dots,19. \tag{103}$$

Thus (98), (102) and (103) imply

$$\sum_{A=1}^{19} |K_A| \leqslant C\Big(|\delta\Xi^j|^2 + \left(1+|F^{j-1}|^2+|F^j|^2\right)|\delta F^j|^2\Big). \tag{104}$$

Similarly, $(31)_1$, (74) and (103) imply

$$\begin{aligned} &\big|\partial_\alpha \bar V_i\left(G_{,A}(\xi) - G_{,A}(\Xi)\right)\big(\Phi^A_{,i\alpha}(F) - \Phi^A_{,i\alpha}(\bar F)\big)\big| \\ &\qquad \leqslant C\left(|\delta\Xi^j|^2 + \left(1+|F|^2\right)|F-\bar F|^2\right). \end{aligned} \tag{105}$$

Also, by $(31)_1$, (77) and (98), we find that

$$\begin{aligned}\big|\partial_\alpha \bar{V}_i \left(G_{,A}(\Xi) - G_{,A}(\bar{\Xi})\right)\left(\Phi^A_{,i\alpha}(\tilde{f}) - \Phi^A_{,i\alpha}(F)\right)\big| \\ \leqslant C\Big(|\Xi - \bar{\Xi}|^2 + \big(1 + |F^j|^2 + |F^{j-1}|^2\big)|\delta F^j|^2\Big).\end{aligned} \tag{106}$$

Consider remaining linear terms. By $(83)_1$ and (97) we have

$$\big|\partial_\alpha (G_{,A}(\bar{\Xi}))\Phi^A_{,i\alpha}(\bar{F})\,(v_i - V_i)\big| \,\leqslant\, C\,|\delta v^j| \,\leqslant\, C\bigg(\frac{h}{\varepsilon} + \frac{\varepsilon}{h}|\delta v^j|^2\bigg) \tag{107}$$

where $\varepsilon > 0$ which will be chosen later.

We next take $A \in \{1, \dots, 9\}$. Then $A = 3(i-1) + \alpha$ for some unique $i, \alpha \in \{1, 2, 3\}$ according to $(9)_1$. Hence by (61)-(62), (65), $(83)_2$ and (97) we obtain for $A = 1, \dots, 9$

$$\begin{aligned}|G_{,A}(\xi) - G_{,A}(\Xi)| &= |H_{,i\alpha}(f) - H_{,i\alpha}(F) + R_{,A}(\xi) - R_{,A}(\Xi)| \\ &\leqslant C\Big(|f - F|\int_0^1 |sf + (1-s)F|^{p-2} ds + |\xi - \Xi|\Big) \\ &\leqslant C\Big(|\delta F^j|\,(|F^{j-1}|^{p-2} + |F^j|^{p-2}) + |\delta \Xi^j|\Big).\end{aligned} \tag{108}$$

Observe now that (H1) implies for $p \in (6, \infty)$

$$\begin{aligned}&(|F^{j-1}|^{p-2} + |F^j|^{p-2})|\delta F^j| \\ &\quad\leqslant \frac{h}{\varepsilon}(|F^{j-1}|^{p-2} + |F^j|^{p-2}) + \frac{\varepsilon}{h}(|F^{j-1}|^{p-2} + |F^j|^{p-2})|\delta F^j|^2 \\ &\quad\leqslant C\,\frac{h}{\varepsilon}\big(1 + G(\Xi^{j-1}) + G(\Xi^j)\big) + \frac{\varepsilon}{h}(|F^{j-1}|^{p-2} + |F^j|^{p-2})|\delta F^j|^2.\end{aligned} \tag{109}$$

and, similarly,

$$|\delta \Xi^j| \leqslant \frac{h}{\varepsilon} + \frac{\varepsilon}{h}|\delta \Xi^j|^2. \tag{110}$$

Hence, by (103) and (108)-(110), we obtain the following estimate

$$\begin{aligned}&\big|\partial_\alpha \bar{V}_i \Phi^A_{,i\alpha}(\bar{F})(G_{,A}(\xi) - G_{,A}(\Xi))\big| \\ &\qquad\leqslant C\,\frac{h}{\varepsilon}\big(1 + G(\Xi^{j-1}) + G(\Xi^j)\big) \\ &\qquad\quad + C\,\frac{\varepsilon}{h}\Big((|F^{j-1}|^{p-2} + |F^j|^{p-2})|\delta F^j|^2 + |\delta \Xi^j|^2\Big).\end{aligned} \tag{111}$$

Thus (54), (99)-(101), (104)-(107) and (111) imply for $p \in (6, \infty)$

$$
\begin{aligned}
|S| &\leqslant C\bigg(\Big(1+\frac{\varepsilon}{h}\Big)\Big(|\delta v^j|^2 + |\delta \Xi^j|^2 + (|F^j|^{p-2} + |F^{j-1}|^{p-2})|\delta F^j|^2\Big) \\
&\qquad + \frac{h}{\varepsilon}\big(1 + G(\Xi^{j-1}) + G(\Xi^j)\big) \\
&\qquad + |V-\bar V|^2 + |\Xi - \bar\Xi|^2 + (1+|F|^2)|F-\bar F|^2\bigg) \\
&\leqslant C_S\bigg(\Big(1+\frac{\varepsilon}{h}\Big)\Big(|\delta\Theta^j|^2 + (1+|F^{j-1}|^{p-2} + |F^j|^{p-2})|\delta F^j|^2\Big) \\
&\qquad + \frac{h}{\varepsilon}\big(1 + G(\Xi^{j-1}) + G(\Xi^j)\big) + d(\Theta, \bar\Theta)\bigg)
\end{aligned}
\tag{112}
$$

for some constant $C_S > 0$ independent of h, j and t.

Before we proceed, notice that, (H1) and (90) imply

$$
\begin{aligned}
&|\delta\Theta^j|^2 + (1+|F^{j-1}|^{p-2} + |F^j|^{p-2})|\delta F^j|^2 \\
&\qquad \leqslant C\big(1 + |v^{j-1}|^2 + |v^j|^2 + G(\Xi^{j-1}) + G(\Xi^j)\big)
\end{aligned}
\tag{113}
$$

with $C > 0$ independent of j. Then (23), (71) and (113) imply that the right hand side of (112) is in $L^\infty\big(I'_j\,; L^1\big)$ and hence

$$
S \in L^\infty\big(I'_j\,; L^1(\Omega)\big) \subset L^1\big(I'_j \times \Omega\big). \tag{114}
$$

We next integrate (112) and, by using (23), conclude

$$
\begin{aligned}
&\int_{(j-1)h}^{\tau}\int_\Omega |S|\,dx\,dt \\
&\quad \leqslant C_S\bigg((h+\varepsilon)\bar a\int_\Omega\Big(|\delta\Theta^j|^2 + (1+|F^{j-1}|^{p-2}+|F^j|^{p-2})|\delta F^j|^2\Big)dx \\
&\qquad\quad + \frac{\bar a h^2}{\varepsilon}(1+2E_0) + \int_{(j-1)h}^{\tau}\int_\Omega d(\Theta,\bar\Theta)\,dx\bigg).
\end{aligned}
\tag{115}
$$

Set $\varepsilon := C_D/(4C_S)$. Then $-C_D + C_S(h+\varepsilon) \leqslant 0$ for $\forall h \in (0,\varepsilon)$. Hence, by (96) and (115), we get

$$
\begin{aligned}
&\int_{(j-1)h}^{\tau}\int_\Omega\Big(-\frac{1}{h}D^j + |S|\Big)dx\,dt \\
&\qquad \leqslant C_S\bigg(\frac{\bar a h^2}{\varepsilon}(1+2E_0) + \int_{(j-1)h}^{\tau}\int_\Omega d(\Theta,\bar\Theta)\,dx\,dt\bigg), \ \ h \in (0,\varepsilon)
\end{aligned}
\tag{116}
$$

where we remind the reader that $\tau \in \bar I'_j$ and $\bar a = a(\tau) = \frac{\tau - h(j-1)}{h} \in [0,1]$.

4.4 Conclusion of the proof via Gronwall's inequality

Observe that (51), (73), (93), (114) and Lemma 8 imply

$$\Big(\partial_t \eta^r - \operatorname{div} q^r\Big) \in L^\infty\big([0,T], L^1(\Omega)\big) \subset L^1\big([0,T]\times\Omega\big). \tag{117}$$

Then, by (51), (73) and (116), we conclude for $h \in (0,\varepsilon)$ and $\tau \in \bar{I}'_j$

$$\begin{aligned} &\int_{(j-1)h}^{\tau}\int_\Omega \big(\partial_t \eta^r - \operatorname{div} q^r\big)\,dx\,dt \\ &\qquad \leqslant \int_{(j-1)h}^{\tau}\int_\Omega \Big(-\frac{1}{h}D^j + |S| + |Q|\Big)\,dx\,dt \\ &\qquad\quad \leqslant \; C_I\Big(\big(\tau - h(j-1)\big)h + \int_{(j-1)h}^{\tau}\int_\Omega d(\Theta,\bar\Theta)\,dx\,dt\Big) \end{aligned} \tag{118}$$

where $C_I := \max\big(C_s(1+2E_0)/\varepsilon, C_S + \lambda\big) > 0$ is independent of h, j and τ.

Take now $h \in (0,\varepsilon)$ and $\tau \in (0,T]$. Then $\tau \in ((n-1)h, nh]$, for some $n \geqslant 1$, and we can write

$$\begin{aligned} &\int_0^\tau \int_\Omega \big(\partial_t \eta^r - \operatorname{div} q^r\big)\,dx\,dt \\ &\quad = \sum_{j=1}^{n-1}\int_{(j-1)h}^{jh}\int_\Omega \big(\partial_t \eta^r - \operatorname{div} q^r\big)\,dx\,dt + \int_{(n-1)h}^{\tau}\int_\Omega \big(\partial_t \eta^r - \operatorname{div} q^r\big)\,dx\,dt. \end{aligned}$$

By using (118), we estimate each term on the right hand side of the identity above and conclude

$$\int_0^\tau \int_\Omega \big(\partial_t \eta^r - \operatorname{div} q^r\big)\,dx\,dt \leqslant C_I\Big(\tau h + \int_0^\tau\int_\Omega d(\Theta,\bar\Theta)\,dx\,dt\Big). \tag{119}$$

We next observe that Corollary 1, (36)-(39) and (52) and imply

$$\operatorname{div} q^r \in L^\infty\big([0,T]; L^1\big) \subset L^1\big([0,T]\times\Omega\big). \tag{120}$$

Moreover, due to periodic boundary conditions, (using the density argument) we have $\int_\Omega\big(\operatorname{div} q^r(x,s)\big)\,dx = 0$ for a.e. $s \in [0,T]$. Hence

$$\int_0^\tau\int_\Omega \operatorname{div} q^r\,dx\,dt = 0. \tag{121}$$

Finally, by Lemma 8, (117), (120) and the time-continuity of Θ and $\bar\Theta$, we obtain

$$\int_0^\tau\int_\Omega \partial_t \eta^r\,dx\,dt = \int_\Omega \eta^r(x,\tau)\;dx - \int_\Omega \eta^r(x,0)\,dx. \tag{122}$$

Then (119), (121)-(122) and Lemma 8 imply for $h \in (0,\varepsilon)$

$$\Psi_d(\tau) \;\leqslant\; C\Big(\Psi_d(0) + \int_0^\tau \Psi_d(t)\,dt + h\Big) \tag{123}$$

for some constant $C = C(T, E_0, \gamma, \gamma', \kappa, \kappa', \mu, \mu', \lambda, \varepsilon, \bar{\Theta},) > 0$ independent of τ and h. Since $\tau \in (0, T]$ is arbitrary, by (123) and Gronwall's inequality we conclude

$$\Psi_d(\tau) \leqslant C\,(\Psi_d(0) + h)\, e^{C\tau}, \;\; \tau \in [0, T].$$

Thus if $\Psi(0) \to 0$ as $h \downarrow 0$, then $\sup_{\tau\in[0,T]} |\Psi_d(\tau)| \to 0$, as $h \downarrow 0$.

References

[1] J.M. Ball, Convexity conditions and existence theorems in nonlinear elasticity, *Arch. Rational Mech. Anal.* **63** (1977), 337-403.

[2] J.M. Ball, J.C. Currie, P.J. Olver, Null Lagrangians, weak continuity and variational problems of arbitrary order, *J. Funct. Anal.* **41** (1981), 135-174.

[3] J.M. Ball, Some open problems in elasticity. In *Geometry, mechanics, and dynamics. Volume in honor of the 60th birthday of J. E. Marsden.* P. Newton, Ph. Holmes and A. Weinstein, eds; Springer-Verlag, New York, 2002, pp 3-59.

[4] Y. Brenier, Hydrodynamic structure of the augmented Born-Infeld equations, *Arch. Rational Mech. Anal.* **172** (2004), 65-91

[5] P.G. Ciarlet, Mathematical Elasticity, Vol. **1**, North Holland, (1993).

[6] C. Dafermos, The second law of thermodynamics and stability, *Arch. Rational Mech. Anal.* **70** (1979), 167-179.

[7] C. Dafermos, Quasilinear hyperbolic systems with involutions, *Arch. Rational Mech. Anal.* **94** (1986), 373-389.

[8] C. Dafermos, *Hyperbolic conservation laws in continuum physics.* Third edition. Grundlehren der Mathematischen Wissenschaften, 325. Springer-Verlag, Berlin, (2010).

[9] C. Dafermos and W. Hrusa, Energy methods for quasliniear hyperbolic initial-boundary value problems. Applications to elastodynamics. *Arch. Rational Mech. Anal.* **87** (1985), 267-292.

[10] S. Demoulini, D.M.A. Stuart, A.E.Tzavaras, A variational approximation scheme for three dimensional elastodynamics with polyconvex energy, *Arch. Rational Mech. Anal.* **157** (2001), 325–344.

[11] R. DiPerna, Uniqueness of solutions to hyperbolic conservation laws, *Indiana U. Math. J.* **28** (1979), 137-188.

[12] R. DiPerna, Convergence of approximate solutions to conservation laws, *Arch. Rational Mech. Analysis* **82** (1983), 27-70.

[13] C. Lattanzio, A.E. Tzavaras, Structural properties of stress relaxation and convergence from viscoelasticity to polyconvex elastodynamics, *Archive for Rational Mechanics and Analysis* **180** (2006), 449–492.

[14] T. Qin, Symmetrizing nonlinear elastodynamic system, *J. Elasticity* **50** (1998), 245-252.

[15] C. Truesdell, W. Noll, *The non-linear field theories of mechanics.* Handbuch der Physik **III**, 3 (Ed. S.Flügge), Berlin: Springer, (1965).