THE FRACTIONAL DIFFUSION LIMIT OF A KINETIC MODEL WITH BIOCHEMICAL PATHWAY

BENOÎT PERTHAME, WEIRAN SUN, AND MIN TANG

ABSTRACT. Kinetic-transport equations that take into account the intra-cellular pathways are now considered as the correct description of bacterial chemotaxis by run and tumble. Recent mathematical studies have shown their interest and their relations to more standard models. Macroscopic equations of Keller-Segel type have been derived using parabolic scaling. Due to the randomness of receptor methylation or intra-cellular chemical reactions, noise occurs in the signaling pathways and affects the tumbling rate. Then, comes the question to understand the role of an internal noise on the behavior of the full population. In this paper we consider a kinetic model for chemotaxis which includes biochemical pathway with noises. We show that under proper scaling and conditions on the tumbling frequency as well as the form of noise, fractional diffusion can arise in the macroscopic limits of the kinetic equation. This gives a new mathematical theory about how long jumps can be due to the internal noise of the bacteria.

To appear in ZAMP

2010 *Mathematics Subject Classification.* 35B25; 35R11; 82C40; 92C17
Keywords and phrases. Mathematical biology; kinetic equations; chemotaxis; asymptotic analysis; run and tumble; biochemical pathway; Fractional Laplacian ; Lévy walk.

INTRODUCTION

Kinetic-transport equations are often used to describe the population dynamics of bacteria moving by run-and-tumble. One of the key biological properties relating to bacteria movement is how a bacterium determines its tumbling frequency. The tumbling frequency is the rate for a running bacterium to stop and change its moving direction. Recently it has been found that, for a large class of bacteria, the tumbling frequencies depend on the level of the external chemotactic signal as well as the internal states of the bacteria. Based on this observation, kinetic models incorporating the intracellular chemo-sensory system are introduced in [11, 26], which write

$$\partial_t q + v \cdot \nabla_x q + \partial_y \left(f(y,S) q \right) = \Lambda(y,S)(\langle q \rangle - q) \,. \tag{0.1}$$

Here $q(t, x, \mathbf{v}, y)$ denotes the probability density function of bacteria at time t, position $x \in \mathbb{R}^d$, velocity $\mathbf{v} \in \mathbb{V}$ with $\mathbb{V}$ the sphere (or the ball) with radius V_0, and the intra-cellular molecular content $y \in \mathbb{R}$. The function $f(y, S)$ takes into account the slowest reaction in the chemotactic signal transduction pathways for a given external effective signal S. The right hand side terms in (0.1) describes the velocity jump process where $\Lambda(y, S)$ is the tumbling frequency. The specific forms of $f(y, S)$ and $\Lambda(y, S)$ depend on different types of bacteria, where a linear cartoon description for $f(y, S)$ is used in [11] and more sophisticated forms for E.coli chemotaxis have been studied in [16, 22]. The frequency $\Lambda(y, S)$ is determined by the regulation of the flagellar motors by biochemical pathways [16] and it usually has steep transition with respect to y.

Date: 15/02/2018.

In the case when the external signal S is absent, macroscopic models have been derived from (0.1) in the diffusion regime. For example, in [10–12, 25, 29] the authors have recovered the Keller-Segel type of equations that govern the dynamics of cell density as the diffusion limit of (0.1). These results indicate the underlying microscopic dynamics of the bacteria follow the Brownian motion.

Recent experiments of tracking individual cell trajectories, however, showed that some bacteria actually adopts a Lévy-flight type movement instead of the Brownian motion [4, 7]. Lévy flight is a random process whose path length distribution obeys a power-law decay, as opposed to the Brownian motion whose path length distribution decays exponentially. Therefore, a Lévy flight exhibits a non-negligible probability of "long jumps". Various explanations have been proposed to understand the origin of the long jumps. For example, the works in [17, 28] relate molecular noise to power-law switching in bacterial flagellar motors. The model in [18] suggests that the fluctuation in CheR (a protein which regulates the receptor activity) can induce the power-law distribution of the path length.

Motivated by the aforementioned experimental and theoretical work, we study in this paper a kinetic model that incorporates noise in the intra-cellular molecular content y in equation (0.1). Similar equation has appeared in [24]. Our main goal is to rigorously derive fractional diffusion equations (which correspond to Lévy processes) from the new kinetic equation. The particular equation we consider is as follows:

$$\epsilon^{1+\mu}\partial_t q_\epsilon + \epsilon v\cdot\nabla_x q_\epsilon - \epsilon^s\partial_y\left(D(y)Q_0(y)\partial_y\frac{q_\epsilon}{Q_0}\right) = \Lambda(y)(\langle q_\epsilon\rangle - q_\epsilon)\,, \tag{0.2}$$

$$q_\epsilon(0,x,v,y) = q^{\text{in}}(x,v,y) := \rho^{\text{in}}(x)Q_0(y) \geq 0\,, \tag{0.3}$$

where $0<\mu<1$, $0<s<1+\mu$, and

$$\langle q_\epsilon\rangle(t,x,y) := \int_{\mathbb{V}} q_\epsilon(t,x,v,y)\,\mathrm{d}v\,,$$

with $\mathbb{V}$ being the sphere $\partial B(0,V_0)\subseteq\mathbb{R}^d$ and $\mathrm{d}v$ is the normalized surface measure. The precise range of μ, s is specified in (1.5) and (1.6) in Section 1. For later purpose, we also introduce the notation

$$\rho_\epsilon(t,x) = \int_{\mathbb{R}}\langle q_\epsilon\rangle(t,x,y)\,\mathrm{d}y.$$

The given function $Q_0(y)$ can be viewed as the equilibrium distribution in y in absence of outside signal. One can decompose the y derivative term on the left hand side of (0.2) into two terms

$$\epsilon^s\partial_y\left(D(y)Q_0(y)\partial_y\frac{q_\epsilon}{Q_0}\right) = \epsilon^s\partial_y\left(D(y)\partial_y q_\epsilon\right) - \epsilon^s\partial_y\left(D(y)\frac{\partial_y Q_0}{Q_0}q_\epsilon\right). \tag{0.4}$$

Therefore, $D(y)$ turns out to be the diffusion coefficient in y. Compared with the model in (0.1), the diffusion term in y takes into account the intrinsic noise of the signally pathway. For technical reasons we consider a specific form of noise and leave open the derivation with more general types. The initial datum $q^{\text{in}}(x,y,v)$ is assumed to be independent of ϵ and takes a separated form for simplicity. One can also consider the more general case where the sequence of initial data converges as $\epsilon\to 0$.

We identify conditions on the parameters and coefficients that give rise to a fractional diffusion limit as $\epsilon\to 0$. We show that under these conditions, there exists $\rho(t,x)$ such that the density function q_ϵ satisfies

$$q_\epsilon(t,x,v,y)\to\rho(t,x)Q_0(y) \qquad \text{as } \epsilon\to 0 \tag{0.5}$$

and ρ solves

$$\begin{cases} \partial_t\rho(t,x) + \nu\left(-\Delta\right)^{\frac{1+\mu}{2}}\rho = 0, \\ \rho(0,x) = \rho^{\text{in}}(x)\,, \end{cases} \tag{0.6}$$

where the constant $\nu>0$ can be computed explicitly.

Deriving fractional diffusion models from a classical kinetic model (where the density function only depends on (t, x, v)) is initiated in [15] by probabilistic methods and [1, 9, 20] by analytic methods. The case of boundary conditions is treated in [5]. In these works, the fractional diffusion arises either from a fat-tail equilibrium distribution in the velocity v [1, 9, 20] or the degeneracy of the collision frequency for small velocities [1, 15]. In some recent works in [2, 8], similar results have been extended to kinetic models for chemotaxis, where a fractional diffusion equation with advection is derived when there exist small bias along the direction of the chemical gradient. We note that, in all previous works for chemotaxis, the fractional diffusion occurs from fat tail distribution with unbounded velocities v, while in chemotaxis it is more realistic to consider bounded bacteria velocities. This is our main contribution, to perform a rigorous derivation with the more physical assumption of bounded velocities. There are also works deriving fractional diffusion limits from kinetic equations with extended variables. For example, the models in [13, 14] have the free path length as an independent variable and fraction diffusion limits are derived under the condition that the second moments of the path length distribution functions are unbounded. The models in [13, 14] phenomenologically incorporate occasional long jumps in the tumbling frequency, while $\Lambda(y)$ in our model depends on the internal state. We also note that , with bounded velocities, degeneracy of the collision kernel is also used to obtain abnormal diffusion in [19].

In proving the fraction diffusion limit, we note two main differences in our methodology compared with earlier works. First, unlike in the (fractional) diffusion limits of classical kinetic equations (with only (t, x, v) as their independent variables), the mass conservation equation in terms of $\rho_\epsilon = \int_{\mathbb{R}} \int_{\mathbb{V}} q_\epsilon \,\mathrm{d}v \,\mathrm{d}y$ does not seem to be the proper setting for deriving the limiting equation. This is indeed due to the appearance of the extended variable y and the additional noise term. Instead, we need to consider a properly weighted quantity $\int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_{\mathbb{V}} \chi_0 q_\epsilon \,\mathrm{d}v \,\mathrm{d}y \,\mathrm{d}x$ where χ_0 satisfies the dual equation given by (3.2). This weighted quantity thus encodes the effect of the noise. We note that working with a weighted density seems to be a general setting when deriving (fractional) diffusion limits of kinetic equations with extended variables. See for example in [14], where the macroscopic equations for a non-classical kinetic equation are derived for the weighted density function against the path length distribution. Compared with [14], the choice of the weight function χ_0 in this paper is much less obvious. Second, the derivation of the fractional diffusion equations in [1, 9, 20] relies on the method of auxiliary functions or a related Hilbert expansion. In the current paper, we use the method of moments [6] which leads to reformulate the equation for q_ϵ in a convenient way (see (3.6)) and apply it in the flux term of the conservation law. This framework is more standard, intuitive and consistent with the classical Chapman-Enskog method of deriving macroscopic limits of kinetic equations.

The paper is organized as follows. In Section 1 we state our assumptions on the parameter ranges and the main result, i.e., the validity of (0.6). The proof of the main result is presented in Section 2 and Section 3. Specifically, we state several a priori bounds and estimates in Section 2 and give the main core of the proof the main result in Section 3. Finally we conclude in Section 4.

1. Assumptions and main results

Assumptions on the coefficients. Let $M_0 > 1$, A_0, A_1 be fixed numbers.. We are given a smooth function $Q_0(y)$ which describes the equilibrium in the internal state y,

$$Q_0(y) = \begin{cases} c^+ |y|^{-\sigma}, & y > M_0, \\ c^- |y|^{-\sigma}, & y < -M_0, \end{cases} \qquad \sigma > 1, \qquad Q_0(y) > 0, \qquad \int_{\mathbb{R}} Q_0 \, dy = 1. \tag{1.1}$$

The mechanism at work here is the degeneracy of the tumbling rate Λ, a smooth function on $\mathbb{R}$, namely

$$\Lambda(y) = \begin{cases} \mathcal{O}(1)\,, & y \geq M_0\,, \\ |y|^{-\beta}\,, & y \leq -M_0\,, \end{cases} \qquad |\Lambda'(y)| \leq \frac{A_0}{y^\gamma} \quad \text{for } y > M_0\,, \tag{1.2}$$

Assume that the diffusion coefficient D is a smooth functions on $\mathbb{R}$ such that

$$D(y) = \begin{cases} \mathcal{O}(1)\,, & y \in [-M_0, M_0]\,, \\ A_1|y|^{n+1}\,, & |y| \geq M_0\,. \end{cases} \tag{1.3}$$

for some $n > 0$ whose range is specified in (1.5). The conditions on σ, β, γ are also described in (1.5).

Assumptions on the initial data. We assume that, for some constant B,

$$q^{\text{in}} \leq BQ_0, \qquad \int_{\mathbb{R}^d}\int_{\mathbb{R}}\int_{\mathbb{V}} \frac{(q^{\text{in}})^2}{Q_0}(x,v,y)\,\mathrm{d}v\,\mathrm{d}y\,\mathrm{d}x \leq B, \qquad \int_{\mathbb{R}^d}\int_{\mathbb{R}}\int_{\mathbb{V}} q^{\text{in}}(x,v,y)\,\mathrm{d}v\,\mathrm{d}y\,\mathrm{d}x \leq B\,. \tag{1.4}$$

Parameter range. The main assumptions of the parameters are

$$n > \sigma > 1\,, \qquad s > 1\,, \qquad \gamma > \frac{n-\sigma}{2} + 1\,, \qquad \beta > n-1\,, \qquad \beta + n - 1 > s\beta > \beta + \sigma - 1\,. \tag{1.5}$$

The analysis below leads to the relation

$$\mu = \frac{n-1}{\beta} \in (0,1)\,. \tag{1.6}$$

Therefore, we observe that

$$\beta + n - 1 > s\beta \Longleftrightarrow 1 + \mu > s\,,$$

which makes the time-derivative term in equation (0.2) a (formally) high-order term.

Remark 1.1. Note that μ denotes the strength of the fractional diffusion for ρ. It can be any number in $(0,1)$ by proper choices of n, β, s satisfying (1.5). One family of examples of $(\mu, n, \sigma, \beta, \gamma, s)$ are: let $\mu \in (0,1)$ and $n > 1$ be arbitrary. Let Λ be a constant for $y > M_0$ such that $\gamma = \infty$. Then $\beta = \frac{n-1}{\mu}$ and the region for (σ, s) is

$$1 + \mu > s > 1 + \mu\frac{\sigma-1}{n-1}.$$

Our main result of this paper is

Main Theorem. *Let q_ϵ be the solution of (0.2) with the above assumptions (1.1)–(1.4). Suppose the parameters $n, \sigma, s, \beta, \gamma, \mu$ satisfy the parameter range (1.5) and (1.6). Then, as $\epsilon \to 0$, the limit (0.5) holds in the sense that $\frac{q_\epsilon}{Q_0}$ converges $L^\infty - w*$ to $\rho \in L^\infty(\mathbb{R}^+; L^1 \cap L^\infty(\mathbb{R}^d))$ and ρ satisfies the fractional Laplacian equation (0.6) with $\nu = B_0/\nu_0$ where B_0, ν_0 are defined in (3.4) and (3.11) respectively.*

This theorem establishes a possible explanation of the observed Lévy flights, $\mu < 1$, in bacterial trajectories based on internal signaling pathways under the biologically relevant condition that velocities during jumps are bounded. The assumption on Λ indicates that for some extreme values of the internal state, the bacteria should rarely tumble, this requirement is necessary for long jumps. The role of noise is also strongly underlined with our assumption on D. It is likely that several different regimes might lead to interesting macroscopic limits. For instance, standard diffusion, a widely studied in the biological context, can be achieved and, not surprizingly, this regime is much simpler to analyze. It arises with a different range of parameters when $\Lambda = \mathcal{O}(1), D = \mathcal{O}(1)$. Then, the equilibrium state Q_0 has exponential tails for $y \sim \pm\infty$ and the limit turns out to be the classical diffusion, i.e., $\mu = 1$.

The proof of the main theorem is carried out using the Fourier method combined with the moment method. We first show that due to the structure of Λ, the equilibrium state Q_0 decays algebraically near $y = -\infty$ and exponentially near $y=\infty$. The key idea to prove the main theorem is to show that the fractional diffusion in x is generated by the (slow) algebraic decay of Q_0 in y near $-\infty$. It is then natural to separate the analysis for the part where y is close to $-\infty$ from the rest. The Fourier method makes it clear that the fractional diffusion arises from the transport term $v\cdot\nabla_x q_\epsilon$, similar as in the regular diffusion case. The rest of this paper is devoted to the proof of the main theorem.

2. Estimates and a priori bounds

2.1. **Relative entropy estimates.** The method of relative entropy (see for instance [21,23]) can be applied to provide us with useful a priori bounds for all $t\geq 0$, which is summarized as

Lemma 2.1. *Let q_ϵ be the solution to equation (0.2) with an initial data $q^{\rm in}$ satisfying*

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}}\int_{\mathbb{V}} q^{\rm in}\,{\rm d}v\,{\rm d}y\,{\rm d}x = 1 \quad \textit{and} \quad 0\leq q^{\rm in}\leq BQ_0 \quad \textit{for some constant } B>0.$$

Then, q_ϵ satisfies

$$0\leq q_\epsilon\leq BQ_0, \qquad \int_{\mathbb{R}^d}\int_{\mathbb{R}}\int_{\mathbb{V}}\frac{q_\epsilon^2}{Q_0}(t,x,v,y)\,{\rm d}v\,{\rm d}y\,{\rm d}x\leq B, \qquad \int_{\mathbb{R}^d}\int_{\mathbb{R}}\int_{\mathbb{V}} q_\epsilon(t,x,v,y)\,{\rm d}v\,{\rm d}y\,{\rm d}x\leq B\,, \tag{2.1}$$

and

$$\int_0^\infty\int_{\mathbb{R}^d}\int_{\mathbb{R}}\int_{\mathbb{V}} D(y)Q_0(y)\left(\partial_y\left(\frac{q_\epsilon}{Q_0}\right)\right)^2\leq B\epsilon^{1+\mu-s}\,, \qquad \int_0^\infty\int_{\mathbb{R}^d}\int_{\mathbb{R}}\int_{\mathbb{V}}\Lambda(y)\frac{(q_\epsilon-\langle q_\epsilon\rangle)^2}{Q_0}\leq B\epsilon^{1+\mu}\,. \tag{2.2}$$

Proof. We multiply equation (0.2) by $\dfrac{q_\epsilon}{Q_0}$ and integrating in x,v,y. The resulting equation is

$$\frac{1}{2}\epsilon^{1+\mu}\frac{\rm d}{{\rm d}t}\int_{\mathbb{R}^d}\int_{\mathbb{R}}\int_{\mathbb{V}}\frac{q_\epsilon^2}{Q_0}+\epsilon^s\int_{\mathbb{R}^d}\int_{\mathbb{R}}\int_{\mathbb{V}} D(y)Q_0\left(\partial_y\left(\frac{q_\epsilon}{Q_0}\right)\right)^2+\int_{\mathbb{R}^d}\int_{\mathbb{R}}\int_{\mathbb{V}}\Lambda(y)\frac{(q_\epsilon-\langle q_\epsilon\rangle)^2}{Q_0}=0\,.$$

Integrating the above equation in t then gives the integral bounds in (2.1)-(2.2), since the initial data satisfies the bound

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}}\int_{\mathbb{V}}\frac{(q^{\rm in})^2}{Q_0}\,{\rm d}v\,{\rm d}y\,{\rm d}x\leq B\int_{\mathbb{R}^d}\int_{\mathbb{R}}\int_{\mathbb{V}} q^{\rm in}\,{\rm d}v\,{\rm d}y\,{\rm d}x = B\,.$$

To show that $0\leq q_\epsilon\leq BQ_0$, by lenearity, we only need to prove that if $q^{\rm in}\geq 0$, then the solution $q_\epsilon\geq 0$. Indeed, let $s_\delta\in C^2(\mathbb{R})$ be a convex, decreasing function such that

$$s_\delta(x)\to\max\{0,-x\} \quad \text{as } \delta\to 0.$$

In other words, $s_\delta(x)$ is a smooth approximation of the function $x_- := \max\{0,-x\}$. Multiplying equation (0.2) by $s'_\delta(\frac{q_\epsilon}{Q_0})$ and integrating in x,v,y, the transport term $v\cdot\nabla_x q_\epsilon$ vanishes. To treat the term with y-derivatives we use the convexity of s_δ and equation (0.4), we have

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}} s'_\delta(\frac{q_\epsilon}{Q_0})\partial_y\left(DQ_0\partial_y\frac{q_\epsilon}{Q_0}\right)=-\int_{\mathbb{R}^d}\int_{\mathbb{R}} s''_\delta(\frac{q_\epsilon}{Q_0})D(y)Q_0\left|\partial_y\frac{q_\epsilon}{Q_0}\right|^2\,{\rm d}y\,{\rm d}x\leq 0\,, \tag{2.3}$$

Moreover, the tumbling term reads

$$\Lambda(y)Q_0(y)\int_{\mathbb{V}} s'_\delta(\frac{q_\epsilon}{Q_0})\,\left[\left\langle\frac{q_\epsilon}{Q_0}\right\rangle-\frac{q_\epsilon}{Q_0}\right]\,{\rm d}v=\int_{\mathbb{V}}\left[s'_\delta(\frac{q_\epsilon}{Q_0})-s'_\delta\left(\left\langle\frac{q_\epsilon}{Q_0}\right\rangle\right)\right]\left[\left\langle\frac{q_\epsilon}{Q_0}\right\rangle-\frac{q_\epsilon}{Q_0}\right]\,{\rm d}v\leq 0\,. \tag{2.4}$$

Combining (2.3) and (2.4), we get

$$\epsilon^{1+\mu}\frac{\rm d}{{\rm d}t}\int_{\mathbb{R}^d}\int_{\mathbb{R}}\int_{\mathbb{V}} q_\epsilon^-\leq 0\,.$$

Since $q_-^{\rm in} = 0$, integrating in t, x, y gives

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}} q_\epsilon^- \,\mathrm{d}y\,\mathrm{d}x \le 0 \qquad \text{for all } t \ge 0.$$

Since by definition $q_\epsilon^- \ge 0$, we have that $q_\epsilon^- = 0$ for all time, which is equivalent to $q_\epsilon \ge 0$ for all time. Note that since BQ_0 is a solution to equation (0.2), the upper bound follows by the positivity of $q_\epsilon - BQ_0$. □

A first and immediate consequence of Lemma 2.1 is the weak convergence of q_ϵ:

Lemma 2.2. *After extraction of a subsequence, still denoted by q_ϵ, we have*

$$\frac{q_\epsilon}{Q_0}(t,x,v,y) \to \rho(t,x), \qquad \textit{in } L^\infty(\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{V}) - w^*\,,$$

where $\rho(t,x) \in L^\infty(\mathbb{R}^+; L^1 \cap L^\infty(\mathbb{R}^d))$.

2.2. **A priori bounds.** Another consequence of the a priori estimate is the following lemma:

Lemma 2.3. *Suppose q_ϵ satisfies the a priori bound (2.2). Denote*

$$R_\epsilon = \int_{\mathbb{R}} q_\epsilon \,\mathrm{d}y\,.$$

Then there exists a constant $C > 0$ independent of t, x, y and ϵ such that for all $y \in \mathbb{R}$, we have

$$\left|\frac{q_\epsilon}{Q_0}(t,x,v,y) - R_\epsilon(t,x,v)\right| \le C H^{1/2}(t,x,v)\,, \qquad \forall y \in \mathbb{R},\ v \in \mathbb{V}, \tag{2.5}$$

where

$$H(t,x,v) = \int_{\mathbb{R}} Q_0(y) D(y) \left(\partial_{y'}\Big(\frac{q_\epsilon(t,x,y,v)}{Q_0(y)}\Big)\right)^2 \mathrm{d}y\,. \tag{2.6}$$

Proof. By the a priori bound (2.2), it holds that

$$\begin{aligned}
\left|\frac{q_\epsilon}{Q_0} - \rho_\epsilon\right| &= \left|\frac{q_\epsilon(y)}{Q_0(y)} - \int \frac{q_\epsilon(z)}{Q_0(z)} Q_0(z)\,\mathrm{d}z\right| \le \int_{\mathbb{R}} \left|\frac{q_\epsilon(y)}{Q_0(y)} - \frac{q_\epsilon(z)}{Q_0(z)}\right| Q_0(z)\,\mathrm{d}z \\
&= \int_{\mathbb{R}} \left(\int_z^y \left|\partial_{y'}\left(\frac{q_\epsilon(y')}{Q_0(y')}\right)\right| \mathrm{d}y'\right) Q_0(z)\,\mathrm{d}z \\
&\le \int_{\mathbb{R}} \left(\left|\int_z^y Q_0(y')D(y')\left(\partial_{y'}\left(\frac{q_\epsilon(y')}{Q_0(y')}\right)\right)^2 \mathrm{d}y'\right|\right)^{1/2} \left(\left|\int_z^y \frac{1}{Q_0(y')D(y')}\,\mathrm{d}y'\right|\right)^{1/2} Q_0(z)\,\mathrm{d}z \\
&\le \left(\left|\int_{\mathbb{R}} \frac{1}{Q_0(y')D(y')}\,\mathrm{d}y'\right|\right)^{1/2} H^{1/2}(t,x,v)\,.
\end{aligned}$$

Near $y = \pm\infty$, we have

$$Q_0(y) \sim |y|^{-\sigma}\,, \qquad D(y) \sim |y|^{n+1}, \qquad \frac{1}{Q_0(y')D(y')} \sim \frac{1}{|y|^{n+1-\sigma}}\,,$$

which is integrable on $\mathbb{R}$ by the assumption that $n > \sigma$. Hence (2.5) holds with the constant $C = \left(\int_{\mathbb{R}} \frac{1}{Q_0(y')D(y')}\,\mathrm{d}y'\right)^{1/2}$. □

2.3. **From the Fourier side.** In fact, we need Fourier versions of the a priori bounds and thus we denote the Fourier transform in x of u with a $\widehat{u}$, in particular

$$\widehat{q}(t,\xi,v,y) = \int_{\mathbb{R}^d} q(t,x,v,y) e^{ix.\xi} dx.$$

For instance, from (2.2), we conclude, using Parseval identity,

$$\int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_{\mathbb{V}} \Lambda(y) \frac{|\widehat{q}_\epsilon - \langle \widehat{q}_\epsilon \rangle|^2}{Q_0} \leq B\epsilon^{1+\mu}. \tag{2.7}$$

Also, following the same calculations as in Lemma 2.3, we have

$$\left| \frac{\widehat{q}_\epsilon}{Q_0}(t,\xi,v,y) - \widehat{R}_\epsilon(t,\xi,v) \right| \leq C K^{1/2}(t,\xi,v), \qquad \forall y \in \mathbb{R},\ v \in \mathbb{V}, \tag{2.8}$$

with

$$K(t,\xi,v) = \int_{\mathbb{R}} Q_0(y) D(y) \left| \partial_y \Big(\frac{\widehat{q}_\epsilon(t,\xi,y,v)}{Q_0(y)} \Big) \right|^2 \mathrm{d}y. \tag{2.9}$$

And Parseval identity gives

$$\int_0^\infty \int_{\mathbb{V}} \int_{\mathbb{R}^d} K(t,\xi,v)\,\mathrm{d}\xi\,\mathrm{d}v\,\mathrm{d}t = \int_0^\infty \int_{\mathbb{V}} \int_{\mathbb{R}^d} H(t,x,v)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t \leq B\epsilon^{1+\mu-s}. \tag{2.10}$$

Because, for any $M_1 > 0$

$$\frac{1}{2}\int_0^\infty \int_{\mathbb{R}^d} \int_{y>-M_1} \int_{\mathbb{V}} \frac{|\widehat{q}_\epsilon - \widehat{\rho}_\epsilon Q_0|^2}{Q_0} \leq \int_0^\infty \int_{\mathbb{R}^d} \int_{y>-M_1} \int_{\mathbb{V}} Q_0 \left| \frac{\widehat{q}_\epsilon}{Q_0} - \frac{\langle \widehat{q}_\epsilon \rangle}{Q_0} \right|^2 + \int_0^\infty \int_{\mathbb{R}^d} \int_{y>-M_1} Q_0 \left| \frac{\langle \widehat{q}_\epsilon \rangle}{Q_0} - \left\langle \widehat{R}_\epsilon \right\rangle \right|^2.$$

Finally, combining (2.7), (2.8) and (2.10), we also infer that, in Fourier variable, we have for all $M_1 > 0$,

$$\int_0^\infty \int_{\mathbb{R}^d} \int_{y>-M_1} \int_{\mathbb{V}} \frac{(\widehat{q}_\epsilon - \widehat{\rho}_\epsilon Q_0)^2}{Q_0} \leq C\epsilon^{1+\mu-s}. \tag{2.11}$$

In the later proof we simply choose $M_1 = M_0$.

2.4. **Useful calculations.** Two integrals repeatedly appear in the rest of this note. We list them out as a lemma:

Lemma 2.4. *Suppose*

$$0 < \alpha + 1 < 2\beta_1, \qquad 0 < \alpha + 1 < \beta_2, \qquad \beta_1, \beta_2 > 0.$$

Then the following integrals are well-defined and there exists a constant $c_1 > 0$ such that

$$\int_{-\infty}^0 \frac{|y|^\alpha}{1 + (\epsilon|\xi\cdot v||y|^{\beta_1})^2}\,\mathrm{d}y = c_1 (\epsilon|\xi\cdot v|)^{-\frac{\alpha+1}{\beta_1}}, \qquad \int_{-\infty}^0 \frac{|y|^\alpha}{\sqrt{1 + (\epsilon|\xi\cdot v||y|^{\beta_2})^2}}\,\mathrm{d}y = c_2 (\epsilon|\xi\cdot v|)^{-\frac{\alpha+1}{\beta_2}}.$$

Proof. Make a change of variable $z = \epsilon|\xi\cdot v||y|^{\beta_1}$ in the first integral and $z = \epsilon|\xi\cdot v||y|^{\beta_2}$ in the second one. Then

$$\int_{-\infty}^0 \frac{|y|^\alpha}{1 + (\epsilon|\xi\cdot v||y|^{\beta_1})^2}\,\mathrm{d}y = \frac{1}{\beta_1}(\epsilon|\xi\cdot v|)^{-\frac{\alpha+1}{\beta_1}} \int_0^\infty \frac{z^{\frac{\alpha+1}{\beta_2}-1}}{1+z^2}\,\mathrm{d}z = c_1 (\epsilon|\xi\cdot v|)^{-\frac{\alpha+1}{\beta_1}},$$

$$\int_{-\infty}^0 \frac{|y|^\alpha}{\sqrt{1 + (\epsilon|\xi\cdot v||y|^{\beta_2})^2}}\,\mathrm{d}y = \frac{1}{\beta_2}(\epsilon|\xi\cdot v|)^{-\frac{\alpha+1}{\beta_2}} \int_0^\infty \frac{z^{\frac{\alpha+1}{\beta}-1}}{\sqrt{1+z^2}}\,\mathrm{d}z = c_2 (\epsilon|\xi\cdot v|)^{-\frac{\alpha+1}{\beta_2}},$$

where the integrability of the z-integral is guaranteed respectively by the condition $0 < \frac{\alpha+1}{\beta_1} < 2$ and $0 < \frac{\alpha+1}{\beta_2} < 1$, or equivalently, $0 < \alpha+1 < 2\beta_1$ and $0 < \alpha+1 < \beta_2$. □

3. Asymptotics

3.1. **A solution of the dual problem.** We are going to make use of a weight in the variable y that is built by duality. Let $\chi_0(y)$ be given by

$$\chi_0(y) = \int_{-\infty}^{y} \frac{1}{D(z)Q_0(z)}\,\mathrm{d}z\,. \tag{3.1}$$

It is a solution of the dual problem in y because

$$\partial_y(D(y)Q_0(y)\partial_y\chi_0) = 0. \tag{3.2}$$

The properties of χ_0 are summarized in the following lemma:

Lemma 3.1. *With Q, D as in (1.1), (1.3) and with the parameter range (1.5), $\chi_0 \in C_b(\mathbb{R})$ is nonnegative, increasing and*

$$\chi_0 = \begin{cases} \mathcal{O}(1)\,, & y > -M_0\,, \\ C^-|y|^{\sigma-n}\,, & y < -M_0\,. \end{cases}$$

Proof. The non-negativity and monotonicity are both clear by the positivity of D and Q_0. We check the behaviour of χ_0 near $y = \pm\infty$. Recall that $\sigma < n$. Thus for $y < -M_0$,

$$\int_{-\infty}^{y} \frac{1}{D(z)Q_0(z)}\,\mathrm{d}z = \frac{1}{c^- A_1}\int_{-\infty}^{y} \frac{\mathrm{d}z}{z^{n+1-\sigma}} = C^-|y|^{\sigma-n}\,.$$

For $y > M_0$, the same decay holds for D and Q_0, and thus $\frac{1}{D(z)Q_0(z)}$ is integrable and it proves that χ_0 is bounded. □

3.2. **The proof of Theorem 1.** We derive the limiting equation by multiplying both sides of (0.2) by the weight function $\chi_0(y)$ and integrate in y and v. Thanks to the property that χ_0 solves the dual problem in y, we find

$$\partial_t \int_{\mathbb{R}}\int_{\mathbb{V}} q_\epsilon\chi_0\,\mathrm{d}y\,\mathrm{d}v + \mathrm{div}_x J_\epsilon = 0, \qquad J_\epsilon := \frac{1}{\epsilon^\mu}\int_{\mathbb{R}}\int_{\mathbb{V}} v q_\epsilon \chi_0\,\mathrm{d}y\,\mathrm{d}v. \tag{3.3}$$

We only need to show the convergence in the distributional sense of the equation in (3.3). The initial condition of ρ follows through the weak formulation that incorporates the initial data. We observe that, using Lemma 2.2, the weak limit of the first term is

$$\int_{\mathbb{R}}\int_{\mathbb{V}} q_\epsilon\chi_0\,\mathrm{d}y\,\mathrm{d}v \to \int_{\mathbb{R}}\int_{\mathbb{V}} \rho(t,x)Q_0(y)\chi_0\,\mathrm{d}y\,\mathrm{d}v = B_0\rho(t,x), \qquad B_0 = \int_{\mathbb{R}} Q_0(y)\chi_0\,\mathrm{d}y\,\mathrm{d}v. \tag{3.4}$$

It remains to identify the limit of the flux J_ϵ. Notice that the a priori estimates do not provide any L^p bound on J_ϵ and it turns out that this term is a fractional derivative in x. This motivates to work in the Fourier variable.

We are going to prove that, for some constant ν_0 defined in (3.11), as $\epsilon \to 0$,

$$\widehat{\mathrm{div}_x J_\epsilon} \to \nu_0|\xi|^{\frac{n-1}{\beta}+1}\widehat{\rho}, \qquad \text{in the sense of distributions (or in } \mathcal{D}'(\mathbb{R}^+\times\mathbb{R}^d)) \tag{3.5}$$

and thus conclude the proof of Theorem 1.

3.3. **Identifying the flux J_ϵ.** We apply Fourier transform in x for (0.2), and denote by ξ the Fourier variable. We obtain

$$\epsilon^{1+\mu}\partial_t\widehat{q}_\epsilon + i\epsilon\xi\cdot v\,\widehat{q}_\epsilon - \epsilon^s\partial_y\left(D(y)Q_0(y)\partial_y\frac{\widehat{q}_\epsilon}{Q_0(y)}\right) = \Lambda(y)(\langle\widehat{q}_\epsilon\rangle - \widehat{q}_\epsilon)\,,$$

from which, combining the terms including $\widehat{q}_\epsilon$, we get

$$\widehat{q}_\epsilon - \langle\widehat{q}_\epsilon\rangle = -\frac{i\epsilon\xi\cdot v}{i\epsilon\xi\cdot v+\Lambda}\langle\widehat{q}_\epsilon\rangle + \frac{\epsilon^s}{i\epsilon\xi\cdot v+\Lambda}\partial_y\left(D(y)Q_0(y)\partial_y\frac{\widehat{q}_\epsilon}{Q_0(y)}\right) - \epsilon^{1+\mu}\frac{1}{i\epsilon\xi\cdot v+\Lambda}\partial_t\widehat{q}_\epsilon\,. \tag{3.6}$$

Therefore, we may also decompose $\widehat{\mathrm{div}_x J_\epsilon} = \frac{1}{\epsilon^\mu}\int_{\mathbb{R}}\int_{\mathbb{V}}(i\xi\cdot v)\,\chi_0\widehat{q}_\epsilon\,\mathrm{d}y\,\mathrm{d}v$ according to the three terms on the right hand side as

$$\widehat{\mathrm{div}_x J_\epsilon}(t,\xi) = \frac{1}{\epsilon^\mu}\int_{\mathbb{R}}\int_{\mathbb{V}}(i\xi\cdot v)\,\chi_0\,(\widehat{q}_\epsilon - \langle\widehat{q}_\epsilon\rangle)\,\mathrm{d}y\,\mathrm{d}v = i\xi\cdot\widehat{J}^1_\epsilon + \widehat{J}^2_\epsilon + \partial_t\widehat{J}^3_\epsilon. \tag{3.7}$$

We show in the following subsections that the last two contributions vanish as $\epsilon\to 0$ and the fractional Laplacian stems from the first term.

3.4. **The term $\widehat{J}^1_\epsilon$.** Separate the imaginary and real part such that

$$\frac{i\epsilon\xi\cdot v}{i\epsilon\xi\cdot v+\Lambda} = \frac{(\epsilon\xi\cdot v)^2}{(\epsilon\xi\cdot v)^2+\Lambda^2} + \frac{(i\epsilon\xi\cdot v)\,\Lambda}{(\epsilon\xi\cdot v)^2+\Lambda^2}\,.$$

Using the symmetry of $\mathbb{V}$, contribution from the real part above vanishes and we have

$$\widehat{J}^1_\epsilon(t,\xi) = \frac{-1}{\epsilon^\mu}\int_{\mathbb{V}}\int_{\mathbb{R}} v\chi_0\frac{i\epsilon\xi\cdot v}{i\epsilon\xi\cdot v+\Lambda}\langle\widehat{q}_\epsilon\rangle\,\mathrm{d}y\,\mathrm{d}v = \frac{-i}{\epsilon^\mu}\int_{\mathbb{V}}\int_{\mathbb{R}} v\chi_0\frac{\Lambda\epsilon\xi\cdot v}{(\epsilon\xi\cdot v)^2+\Lambda^2}\langle\widehat{q}_\epsilon\rangle\,\mathrm{d}y\,\mathrm{d}v.$$

Therefore we may write (notice that $\widehat{\rho}_\epsilon$ is bounded in L^2)

$$i\xi\cdot\widehat{J}^1_\epsilon = \widehat{\rho}_\epsilon\frac{1}{\epsilon^\mu}\int_{\mathbb{V}}\int_{\mathbb{R}}\chi_0\frac{\Lambda\epsilon\,(\xi\cdot v)^2}{(\epsilon\xi\cdot v)^2+\Lambda^2}Q_0(y)\,\mathrm{d}y\,\mathrm{d}v + i\xi\cdot\widehat{RJ^1_\epsilon}\,. \tag{3.8}$$

In order to derive the limit for the first term on the right-hand side, we divide the integration domain for y into two parts: $y\geq -M_0$ and $y<-M_0$. Note that by the definition of Λ, we have $\Lambda\sim\mathcal{O}(1)$ for $y\geq -M_0$. Hence,

$$\left|\frac{-i}{\epsilon^\mu}\int_{\mathbb{V}}\int_{y>-M_0} v\chi_0\frac{\Lambda\epsilon\xi\cdot v}{(\epsilon\xi\cdot v)^2+\Lambda^2}Q_0(y)\,\mathrm{d}y\,\mathrm{d}v\right| \leq C\epsilon^{1-\mu}|\xi|\int_{\mathbb{R}}\chi_0 Q_0\,\mathrm{d}y \leq C\epsilon^{1-\mu}|\xi|\,,$$

since $\chi_0 Q_0\in L^1(\mathbb{R})$. Then,

$$\widehat{\rho}_\epsilon\frac{-i}{\epsilon^\mu}\int_{\mathbb{V}}\int_{y>-M_0} v\chi_0\frac{\Lambda\epsilon\xi\cdot v}{(\epsilon\xi\cdot v)^2+\Lambda^2}Q_0(y)\,\mathrm{d}y\,\mathrm{d}v \to 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^+\times\mathbb{R}^d)\,. \tag{3.9}$$

Therefore, the contribution in the the integral term in (3.8) comes from the values $y\to-\infty$ where $\Lambda(y)$ vanishes. Using Lemma 2.4 and with $v_1 = v\cdot\xi/|\xi|$, one has

$$\begin{aligned}\widehat{\rho}\lim_{\epsilon\to 0}\frac{1}{\epsilon^\mu}\int_{\mathbb{V}}\int_{y\leq -M_0}\xi\cdot v\chi_0\frac{|y|^{\beta-\sigma}\epsilon\xi\cdot v}{(\epsilon\xi\cdot v|y|^\beta)^2+1} &= \widehat{\rho}\lim_{\epsilon\to 0}\frac{1}{\epsilon^\mu}\int_{\mathbb{V}}\int_{-\infty}^0\xi\cdot v\chi_0\frac{|y|^{\beta-\sigma}\epsilon\xi\cdot v}{(\epsilon\xi\cdot v|y|^\beta)^2+1}\\ &= \widehat{\rho}\frac{1}{\epsilon^\mu}\int_{\mathbb{V}}c_1|v_1||\xi|(|v_1|\epsilon|\xi|)^{\frac{n-1}{\beta}} = \nu_0|\xi|^\mu\widehat{\rho}.\end{aligned} \tag{3.10}$$

where the diffusion coefficient ν_0 is given by

$$\nu_0 = \int_{\mathbb{V}}c_1|v_1|^{1+\frac{n-1}{\beta}}\,\mathrm{d}v\,. \tag{3.11}$$

with c_1 defined in Lemma 2.4. Note that in the above calculations, the integration limit changes from $y\leq -M_0$ to $(-\infty,0)$ in the limit since one applies the same change variables $z=\epsilon|\xi\cdot v||y|^\beta$ as in Lemma 2.4. This calculation gives the announced scale $\mu=\frac{n-1}{\beta}$ and the fractional derivative in (0.6).

Next we prove that the remainder term $\widehat{RJ^1_\epsilon}$ vanishes. Recall that the remainder term is

$$\widehat{RJ^1_\epsilon} = \frac{-i}{\epsilon^\mu} \int_{\mathbb{V}} \int_{\mathbb{R}} v\chi_0 \frac{\Lambda \epsilon \xi \cdot v}{(\epsilon\xi\cdot v)^2 + \Lambda^2} Q_0(y) \left(\frac{\langle \widehat{q}_\epsilon \rangle}{Q_0(y)} - \widehat{\rho}_\epsilon \right) \mathrm{d}y \, \mathrm{d}v.$$

For $y > -M_0$, using the the L^2 bound (2.11) together with a similar estimate for deriving (3.9), we conclude that the corresponding part vanishes. Therefore we may again consider only the tail $y < -M_0$. We control the corresponding term using estimates similar to (3.10), by

$$\begin{aligned}
&\frac{1}{\epsilon^\mu} \left(\int_{-\infty}^{0} \int_{\mathbb{V}} |v| \chi_0 \frac{|y|^\beta \epsilon |\xi\cdot v|}{(\epsilon\xi\cdot v|y|^\beta)^2 + 1} Q_0(y) \, \mathrm{d}y \, \mathrm{d}v \right) \left(\sup_y \left| \frac{\langle \widehat{q}_\epsilon(t,\xi,y) \rangle}{Q_0(y)} - \widehat{\rho}_\epsilon(t,\xi) \right| \right) \\
&= C \int_{\mathbb{V}} |v| |\xi\cdot v|^{\frac{n-1}{\beta}} \, \mathrm{d}v \ \sup_y \left| \int_{\mathbb{V}} \frac{\widehat{q}_\epsilon(t,\xi,y,v)}{Q_0(y)} \, \mathrm{d}v - \int_{\mathbb{V}} \widehat{R}_\epsilon(t,\xi,v) \, \mathrm{d}v \right| \\
&\le C|\xi|^{\frac{n-1}{\beta}} \int_{\mathbb{V}} \sup_y \left| \frac{\widehat{q}_\epsilon}{Q_0(y)} - \widehat{R}_\epsilon \right| \mathrm{d}v \le C|\xi|^{\frac{n-1}{\beta}} \int_{\mathbb{V}} K^{1/2}(t,\xi,v) \, \mathrm{d}v \le C|\xi|^{\frac{n-1}{\beta}} \left(\int_{\mathbb{V}} K(t,\xi,v) \, \mathrm{d}v \right)^{1/2}.
\end{aligned}$$

and we conclude, using (2.10) because we assume $1+\mu > s$ in (1.5)-(1.6), that $i\xi \cdot \widehat{RJ^1_\epsilon}$ vanishes in $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d)$. Finally, noting that $\widehat{\rho}_\epsilon$ converges to $\widehat{\rho}$ weakly in L^2, we obtain

$$\widehat{J}^1_\epsilon(t,\xi) \to \nu_0 |\xi|^{\frac{n-1}{\beta}+1} \widehat{\rho}.$$

3.5. **The term $\widehat{J}^2_\epsilon$.** Back to (3.7), we show that $\widehat{J}^2_\epsilon$ vanishes as $\epsilon \to 0$. The term $\widehat{J}^2_\epsilon$ is given by

$$\begin{aligned}
\widehat{J}^2_\epsilon &= \epsilon^{s-\mu} \int_{\mathbb{V}} \int_{\mathbb{R}} \frac{(i\xi\cdot v)\,\chi_0}{i\epsilon\xi\cdot v + \Lambda} \partial_y \left(D(y) Q_0(y) \partial_y \frac{\widehat{q}_\epsilon}{Q_0(y)} \right) \mathrm{d}y \, \mathrm{d}v \\
&= -\epsilon^{s-\mu} \int_{\mathbb{V}} \int_{\mathbb{R}} \left[\frac{(i\xi\cdot v)\,\partial_y \chi_0}{i\epsilon\xi\cdot v + \Lambda} - \frac{(i\xi\cdot v)\,\chi_0 \partial_y \Lambda}{(i\epsilon\xi\cdot v + \Lambda)^2} \right] D(y) Q_0(y) \partial_y \frac{\widehat{q}_\epsilon}{Q_0(y)} \, \mathrm{d}y \, \mathrm{d}v
\end{aligned}$$

after integrating by parts.

Recalling the definition of K in (2.9), and using the Cauchy-Schwarz inequality, we can get the upper bound

$$\begin{aligned}
|\widehat{J}^2_\epsilon|^2 &\le C\epsilon^{2(s-\mu)} \int_{\mathbb{R}} \int_{\mathbb{V}} D(y) Q_0(y) \left[\frac{|\xi\cdot v|^2 (\partial_y \chi_0)^2}{|\epsilon\xi\cdot v|^2 + \Lambda^2} + \frac{|\xi\cdot v|^2 \chi_0^2 (\partial_y \Lambda)^2}{((\epsilon\xi\cdot v)^2 + \Lambda^2)^2} \right] \mathrm{d}v \, \mathrm{d}y \int_{\mathbb{V}} K(t,\xi,v) \, \mathrm{d}v \\
&= C\epsilon^{2(s-\mu)} \left[G^1(t,\xi) + G^2(t,\xi) \right] \int_{\mathbb{V}} K(t,\xi,v) \, \mathrm{d}v.
\end{aligned}$$

We begin with the term G^1. Using the definitions of χ_0 in (3.1), we have

$$G^1(t,\xi) = \int_{\mathbb{R}} \int_{\mathbb{V}} \frac{1}{D(y) Q_0(y)} \frac{|\xi\cdot v|^2}{|\epsilon\xi\cdot v|^2 + \Lambda^2} \, \mathrm{d}v \, \mathrm{d}y.$$

Because, for $|y| \gg 1$, $\frac{1}{D(y)Q_0(y)} \approx |y|^{-n-1+\sigma}$ is integrable, the values $y > -M_0$ contribute to a small term and the difficulty is for $y < -M_0$. The corresponding contribution to G^1 is, using Lemma 2.4,

$$\int_{\mathbb{R}} \int_{\mathbb{V}} |y|^{-n-1+\sigma} \frac{|y|^{2\beta} |\xi\cdot v|^2}{1 + |\epsilon\xi\cdot v|^2 |y|^{2\beta}} \, \mathrm{d}v \, \mathrm{d}y = c \int_{\mathbb{V}} |\epsilon\xi\cdot v|^{\frac{n-\sigma-2\beta}{\beta}} |\xi\cdot v|^2 \, \mathrm{d}v.$$

Integrability in v is immediate since $n > \sigma$. The resulting power in ϵ in the corresponding expression of $|\widehat{J}^2_\epsilon|^2$ is, taking into account (2.10),

$$2(s-\mu) \ + \frac{n-\sigma-2\beta}{\beta} + 1 \ + \mu - s = s + \frac{1-\sigma}{\beta} - 1 > 0$$

thanks to the last condition in the parameter range (1.5). Hence the contribution of $G1$ vanishes in $L^2(\mathbb{R}^d)$.

The term with G^2 is treated with different arguments depending on the values of y and, because the middle range is easy we treat separately $y > M_0$ and $y < -M_0$. For $y > M_0$, we use the condition for Λ' in (1.2) and obtain the bound by

$$C \int_{y>M_0} \int_{\mathbb{V}} D(y) Q_0(y) (\partial_y \Lambda)^2 \, \mathrm{d}v \, \mathrm{d}y \leq C \int_{y>M_0} \int_{\mathbb{V}} |y|^{n+1-\sigma} |y|^{-2\gamma} \, \mathrm{d}v \, \mathrm{d}y < \infty$$

thanks to the parameter range $2\gamma > n + 2 - \sigma$ in (1.5). Therefore this contribution to G^2 vanishes.

Finally, the contribution to G^2 for $y < -M_0$ is more elaborate. We have

$$\begin{aligned}\int_{y<-M_0} \int_{\mathbb{V}} D(y) Q_0(y) \chi_0^2 \frac{(\partial_y \Lambda)^2 |\xi \cdot v|^2}{((\epsilon \xi \cdot v)^2 + \Lambda^2)^2} \mathrm{d}v \, \mathrm{d}y &\leq C \int_{y<-M_0} \int_{\mathbb{V}} |y|^{1-n+\sigma} \frac{|y|^{-2(1+\beta)} |y|^{4\beta} |\xi \cdot v|^2}{(1 + (\epsilon \xi \cdot v \, |y|^\beta)^2)^2} \mathrm{d}v \, \mathrm{d}y \\ &\leq C \int_{\mathbb{V}} (\epsilon \xi \cdot v)^{\frac{n-\sigma-2\beta}{\beta}} |\xi \cdot v|^2 \, \mathrm{d}v = C \epsilon^{\frac{n-\sigma-2\beta}{\beta}} |\xi|^{\frac{n-\sigma}{\beta}}.\end{aligned}$$

Therefore, in G^2, the power of ϵ stemming from this is

$$2(s-\mu) \; + \frac{n-\sigma-2\beta}{\beta} \; + 1 + \mu - s = s + \frac{1-\sigma}{\beta} - 1 > 0$$

using again the assumption (1.5).

3.6. **The term $\widehat{J}^3_\epsilon$.** This term is

$$\widehat{J}^3_\epsilon(t,\xi) = -\epsilon \int_{\mathbb{V}} \int_{\mathbb{R}} \frac{(i\xi \cdot v)\, \chi_0}{i\epsilon \xi \cdot v + \Lambda} \widehat{q}_\epsilon \, \mathrm{d}y \, \mathrm{d}v \,,$$

and we show that, for all $T > 0$, this term vanishes strongly in $L^2((0,T) \times \mathbb{R}^d)$ as $\epsilon \to 0$. To this end, we separate the integral as

$$-\widehat{J}^3_\epsilon(t,\xi) = \epsilon \int_{\mathbb{V}} \int_{y>-M_0} \frac{(i\xi \cdot v)\, \chi_0}{i\epsilon \xi \cdot v + \Lambda} \widehat{q}_\epsilon \, \mathrm{d}y \, \mathrm{d}v + \epsilon \int_{\mathbb{V}} \int_{y<-M_0} \frac{(i\xi \cdot v)\, \chi_0}{i\epsilon \xi \cdot v + \Lambda} \widehat{q}_\epsilon \, \mathrm{d}y \, \mathrm{d}v \,.$$

Using the Cauchy-Schwarz inequality, we bound the term with the integration over $y > -M_0$ by

$$C\epsilon |\xi| \int_{\mathbb{V}} \int_{\mathbb{R}} Q_0^{1/2} \frac{|\widehat{q}_\epsilon|}{Q_0^{1/2}} \mathrm{d}y \, \mathrm{d}v \leq \epsilon |\xi| \left(\int_{\mathbb{V}} \int_{\mathbb{R}} \frac{|\widehat{q}_\epsilon|^2}{Q_0} \mathrm{d}y \, \mathrm{d}v \right)^{1/2} .$$

This term is of order ϵ in $L^2(\mathbb{R}^d)$ uniformly in time thanks to the second bound in (2.1) which holds in Fourier variable as well.

The term with the integral over $y < -M_0$ has to be treated more carefully. Using the Cauchy-Schwarz inequality, we have

$$\left| \epsilon \int_{\mathbb{V}} \int_{y<-M_0} \frac{(i\xi \cdot v)\, \chi_0}{i\epsilon \xi \cdot v + \Lambda} \widehat{q}_\epsilon \, \mathrm{d}y \, \mathrm{d}v \right|^2 \leq \epsilon^2 \int_{\mathbb{V}} \int_{y<-M_0} \frac{|\xi \cdot v|^2 \chi_0^2}{(\epsilon \xi \cdot v)^2 + \Lambda^2} Q_0 \, \mathrm{d}y \, \mathrm{d}v \; \int_{\mathbb{V}} \int_{\mathbb{R}} \frac{|\widehat{q}_\epsilon|^2}{Q_0} \mathrm{d}y \, \mathrm{d}v.$$

Using the assumptions in Section 1 and Lemma 2.4, this is also upper bounded by

$$\begin{aligned}&C\epsilon^2 \int_{\mathbb{V}} \int_{y<-M_0} \frac{|\xi \cdot v|^2 |y|^{\sigma-2n+2\beta}}{1 + (\epsilon |\xi \cdot v| \, |y|^\beta)^2} \mathrm{d}y \mathrm{d}v \; \int_{\mathbb{V}} \int_{\mathbb{R}} \frac{|\widehat{q}_\epsilon|^2}{Q_0} \mathrm{d}y \, \mathrm{d}v \\ &\leq C\epsilon^2 \left(\int_{\mathbb{V}} (\epsilon |\xi \cdot v|)^{-\frac{\sigma-2n+2\beta+1}{\beta}} |\xi \cdot v|^2 \, \mathrm{d}v \right) \int_{\mathbb{V}} \int_{\mathbb{R}} \frac{|\widehat{q}_\epsilon|^2}{Q_0} \mathrm{d}y \, \mathrm{d}v \\ &\leq C(\epsilon |\xi|)^{\frac{2n-\sigma-1}{\beta}} \; \int_{\mathbb{V}} \int_{\mathbb{R}} \frac{|\widehat{q}_\epsilon|^2}{Q_0} \mathrm{d}y \, \mathrm{d}v.\end{aligned}$$

Here integrability in y and v are due to the assumption that $n > \sigma > 1$ in (1.5). Therefore, by the same L^2 bound for $\widehat{q}_\epsilon$ as above for the part where $y > -M_0$, we conclude that $\widehat{J}^3_\epsilon$ vanishes in $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d)$ as $\epsilon \to 0$.

4. Conclusion

In this work we give a new rigorous derivation of fractional diffusion limit for a bacterial population, with the remarkable feature that the speed of cells during their jump is bounded and their jumps are controlled by an internal process. The intracellular noise can replace the infinite speed assumption in [2, 3], and thus plays an important role on the population-level behaviour for *E. coli* chemotaxis. In particular, when the intracellular noise is strong ($n > 1$) and the adaptation process is slow ($s > 1$), the bacteria move with a Lévy walk and their population-level behaviour turns out to satisfy a fractional diffusion equation. This is in contrast to the case when there is no noise involved and the population-level equation is a regular diffusion [11, 25, 29].

Our derivation is obtained rigorously under the assumption that the parameters and coefficients satisfy (1.1)-(1.5). The conditions of the coefficients in (1.1)-(1.3) require that both the equilibrium and tumbling frequency decay polynomially with respect to the internal variable y as $y \to -\infty$. Part of the assumptions for the parameters in (1.5) are for mathematical convenience and it is not yet clear to us whether they are biologically relevant. However, among them, the two major conditions $s > 1$ and $n > 1$ are consistent with those required in biophysics works [18, 28], where with added noise in the chemotactic signally pathways, the authors perform stochastic simulations and obtain path length distributions with polynomial tails that correspond to Lévy processes.

Several points remain to understand. The case where the structuring variable is time between jumps, proposed in [13] is a possible direction. Also, other scalings in the model with internal pathwayl are certainly possible. Finally, our current work does not contain chemical signals. In the presence of this exterior influence, the bacteria move towards their favorite location by advection or advection/diffusion, see [27]. One interesting question is how intracellular noise can affect the advection with the appearance of chemical signals. This is kept for our future investigations.

References

[1] N. B. Abdallah, A. Mellet, and M. Puel, *Anomalous diffusion limit for kinetic equations with degenerate collision frequency*, Math. Models Methods Appl. Sci. **21** (2011), no. 11, 2249–2262.

[2] P. Aceves-Sanchez and C. Schmeiser, *Fractional-diffusion-advection limit of a kinetic model*, SIAM J. Appl. Math. **48** (2016), 2806–2818.

[3] P. Aceves-Sánchez and C. Schmeiser, *Fractional diffusion limit of a linear kinetic equation in a bounded domain*, Kinet. Relat. Models **10** (2017), no. 3, 541–551. MR3591122

[4] G. Ariel, A. Rabani, S. Benisty, J. D. Partridge, R. M. Harshey, and A. BeEr, *Swarming bacteria migrate by Levy walk*, Nature Communications **6** (2015).

[5] C. Bardos, F. Golse, and Y. Moyano, *Linear Boltzmann equation and fractional diffusion*, 2017. Preprint, arXiv:1708.09791.

[6] C. Bardos, R. Santos, and R. Sentis, *Limit theorems for additive functionals of a markov chain*, Transactions of the American mathematical society **284** (1984), no. 2, 617–648.

[7] E. Barkai, Y. Garini, and R. Metzler, *Strange kinetics of single molecules in living cells*, Phys. Today **65** (2012), 29–35.

[8] A. Bellouquid, J. Nieto, and L. Urrutia, *About the kinetic description of fractional diffusion equations modeling chemotaxis*, Mathematical Models and Methods in Applied Sciences **26** (2016), 249–268.

[9] L. Cesbron, A. Mellet, and K. Trivisa, *Anomalous transport of particles in plasma physics*, App. Math. Lett. **25** (2012), 2344–2348.

[10] Y. Dolak and C. Schmeiser, *Kinetic models for chemotaxis: Hydrodynamic limits and spatio-temporal mechanisms*, J. Math. Biol. **51** (2005), 595–615.

[11] R. Erban and H. Othmer, *From individual to collective behaviour in bacterial chemotaxis*, SIAM J. Appl. Math. **65** (2004), no. 2, 361–391.

[12] R. Erban and H. Othmer, *Taxis equations for amoeboid cells*, J. Math. Biol. **54** (2007), 847–885.

[13] G. Estrada-Rodriguez, H. Gimperlein, and K. J. Painter, *Fractional patlak-keller-segel equations for chemotactic superdiffusion* (2017), submitted.

[14] M. Frank and W. Sun, *Fractional diffusion limits of non-classical transport equations* (2017), submitted, arXiv 1607.04028.

[15] M. Jara, T. Komorowski, and S. Olla, *Limit theorems for additive functionals of a markov chain*, The Annals of Applied Probability **19** (2009), no. 6, 2270–2300.

[16] L. Jiang, Q. Ouyang, and Y. Tu, *Quantitative modeling of Escherichia coli chemotactic motion in environments varying in space and time*, PLoS Comput. Biol. **6** (2010), e1000735.

[17] E. Korobkova, T. Emonet, J. M. Vilar, T. S. Shimizu, and P. Cluzel, *From molecular noise to behavioural variability in a single bacterium*, Nature **428** (2004), 574–578.

[18] F. Matthalus, M. JagodicË, and J. Dobnikar, *E. coli superdiffusion and chemotaxis-search strategy, precision, and motility*, Biophysical Journal (2009), 946–957.

[19] A. Mellet and S. Merino-Aceituno, *Anomalous energy transport in fpu-β chain*, J. Stat. Phys.. **160** (2015), no. 3, 583–621.

[20] A. Mellet, C. Mouhot, and S. Mischler, *Fractional diffusion limit for collisional kinetic equations*, Arch. Ration. Mech. Anal. **199** (2011), 493–525.

[21] P. Michel, S. Mischler, and B. Perthame, *General relative entropy inequality: an illustration on growth models*, J.Math.Pures Appl. **84** (2005), 1235–1260.

[22] H. Othmer, X. Xin, and C. Xue, *Excitation and adaptation in bacteria-a model signal transduction system that controls taxis and spatial pattern formation*, Int. J. Mol. Sci. **14** (2013), no. 5, 9205–9248.

[23] B. Perthame, *Transport equations in biology*, Frontiers in Mathematics, Birkhäuser Verlag, Basel, 2007. MR2270822 (2007j:35004)

[24] B. Perthame, M. Tang, and N. Vauchelet, *Derivation of the bacterial run-and-tumble kinetic equation from a model with biological pathway*, J. Math. Biol. **73** (2016), 1161–1178.

[25] G. Si, M. Tang, and X. Yang, *A pathway-based mean-field model for E.coli chemotaxis: mathematical derivation and Keller-Segel limit*, Multiscale Model Simulation **12** (2014), no. 2, 907–926.

[26] G. Si, T. Wu, Q. Ouyang, and Y. Tu, *pathway-based mean-field model for Escherichia coli chemotaxis*, Phys. Rev. Lett. **109** (2012), 048101.

[27] W. Sun and M. Tang, *Macroscopic limits of pathway-based kinetic models for E. coli chemotaxis in large gradient environments*, Multiscale Model. Simul. **15** (2017), no. 2, 797–826.

[28] Y. Tu and G. Grinstein, *How white noise generates power-law switching in bacterial flagellar motors*, Phys. Rev. Lett. **94** (2005), 208101.

[29] C. Xue, *Macroscopic equations for bacterial chemotaxis: integration of detailed biochemistry of cell signaling*, J. Math. Biol. **70** (2015), 1–44.

Sorbonne Université, Université Paris-Diderot SPC, CNRS, INRIA, Laboratoire Jacques-Louis Lions, F75005 Paris, France.

This author has received funding from the European Research Council (ERC) under the European Unions Horizon 2020 research and innovation programme (grant agreement No 740623).

Department of Mathematics, Simon Fraser University, 8888 University Dr., Burnaby, BC V5A 1S6, Canada.

This author is partially supported by NSERC Discovery Grant No. R611626.

Institute of natural sciences and department of mathematics, Shanghai Jiao Tong University, Shanghai, 200240, China.

This author is partially supported by NSFC 11301336 and 91330203.