SUPPRESSING CHEMOTACTIC BLOW-UP THROUGH A FAST SPLITTING SCENARIO ON THE PLANE

SIMING HE AND EITAN TADMOR

Abstract. We revisit the question of global regularity for the Patlak-Keller-Segel (PKS) chemotaxis model. The classical 2D parabolic-elliptic model blows up for initial mass $M > 8\pi$. We consider more realistic scenario which takes into account the flow of the ambient environment induced by *harmonic* potentials, and thus retain the identical elliptic structure as in the original PKS. Surprisingly, we find that already the simplest case of *linear* stationary vector field, $Ax^\top$, with large enough amplitude A, prevents chemotactic blow-up. Specifically, the presence of such an ambient fluid transport creates what we call a 'fast splitting scenario', which competes with the focusing effect of aggregation so that 'enough mass' is pushed away from concentration along the x_1-axis, thus avoiding a finite time blow-up, at least for $M < 16\pi$. Thus, the enhanced ambient flow doubles the amount of allowable mass which evolve to global smooth solutions.

Contents

1. Introduction

The Patlak-Keller-Segel (PKS) model describes the time evolution of colony of bacteria with density $n(x,t)$ subject to two competing mechanisms — aggregation triggered by the concentration of chemo-attractant driven by velocity field $\mathbf{u} := \nabla(-\Delta)^{-1}n(\cdot,t)$, and diffusion due to run-and-tumble effects,

$$n_t + \nabla\cdot(n\mathbf{u}) = \Delta n, \qquad \mathbf{u} := \nabla(-\Delta)^{-1}n.$$

We focus on the two-dimensional case where $(-\Delta)^{-1}n$ takes the general form of a fundamental solution together with an arbitrary harmonic function

$$(-\Delta)^{-1}n(x,t) = (K*n)(x,t) + H(x,t), \qquad K(x) := -\frac{1}{2\pi}\log|x|, \quad \Delta H(\cdot,t)\equiv 0.$$

Date: October 21, 2017.

The resulting PKS equation then reads

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\nabla c) + \mathbf{b} \cdot \nabla n = \Delta n, \quad x = (x_1, x_2) \in \mathbb{R}^2, \tag{1.1}$$

subject to prescribed initial conditions $n(x, 0) = n_0(x)$. Here the divergence free vector field $\mathbf{b}(\cdot)$ represents the environment of an background fluid transported with velocity $\mathbf{b}(x,t) := \nabla H(x,t)$. When $\mathbf{b} \equiv 0$, the system is the classical parabolic-elliptic PKS equation modeling chemotaxis in a *static* environment [Pat1953, KS1970]. We recall the large literature on the static case $\mathbf{b} = 0$, referring the interested reader to the review [Hor2003] and the follow-up representative works [Bil1995, BDP2006, BCM2008, BCC2008, CC2006, CC2008, CR2010, JL1992, BM2014, BK2010]. It is well-known that the large-time behavior of the static case $(1.1)_{\mathbf{b}\equiv 0}$ depends on whether the initial total mass[1] $M := |n_0|_1$ crosses the critical threshold of 8π: the equation admits global smooth solution in the sub-critical case $M < 8\pi$ and it experiences a finite time blow-up if $M > 8\pi$ [BDP2006] (when $M = 8\pi$, aggregation and diffusion exactly balance each other and solutions with finite second moments form Dirac mass as time approach infinity [BCM2008]).

In this paper we study a more realistic scenario of the PKS model (1.1) where we take into account an ambient environment due to the fluid transport by vector field $\mathbf{b}(\cdot, t)$. Surprisingly, we find that already the simplest case of linear stationary vector field, $\mathbf{b} = A(-x_1, x_2)$, corresponding to $H(x) = \frac{1}{2}A(x_2^2 - x_1^2)$, prevents chemotactic blow-up for $M < 16\pi$. As we shall see, the presence of such an ambient fluid transport creates what we call a 'fast splitting scenario' which competes with the focusing effect of aggregation so that 'enough mass' is able to escape a finite time blow-up, at least for $M < 16\pi$. This scenario is likely to be enhanced even further when larger amount of mass can be transported by a more pronounced ambient field $\mathbf{b}(x,t) = \nabla H(\cdot, t) \gg |x|^q$ at $|x| \gg 1$.

We mention two other scenarios of non-static PKS with strong enough transport preventing blow-up for $M > 8\pi$. In [KX2015] the author exploit relaxation enhancing of a vector field $\mathbf{b}$ with a large enough amplitude in order to enforces global smooth solution. Here regularity follows due to a *mixing property* over $\mathbb{T}^2$ and $\mathbb{T}^3$. In [BH2016] it was shown how to exploit the enhanced dissipation effect of non-degenerate *shear flow* with large enough amplitude, [BCZ2015], in order to suppress the blow up in (1.1) on $\mathbb{T}^2, \mathbb{T}^3, \mathbb{T} \times \mathbb{R}, \mathbb{T} \times \mathbb{R}^2$.

1.1. **A fast splitting scenario.** Here, we exploit yet another mechanism that suppress the possible chemotactic blow up of the equation (1.1), where the underlying fluid flow splits the population of bacteria with density n exponentially fast, resulting in several isolated subgroups with mass smaller than the critical 8π. In this manner, an initial total mass greater than 8π is able to escape the finite-time concentration of Dirac mass. This provides a first no blow-up scenario over $\mathbb{R}^2$, at least for M up to 16π.

We now fix the vector field driving a hyperbolic flow as the strain flow in [KJCY2017]):

$$\mathbf{b}(x) := A(-x_1, x_2). \tag{1.2}$$

Our aim is to show that a large enough amplitude, $A \gg 1$, guarantees the global existence of solution of PKS (1.1) subject to initial mass $M < 16\pi$. Observe that an initial center of mass at the origin is an invariant of the flow. Intuitively, the large enough amplitude $A \gg 1$ is required so that the ambient field $A(-x_1, x_2)$ *'pushes away' highly concentrated mass near the x_1-axis*, namely, $\displaystyle\int_{|x_2|\leqslant\epsilon} n_0(x)dx \gg 1$. With this we can state the main theorem of the paper.

Theorem 1.1. *Consider the PKS equation (1.1),(1.2) subject to initial data, $n_0 \in H^s (s \geqslant 2)$ with total mass, $M := |n_0|_1 < 16\pi$, such that $(1 + |x|^2)n_0 \in L^1(\mathbb{R}^2)$ and $n_0 \log n_0 \in L^1(\mathbb{R}^2)$. Assume n_0*

[1] We let $|x|$ denote the ℓ^2-size of vector x and let $|f|_p$ denote the L^p-norm of a vector function $f(\cdot)$.

is symmetric about the x_1-axis, and that the "y-component" of its center of mass in the upper half plane

$$y_+(t) := \frac{1}{M_+}\int_{x_2\geqslant 0} n(x,t)x_2dx, \quad M_+ := \int_{x_2\geqslant 0} n(x,t)dx \equiv \frac{M}{2},$$

is not too close to the x_1-axis in the sense that

$$y_+^2(0) > \frac{2}{M_+}V_+(0), \qquad V_+(t) := \int_{x_2\geqslant 0} n(x,t)|x_2-y_+|^2dx. \tag{1.3}$$

Then there exists a large enough amplitude, $A = A(M, y_+(0), V_+(0))$, such that the weak solution of (1.1),(1.2) exists for all time and the free energy

$$E[n](t) := \int \big(n\log n - \frac{1}{2}cn - H(x)n(x,t)\big)dx, \qquad H(x) = \frac{A}{2}(x_2^2 - x_1^2), \tag{1.4}$$

satisfies the dissipation relation

$$E[n](t) + \int_0^t\int_{\mathbb{R}^2} n|\nabla\log n - \nabla c - \mathbf{b}|^2dxds \leqslant E[n_0]. \tag{1.5}$$

We conclude the introduction with three remarks.

Remark 1.1 (Why large enough stationary field prevents blow-up)**.** *Our main theorem extends the amount of critical mass, so that global regularity of (1.1),(1.3) prevails for $M < 16\pi$, provided A is large enough. To realize the how large the amplitude A should be and thus clarifying the reason behind this doubling the initial mass threshold for global regularity, we express (1.3) as*

$$R^2 := M_+\frac{y_+^2(0)}{2V_+(0)} > 1.$$

Then we can choose $A = M_+\delta^{-2}$ with small enough δ so that $\delta \leqslant (R-1)\sqrt{\frac{2V_+(0)}{M_+}}$. To gain further insight, we recall that the blow up phenomena in the static case, $\mathbf{b}\equiv 0$ is deduced from the time evolution of the second moment, $V(t) := \int_{\mathbb{R}^2} n(x,t)|x|^2dx$. Indeed, a straightforward computation yields, $\dot{V}(t) = 4M\Big(1-\dfrac{M}{8\pi}\Big) < 0$, which implies that the positive $V(t)$ decreases to zero in a finite time and hence rules out existence of global classical solutions for $M > 8\pi$. In contrast, the second moment of our non-static PKS equation (1.1),(1.2), does not decrease to zero if A is chosen large enough.

Lemma 1.1. *Let $n(x,t)$ be the solution of (1.1) with vector $\mathbf{b}(x) = A(-x_1,x_2)$, subject to initial data n_0 such that $W_0 := \int_{\mathbb{R}^2} n_0(x)(x_2^2 - x_1^2)dx$ is strictly positive. Then, if A is chosen large enough, the second moment of the (classical) solution $V(t) = \int_{\mathbb{R}^2} n(x,t)|x|^2dx$ increases in time.*

Proof. First, the time evolution of $V(t)$ can be calculated as follows:

$$\begin{cases} \dfrac{d}{dt}V = 4M\left(1-\dfrac{M}{8\pi}\right) + 2\displaystyle\int x\cdot\mathbf{b}\, n(x,t)dx \\ \qquad = 4M\left(1-\dfrac{M}{8\pi}\right) + 2AW, \qquad W(t) = \displaystyle\int_{\mathbb{R}^2}(-x_1^2+x_2^2)n(x,t)dx. \end{cases} \tag{1.6}$$

Next, we compute the time evolution of $W(t)$

$$\begin{aligned}\frac{d}{dt}W =& \int(-x_1^2+x_2^2)\nabla\cdot(\nabla n-\nabla cn-\mathbf{b}n)dx\\=&-\frac{1}{2\pi}\iint n(x,t)(-2x_1,2x_2)\cdot\frac{x-y}{|x-y|^2}n(y,t)dxdy+2AV\\=&-\frac{1}{2\pi}\iint\frac{-(x_1-y_1)^2+(x_2-y_2)^2}{(x_1-y_1)^2+(x_2-y_2)^2}n(x,t)n(y,t)dxdy+2AV,\end{aligned}$$

where the last step follows by symmetrization. Since the first term on the right is bounded from below by $-\frac{1}{2\pi}M^2$, we have

$$\frac{d}{dt}W\geqslant-\frac{1}{2\pi}M^2+2AV. \tag{1.7}$$

Finally, notice that since W_0 (and hence V_0) are assumed strictly positive, we can choose A large such that

$$AV_0-\frac{1}{4\pi}M^2\geqslant 0,\quad AW_0+2M\Big(1-\frac{M}{8\pi}\Big)\geqslant 0. \tag{1.8}$$

Combining (1.8), (1.6) and (1.7) yields that $W(t)>0,\ \ V(t)>0$. □

Remark 1.2 (On the free energy)**.** *We note that when* $\mathbf{b}=0$*,* $E[n]$ *becomes the classical dissipative free energy*

$$\mathcal{F}=\int_{\mathbb{R}^2}n\log n dx-\frac{1}{2}\int_{\mathbb{R}^2}ncdx. \tag{1.9}$$

Due to the importance of the property (1.5), a weak solution of (1.1) satisfying (1.5) will be called a free energy solution. *One of the important properties of the PKS equation (1.1) with background flow velocity (1.2) is the dissipation of its free energy* $E[n]$*. The formal computation, indicating the energy dissipation in non-static smooth solutions, is the content of our last lemma in this section.*

Lemma 1.2. *Consider the PKS equation (1.1) with background fluid velocity (1.2). If the solution is smooth enough, the free energy* $E[n](t)$ *is decreasing.*

Proof. The time evolution of the free energy (1.4) can be computed in terms of the potential $H=\frac{1}{2}A(x_2^2-x_1^2)$,

$$\begin{aligned}\frac{d}{dt}E[n](t)=&\int n_t(\log n-c-H)dx=-\int n(\nabla\log n-\nabla c-\mathbf{b})\cdot(\nabla\log n-\nabla c-\nabla H)dx\\=&-\int n|\nabla\log n-\nabla c-\mathbf{b}|^2dx\leqslant 0.\end{aligned}$$

This completes the proof of the lemma. □

Remark 1.3. *Arguing along the lines [EM2016], one should be able to prove that the free energy solution is smooth for all positive time,* $n\in C_c^\infty((0,T];C_x^\infty)$ *for all* $T<\infty$ *and thus our global weak solution is in fact a global strong solution.*

Our paper is organized as follows. In section 2, we introduce the regularized problem to (1.1) which leads to the local existence results. In section 3, we prove the main theorem, and in the appendix, we give detailed proofs of the results stated in section 2.

2. Local existence

2.1. Weak formulation.
It is standard to understand the Keller-Segel equation (1.1) with background fluid velocity (1.2) in the following weak formulation.

Definition 2.1 (weak formulation). *n is said to be the weak solution to (1.1) if for $\forall \varphi \in C_c^\infty(\mathbb{R}^2_+)$, the following equation hold:*

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2}\varphi n dx = &\int_{\mathbb{R}^2}\Delta\varphi n dx - \frac{1}{4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2}\frac{(\nabla\varphi(x)-\nabla\varphi(y))\cdot(x-y)}{|x-y|^2}n(x,t)n(y,t)dxdy\\ &+\int_{\mathbb{R}^2}\nabla\varphi\cdot\mathbf{b}n dx.\end{aligned} \tag{2.1}$$

Taking advantage of the assumed symmetry across the x_1-axis, one can further simplify the notation of a weak formulation adapted to the upper half plane, $\mathbb{R}^2_+ = \{(x_1,x_2)\,|\,x_2 \geqslant 0\}$.

Theorem 2.1. *If n is a weak solution to the equation (1.1), then for $\forall\varphi \in C_c^\infty(\mathbb{R}^2_+)$, the following holds:*

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2_+}\varphi n_+ dx = &\int_{\mathbb{R}^2_+}\Delta\varphi n_+ dx - \frac{1}{4\pi}\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+}\frac{(\nabla\varphi(x)-\nabla\varphi(y))\cdot(x-y)}{|x-y|^2}n_+(x,t)n_+(y,t)dxdy\\ &+\int_{\mathbb{R}^2_+}\nabla c_-\cdot\nabla\varphi n_+ dx + \int_{\mathbb{R}^2_+}\nabla\varphi\cdot\mathbf{b}n_+ dx.\end{aligned} \tag{2.2}$$

Here $\left\{\begin{array}{lcl} n_+ &:=& n\mathbf{1}_{x_2\geqslant 0}\\ n_- &:=& n\mathbf{1}_{x_2\leqslant 0}\end{array}\right\}$, *and* $\nabla c_-(x) := -\int_{y\in\mathbb{R}^2_-}\frac{x-y}{2\pi|x-y|^2}n_-(y)dy,$.

Proof. Rewrite (2.1) as follows:

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2_+}\varphi n_+ dx = &\int_{\mathbb{R}^2_+}\Delta\varphi n_+ dx - \frac{1}{4\pi}\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+}\frac{(\nabla\varphi(x)-\nabla\varphi(y))\cdot(x-y)}{|x-y|^2}n_+(x,t)n_+(y,t)dxdy\\ &-\frac{1}{2\pi}\int_{\mathbb{R}^2_+\times\mathbb{R}^2_-}\frac{\nabla\varphi(x)\cdot(x-y)}{|x-y|^2}n_+(x,t)n_-(y,t)dxdy + \int_{\mathbb{R}^2_+}\nabla\varphi\cdot\mathbf{b}n_+ dx.\end{aligned}$$

The third term can be rewritten as $\int_{\mathbb{R}^2_+}\nabla c_-\cdot\nabla\varphi\, n_+(x)dx$, and we get (2.2). □

2.2. Regularized equation and local existence theorems.
In this section we introduce the local existence theorem and the blow up criterion for the Keller-Segel equation with advection. The theorems are standard, so the proofs are postponed to the appendix. The interested reader are referred to the papers [BCM2008],[BDP2006] for further details.

In order to prove the local existence theorem and the blow up criterion for the Keller-Segel system with advection (1.1), we regularize the system as follows:

$$\frac{\partial n^\epsilon}{\partial t} + \nabla\cdot(n^\epsilon\nabla c^\epsilon) + \mathbf{b}\cdot\nabla n^\epsilon = \Delta n^\epsilon, \quad c^\epsilon := K^\epsilon * n, \qquad x\in\mathbb{R}^2, t>0, \tag{2.3}$$

with the regularized kernel, K^ϵ, given by

$$K^\epsilon(z) := K^1\left(\frac{z}{\epsilon}\right) - \frac{1}{2\pi}\log\epsilon, \qquad K^1(z) := \left\{\begin{array}{ll} -\frac{1}{2\pi}\log|z|, & \text{if} \quad |z|\geqslant 4,\\ 0, & \text{if} \quad |z|\leqslant 1.\end{array}\right. \tag{2.4}$$

Noting that $|\nabla K^\epsilon(z)| \leqslant C_\epsilon$ for all $z\in\mathbb{R}^2$, it follows that the solutions to the equation (2.3) exist for all time. The proof is similar to the corresponding proof in the classical case. We refer the interested reader to the paper [BDP2006] for more details.

Before stating the local existence theorems, we introduce the entropy of the solution

$$S[n] := \int_{\mathbb{R}^2} n \log n dx. \tag{2.5}$$

Now the local existence theorems are expressed as follows:

Proposition 2.1. (*Local Existence Criterion*). *Assume that* $|\mathbf{b}|(x) \leqslant C|x|, \forall x \in \mathbb{R}^2$. *Suppose* $\{n^\epsilon\}_{\epsilon \geqslant 0}$ *are the solutions of the regularized equation* (2.3) *on* $[0, T^*)$. *If* $\{S[n^\epsilon](t)\}_\epsilon$ *is bounded from above uniformly in* ϵ *and in* $t \in [0, T^*)$, *then the cluster points of* $\{n^\epsilon\}_{\epsilon \to 0}$, *in a suitable topology, are non-negative weak solutions of the PKS system with advection* (1.1) *on* $[0, T^*)$ *and satisfies the relation* (1.5).

Proposition 2.2. (*Maximal Free-energy Solutions*). *Assume the boundedness of the vector field* $|\mathbf{b}|(x) \lesssim |x|$ *and the integrability of initial data* $(1+|x|^2)n_0 \in L^1_+(\mathbb{R}^2)$, $n_0 \log n_0 \in L^1(\mathbb{R}^2)$. *Then there exists a maximal existence time* $T^* > 0$ *of a free energy solution to the PKS system with advection* (1.1),(1.5). *Moreover, if* $T^* < \infty$ *then*

$$\lim_{t \to T^*} \int_{\mathbb{R}^2} n \log n dx = \infty.$$

We conclude that if the entropy $S[n](t) = \int n \log n$ is bounded, then the free energy solution of (1.1) exists locally. Moreover, if $S[n](t) < \infty$ for all $t < \infty$, the solution exists for all time.

3. Proof of the main results

3.1. **The three-step 'battle-plan'.** We proceed in three steps. The *first step* carried in section 3.2 below, is to control cell density distribution. From the last section, we see that an entropy bound is essential for derivation of local existence theorems for the PKS equation (1.1),(1.2). To this end, information about the distribution of cell density is crucial. The following lemma is the key to the proof of the main results. It shows that mass cannot concentrate along the the x_1-axis, since we can find a thin enough strip along the x_1-axis with controlled amount of mass.

Lemma 3.1. *Suppose a sufficiently smooth* n_0 *is symmetric about the* x_1 *axis and assume that*

$$R^2 := M_+ \frac{y_+^2(0)}{2V_+(0)} > 1. \tag{3.1}$$

Fix a small enough $0 < \eta \ll 1$. *Then there exists* $\delta = \delta(y_+(0), V_+(0), M, \eta)$ *such that if we choose* $A > \frac{M_+}{\delta^2}$, *the smooth solutions to the regularized* $(2.3)_\epsilon$ *satisfy, uniformly for small enough* ϵ,

$$\int_{|x_2| \leqslant 2\delta} n^\epsilon(x,t) dx \leqslant \frac{(1+\eta)^2}{2R^2} M. \tag{3.2}$$

Condition (3.2) implies, at least for $M < 16\pi$, that the mass inside that δ-strip is less than 8π. On the other hand, it indicates the reason for the limitation $R > 1$: for if $R < 1$, then the bound (3.2) would allow a concentration of mass $\frac{M}{2R^2} \geqslant 8\pi$ inside the strip $|x_2| \leqslant 2\delta$, which in turn leads to a finite-time blow-up.

The proof of lemma 3.1 is based on the following simple observation. Given f with $\mathbb{R}^2_+$-center of mass at $(\cdot, y_f)$ and variation $V_f = \int |x_2 - y_f|^2 f(x) dx$, we find that its total mass *outside* the strip $\mathcal{S}[y_f, r] := \{(x_1, x_2) | |x_2 - y_f| \leqslant r\}$ with radius $r = R\sqrt{2V_f/M_f}$, does not exceed

$$\int_{|x_2 - y_f| > r} f(x) dx = \int_{|x_2 - y_f| > r} f(x) \frac{|x_2 - y_f|^2}{|x_2 - y_f|^2} dx \leqslant \frac{M_f}{2R^2 V_f} \int f(x)|x_2 - y_f|^2 dx = \frac{M_f}{2R^2}$$

If we can find the δ such that our target strip $\mathcal{S}_\delta := \{|x_2| \leqslant 2\delta\}$ is lying below and *outside* the strip $\mathcal{S}[y_f, r]$, then the total mass in the strip $\mathcal{S}_\delta$ would be smaller than $\frac{1}{2R^2} M_f$. When $n^\epsilon(x,t)$ takes the role of $f(x)$ with $(y_f, V_f) \mapsto (y_+(t), V_+(t))$, the aim is to bound the strip $\mathcal{S}[y_+(t), r(t)]$ with radius

$r(t) = R\sqrt{2V_+(t)/M_+}$ away from a fixed strip $\mathcal{S}_\delta$. To this end we collect the necessary estimates on $y_+(t), V_+(t)$ and complete the proof of the lemma in section 3.2.

The *second step*, carried in section 3.3, is to prove the main theorem with moderate mass constraint. Equipped with lemma 3.1 we can control the entropy and prove a weaker form of our main theorem for moderate size mass M (which is still larger than the 8π barrier):

Theorem 3.1. *Consider the PKS equation (1.1) with background fluid velocity (1.2) subject to $H^s(s \geqslant 2)$ initial data with mass $M = |n_0|_1$, such that $(1+|x|^2)n_0 \in L^1(\mathbb{R}^2)$, $n_0 \log n_0 \in L^1(\mathbb{R}^2)$. Furthermore, assume n_0 is symmetric about the x_1-axis, that (1.3) holds. If the total mass does not exceed*

$$M < \frac{1}{1+(1+\eta)^2/R^2}16\pi, \tag{3.3}$$

then there exists an $A = A(M, y_+(0), V_+(0), \eta)$ large such that the free energy solution to PKS (1.1),(1.2) exists for all time.

Thus, as the ratio increases over the range $1 < R < \infty$, (3.1) yields global existence with an increasing amount of mass $8\pi < M < 16\pi$. Although theorem 3.1 is not as sharp as the main theorem, its proof is more illuminating and can be extended easily to prove the main theorem for the 'limiting case' of any $M < 16\pi$. We therefore include its proof in section 3.3.
Finally, the *third step* carried in section 3.4 presents the proof of the main theorem 1.1.
We turn to a detailed discussion of the three steps.

3.2. **Step 1— control of the cell density distribution.** As pointed out before, the proof involves the calculation of $y_+(t)$ and $V_+(t)$, summarized in the following two lemmas. Here and below, we let $A \lesssim B$ denote the relation $A \leqslant CB$ with a constant C which is independent of δ.

Lemma 3.2. *Consider the regularized PKS equation (2.3) with background fluid velocity (1.2). Assume that the initial center of mass $y_+(0)$ is separated from the x_1-axes in the sense that (3.1) holds. Then , there exists a constant such that the time evolution of $y_+(t)$ remains bounded from below*

$$y_+(t) \geqslant [y_+(0) - C\delta]\, e^{At}, \quad . \tag{3.4}$$

Lemma 3.3. *Consider the regularized PKS equation (2.3) with background fluid velocity (1.2). Assume that the initial variation around the center of mass $V_+(0)$ is not too large in the sense that (3.1) holds. Then , there exists a constant $C = C(V_+(0))$ such that the variation $V_+(t)$ remains bounded from above,*

$$V_+(t) \leqslant [CM_+\delta + V_+(0)]\, e^{2At}, \quad . \tag{3.5}$$

We note that all the calculations made below should be carried out at the level of the regularized equation (2.3), but for the sake of simplicity, we proceed at the formal level using the weak formulation (2.2). We explicitly point when there is a technical subtlety in the derivation due to difference between the regularized and weak formations.

We begin with the proof of Lemma 3.2. First of all, as $x_2 \notin C_c^\infty(\mathbb{R}^2_+)$, we introduce an approximate test function φ to x_2:

$$\varphi := \begin{cases} x_2 & x_2 \in (2\delta, \infty), \\ 0 & x_2 \in (-\infty, \delta), \\ smooth & x_2 \in (\delta, 2\delta). \end{cases}$$

Note that there exists a constant C_φ, independent of δ, such that $|\varphi| \leqslant 2\delta$, $\forall x_2 \leqslant 2\delta$ and $|\nabla\varphi| + \delta|\nabla^2\varphi| \leqslant C_\varphi$. Here and below, we use C_φ to denote φ-dependent constants .

By replacing $\int_{\mathbb{R}^2_+} x_2 n dx$ with $\int_{\mathbb{R}^2_+} \varphi n dx$, we lose information on the stripe $\{(x_1,x_2)||x_2|\leqslant 2\delta\}$. However, the contribution of this part is small in the sense that:

$$\left|\int_{0\leqslant x_2\leqslant 2\delta} \varphi n_+ dx - \int_{0\leqslant x_2\leqslant 2\delta} x_2 n_+ dx\right| \leqslant 4M_+\delta. \tag{3.6}$$

Next, one can use φ and the weak formulation (2.2) to extract information about y_+:

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2_+} \varphi n_+ dx =& \int_{\mathbb{R}^2_+} \Delta\varphi n_+ dx - \frac{1}{4\pi}\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+} \frac{(\nabla\varphi(x)-\nabla\varphi(y))\cdot(x-y)}{|x-y|^2} n_+(x,t)n_+(y,t)dxdy\\ &+\int_{\mathbb{R}^2_+} n_+\nabla c_-\cdot\nabla\varphi dx + \int_{\mathbb{R}^2_+}\nabla\varphi\cdot\mathbf{b}n_+dx\\ =&I+II+III+IV. \end{aligned}\tag{3.7}$$

Now we estimate the right hand side of (3.7) term by term. The first and second terms are relatively easy to control from above as follows:

$$I = \left|\int_{\mathbb{R}^2_+}\Delta\varphi n_+ dx\right| \leqslant \frac{C_\varphi M_+}{\delta}, \tag{3.8}$$

$$II \leqslant \frac{1}{4\pi}|\nabla^2\varphi|_\infty \int\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+} n_+(x)n_+(y)dxdy \leqslant \frac{1}{4\pi}\frac{C_\varphi}{\delta}M_+^2. \tag{3.9}$$

In order to estimate the third term in (3.7), we need a pointwise estimate on the $\partial_{x_2}c_-$:

$$\begin{aligned}|\partial_{x_2}c_-(x)| =& \frac{1}{2\pi}\left|\int_{\mathbb{R}^2_+}\frac{(x-y)_2}{|x-y|^2}n_-(y)dy\right|\\ &\leqslant \frac{1}{2\pi}\frac{1}{|x_2|}\int_{\mathbb{R}^2_+}\frac{|x_2|}{|x_2|+|y_2|}n_-(y)dy \leqslant \frac{1}{2\pi}\frac{M_-}{|x_2|} = \frac{1}{2\pi}\frac{M_+}{|x_2|}.\end{aligned}\tag{3.10}$$

Now we can use the above estimate to control the third term in (3.7):

$$III = \left|\int_{\mathbb{R}^2_+}\partial_{x_2}c_- n_+\partial_{x_2}\varphi dx\right| \leqslant \frac{C_\varphi}{2\pi}\frac{M_+}{\delta}\int_{\mathbb{R}^2_+} n_+ dx \leqslant \frac{C_\varphi}{2\pi}\frac{M_+^2}{\delta}. \tag{3.11}$$

Here we have used the fact that $supp(\varphi)$ is δ away from the x_1 axis. As a result, $\frac{1}{|x_2|}\leqslant\frac{1}{\delta}$.

Remark 3.1. *The only difference in estimating the regularized solutions (2.3) vs. the formal calculation we have done above is in terms II and III. In the calculation for the (2.3), we will need the estimate*

$$|\nabla K^\epsilon(z)| \leqslant \frac{1}{2\pi|z|}, \quad \forall z\in\mathbb{R}^2.$$

Here we show how to get a similar estimate for term II in (3.7) for the regularized equation (2.3):

$$\begin{aligned}II =& \left|\iint_{\mathbb{R}^2_+\times\mathbb{R}^2_+}\nabla\varphi(x)\nabla_x[K^\epsilon(|x-y|)]n^\epsilon(y)n^\epsilon(x)dxdy\right|\\ =&\frac{1}{2}\left|\iint_{\mathbb{R}^2_+\times\mathbb{R}^2_+}\frac{(\nabla\varphi(x)-\nabla\varphi(y))\cdot(x-y)}{|x-y|}\nabla K^\epsilon(|x-y|)n^\epsilon(x)n^\epsilon(y)dxdy\right|\\ \leqslant&\frac{1}{4\pi}\left|\iint_{\mathbb{R}^2_+\times\mathbb{R}^2_+}\frac{|\nabla\varphi(x)-\nabla\varphi(y)|}{|x-y|}n^\epsilon(x)n^\epsilon(y)dxdy\right| \leqslant \frac{1}{4\pi}|\nabla^2\varphi|_\infty M_+^2 \leqslant \frac{1}{4\pi}\frac{C_\varphi}{\delta}M_+^2.\end{aligned}$$

The treatment of term III is similar to the one we gave above.

Finally, we need to address additional transport term IV in (3.7) to compete with the focusing effect. Recall that $\mathbf{b} = A(-x_1, x_2)$ with $A = \frac{M_+}{\delta^2}$. First we write IV down explicitly,

$$IV = \int \mathbf{b} \cdot n_+ \nabla\varphi dx = \frac{M_+}{\delta^2} \int_{\mathbb{R}^2_+} x_2 \partial_{x_2} \varphi n_+ dx.$$

Next we replace the right hand side by $\int \varphi n_+ dx$. Due to the fact that $x_2 \partial_{x_2}\varphi = x_2 = \varphi$ for $x_2 > 2\delta$, the error introduced in this process originates from the thin 2δ-strip:

$$\left| \int_{\delta < x_2 < 2\delta} (x_2 \partial_{x_2}\varphi - \varphi) n_+ dx \right| \leqslant \int_{\delta < x_2 < 2\delta} |x_2\partial_{x_2}\varphi - \varphi| n_+ dx \leqslant C_\varphi \delta M_+,$$

and we conclude that

$$IV \geqslant \frac{M_+}{\delta^2} \left(\int_{\mathbb{R}^2_+} \varphi n_+ dx - C_\varphi \delta M_+ \right) \geqslant \frac{M_+}{\delta^2} \int_{\mathbb{R}^2_+} \varphi n_+ dx - \frac{C_\varphi M_+^2}{\delta}. \tag{3.12}$$

Combining the equation (3.7) and estimates (3.8), (3.9), (3.11) and (3.12) yields the following

$$\frac{d}{dt} \int_{\mathbb{R}^2_+} \varphi n_+ dx \geqslant -C_\varphi \frac{M_+^2}{\delta} + A \int_{\mathbb{R}^2_+} \varphi n_+ dx,$$

which implies that

$$\int_{\mathbb{R}^2_+} \varphi n_+ dx \geqslant \left(\int_{\mathbb{R}^2_+} \varphi n_0 dx - C_\varphi M_+ \delta \right) e^{At}. \tag{3.13}$$

Finally, we calculate the center of mass of the upper half plane using the lower bound (3.13) and the error control (3.6):

$$\begin{aligned} y_+(t) =& \frac{1}{M_+} \left(\int_{0 \leqslant x_2 \leqslant 2\delta} x_2 n_+ dx - \int_{0 \leqslant x_2 \leqslant 2\delta} \varphi n_+ dx + \int_{\mathbb{R}^2_+} \varphi n_+ dx \right) \\ \geqslant& -\frac{1}{M_+} \left| \int_{0 \leqslant x_2 \leqslant 2\delta} x_2 n_+ dx - \int_{0 \leqslant x_2 \leqslant 2\delta} \varphi n_+ dx \right| + \frac{1}{M_+} \int_{\mathbb{R}^2_+} \varphi n_+ dx \\ \geqslant& -4\delta + \frac{1}{M_+} \left(\int_{\mathbb{R}^2_+} \varphi n_0 dx - C_\varphi M_+ \delta \right) e^{At} \\ \geqslant& (y_+(0) - C_\varphi \delta) \, e^{At}. \end{aligned}$$

This completes the proof of lemma 3.2.□

Next we address the proof of Lemma 3.3. The main goal is to calculate time evolution of the variation

$$V_+(t) := \int_{\mathbb{R}^2_+} |x_2 - y_+(t)|^2 n(x,t) dx.$$

We again use C to denote constants which may change from line to line but are independent of δ.

The first obstacle is that we cannot choose $|x_2 - y_+|^2$ as a test function due to the fact that $y_+(t)$ depends on the solution. However, by the definition of y_+ we can expand the V_+-integrand, ending up with the usual

$$V_+(t) = \int_{\mathbb{R}^2_+} |x_2|^2 n_+(x,t) dx - M_+ y_+^2(t). \tag{3.14}$$

Since we already know y_+, it is enough to calculate the $\int_{\mathbb{R}^2_+} |x_2|^2 n(x,t) dx$. For simplicity, we plug $|x_2|^2$ inside the weak formulation (2.2) and (3.14) to get the time evolution of V_+. Of course, what one really does is to use a test function to approximate the $|x_2|^2$. Furthermore, when we use the weak formulation, we formally integrated by part twice, but since the value and the first derivative of the function $|x_2|^2$ are zero on the boundary, we will not create extra dangerous boundary term.

First combining (2.2) and (3.14) yields

$$\begin{aligned}\frac{d}{dt}V_+ =&\frac{d}{dt}\int_{\mathbb{R}^2_+}|x_2|^2 n_+(x,t)dx - M_+\frac{d}{dt}y_+^2(t)\\ =&\int_{\mathbb{R}^2_+}\Delta\varphi n_+ dx - \frac{1}{4\pi}\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+}\frac{(\nabla\varphi(x)-\nabla\varphi(y))\cdot(x-y)}{|x-y|^2}n_+(x,t)n_+(y,t)dxdy\\ &+\int_{\mathbb{R}^2_+}\nabla c_- n_+\nabla\varphi dx + \int_{\mathbb{R}^2_+}\nabla\varphi\cdot\mathbf{b}n_+dx - \frac{d}{dt}\left(M_+(y_+)^2\right)\\ =&I+II+III+IV-M_+\frac{d}{dt}y_+^2(t) \end{aligned}\tag{3.15}$$

Next we estimate every term on the right hand side of (3.15). The first two terms are estimated as follows:

$$|I| = \left|\int_{\mathbb{R}^2_+}\Delta(x_2^2)n_+dx\right| = 2M_+,\tag{3.16}$$

and

$$\begin{aligned}|II| =&\left|-\frac{1}{4\pi}\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+}\frac{(\nabla(x_2^2)-\nabla(y_2^2))\cdot(x-y)}{|x-y|^2}n_+(x,t)n_+(y,t)dxdy\right|\\ \leqslant&\frac{1}{4\pi}\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+}2n_+(x)n_+(y)dxdy\leqslant\frac{1}{2\pi}M_+^2.\end{aligned}\tag{3.17}$$

Now for the third term III in (3.15), applying (3.10) yields

$$|III| = \left|\int_{\mathbb{R}^2_+}\partial_{x_2}c_- n_+ 2x_2 dx\right| \leqslant \frac{1}{2\pi}\int_{\mathbb{R}^2_+}2|x_2|\frac{M_+}{|x_2|}n_+dx\leqslant\frac{1}{\pi}M_+^2.\tag{3.18}$$

Note that for the term II and III, we only estimate them formally above, one can prove the estimates explicitly using the same techniques as the one in Remark 3.1. For the IV term in (3.15), we use the (3.14) again to obtain

$$|IV| = \left|\int_{\mathbb{R}^2_+}\nabla x_2^2\cdot\mathbf{b}n_+dx\right| = 2A\left|\int_{\mathbb{R}^2_+}x_2^2 n_+dx\right| = 2A(V_+ + M_+y_+^2)\tag{3.19}$$

Collecting equation (3.15) and all the estimates (3.16), (3.17), (3.18) and (3.19) above, we have the following differential inequality,

$$\frac{d}{dt}\left(\frac{1}{M_+}V_+(t)+y_+^2(t)\right)\leqslant C(1+M_+)+2A\left(\frac{1}{M_+}V_+(t)+y_+^2(t)\right),$$

which yields

$$\frac{1}{M_+}V_+(t)+y_+^2(t)\leqslant C\left(1+\frac{1}{M_+}\right)\delta^2e^{2At}+\frac{1}{M_+}V_+(t)e^{2At}+y_+^2(0)e^{2At}.\tag{3.20}$$

Combining (3.4) with (3.20) yields

$$\begin{aligned}\frac{1}{M_+}V_+(t)+\left[(y_+(0)-C\delta)\,e^{At}\right]^2\leqslant&\frac{1}{M_+}V_+(t)+y_+^2(t)\\ \leqslant&C\left(1+\frac{1}{M_+}\right)\delta^2e^{2At}+\frac{1}{M_+}V_+(0)e^{2At}+y_+^2(0)e^{2At}.\end{aligned}$$

By collecting similar terms, we finally have

$$\frac{1}{M_+}V_+(t) \leqslant \left[2C\delta y_+(0) + C\left(1+\frac{1}{M_+}\right)\delta^2 + \frac{1}{M_+}V_+(0)\right]e^{2At}. \tag{3.21}$$

which completes the proof of Lemma 3.3. □

Equipped with the estimate on $V_+(t)$, we can now conclude the proof of Lemma 3.1.

Proof. (Lemma 3.1) Once $0<\eta\ll 1$ was fixed, we can clearly choose a small enough δ such that by $(3.5)_\delta$, there holds

$$V_+(t) \leqslant (1+\eta)V_+(0)e^{2At}. \tag{3.22}$$

Now recalling that $R = y_+(0)\sqrt{\frac{M_+}{2V_+(0)}} > 1$, then we can use (3.4), (3.22) and further choose δ small enough to get:

$$\begin{aligned} y_+(t) - \frac{R}{1+\eta}\sqrt{\frac{2V_+(t)}{M_+}} &\geqslant \left[y_+(0) - C\delta - \frac{R}{\sqrt{1+\eta}}\sqrt{\frac{2V_+(0)}{M_+}}\right]e^{At} \\ &= \left[\left(1-\frac{1}{\sqrt{1+\eta}}\right)y_+(0) - C\delta\right]e^{At} \\ &\geqslant 2\delta e^{At} \geqslant 2\delta. \end{aligned}$$

Thus, the 'thin' δ-strip along the x_1-axis, $\mathcal{S}_\delta := \{(x_1,x_2)|0 \leqslant x_2 \leqslant 2\delta\}$, lies *outside* the strip centered around $y_+(t)$, uniformly in time,

$$\mathcal{S}_\delta \subset \{(x_1,x_2)||x_2 - y_+(t)| > R_\eta(t)\}, \qquad R_\eta(t) := \frac{R}{1+\eta}\sqrt{\frac{2V_+(t)}{M_+}}.$$

It follows that thanks to our choice of δ, the mass inside the δ-strip $\mathcal{S}_\delta$ does not exceed

$$\begin{aligned} \int_{\mathcal{S}_\delta} n_+(x,t)dx &\leqslant \int_{\mathbb{R}^2_+\cap\{|x_2-y_+|>R_\eta\}} n_+(x,t)dx \leqslant \int_{\mathbb{R}^2_+\cap\{|x_2-y_+|>R_\eta\}} n_+(x,t)\frac{|x_2-y_+|^2}{|x_2-y_+|^2}dx \\ &\leqslant \frac{(1+\eta)^2}{R^2 2V_+/M_+}\int_{\mathbb{R}^2_+} n_+(x,t)|x_2-y_+|^2dx = \frac{(1+\eta)^2}{R^2 2V_+/M_+}V_+ \\ &\leqslant \frac{(1+\eta)^2}{2R^2}M_+. \end{aligned}$$

By symmetry, the mass inside the symmetric δ-strip, $\{(x_1,x_2)||x_2| \leqslant 2\delta\}$ is smaller than $\dfrac{(1+\eta)^2}{2R^2}2M_+ = \dfrac{(1+\eta)^2}{2R^2}M$, uniformly in time, which completes the proof of Lemma 3.1. □

Remark 3.2. *One can do a similar computation to get the evolution for the higher moment estimates* $\int_{\mathbb{R}^2_+} n(x,t)|x|^{2k}dx$*, and derive similar results to Lemma 3.1.*

3.3. **Step 2 — proof of the main theorem with moderate mass constraint.** With the Lemma 3.1 at our disposal, we can now turn to the proof of theorem 3.1 along the lines of [BCM2008]. Note that the actual calculation are to be carried out with the regularized solutions n^ϵ of (2.3), though for the sake of simplicity, we only do the formal calculation on $n(x) = n(\cdot,t)$.

The key is to use the logarithmic Hardy-Littlewood-Sobolev inequality to get a bound on the entropy $S[n]$.

Theorem 3.2 (Logarithmic Hardy-Littlewood-Sobolev Inequality)**.** [CL92] *Let f be a nonnegative function in $L^1(\mathbb{R}^2)$ such that $f\log f$ and $f\log(1+|x|^2)$ belong to $L^1(\mathbb{R}^2)$. If $\int_{\mathbb{R}^2} f dx = M$, then*

$$\int_{\mathbb{R}^2} f\log f dx + \frac{2}{M}\iint_{\mathbb{R}^2\times\mathbb{R}^2} f(x)f(y)\log|x-y|dxdy \geqslant -C(M) \tag{3.23}$$

with $C(M) := M(1+\log\pi-\log M)$.

Remark 3.3. *It is pointed out in [BCM2008] that by multiplying f by indicator functions, one can prove that the inequality (3.23) remains true with $\mathbb{R}^2$ replaced by any bounded domains $\mathcal{D}\subset\mathbb{R}^2$.*

The idea of the proof goes as follows. By observing that the mass in the upper half plane and lower half plane are subcritical ($||n_\pm||_1 < 8\pi$), we plan to use the logarithmic Hardy-Littlewood-Sobolev inequality on these sub-domains to get uniform bound on the entropy. However, without extra information concerning the cell density distribution, naive application of logarithmic Hardy-Littlewood-Sobolev inequality fails. For this approach to work, the density distribution constraint required is that the cells in the upper and lower half plane are well-separated by a 'cell clear strip' in which the total number of cells is sufficiently small. The strip is constructed through applying Lemma 3.1. Combining the logarithmic Hardy-Littlewood-Sobolev inequality and the cell seperation constraint, we can use a 'total entropy reconstruction' trick introduced in [BCM2008] to obtain entropy bound. Now let's start the whole proof.

Proof. First we contruct the 'cell clear strip'. Define the following three regions:

$$\Gamma_1 = \{x_2\,|\,x_2 > 2\delta\},\quad \Gamma_2 = \{x_2\,|\,x_2 < -2\delta\},\quad \Gamma_3 = \mathbb{R}^2\backslash(\Gamma_1\cup\Gamma_2). \tag{3.24}$$

Here region Γ_1 contains points in the upper half plane which are 2δ away from the x_1 axis, whereas region Γ_2 contains points in the lower half plane with the same property. Region Γ_3 is a closed neighborhood of the x_1 axis. The δ neighborhood of the Γ_1, Γ_2 region is denoted as follows:

$$\Gamma_1^{(\delta)} = \{x_2 \mid x_2 > \delta\},\quad \Gamma_2^{(\delta)} = \{x_2\,|\,x_2 < -\delta\}. \tag{3.25}$$

We further decompose Γ_3 into subdomains:

$$S_1 = \{x_2|\delta < x_2 \leqslant 2\delta\},\quad S_2 = \{x_2|-\delta > x_2 \geqslant -2\delta\},\quad S_3 = \{x_2 \mid |x_2| \leqslant \delta\}. \tag{3.26}$$

Applying Lemma 3.1 yields that the total mass inside Γ_3 is small, i.e.

$$\int_{\Gamma_3} ndx = \int_{\{x_2||x_2|\leqslant 2\delta\}} ndx \leqslant \frac{(1+\eta)^2}{2R^2}M. \tag{3.27}$$

Therefore, the Γ_3 strip is the 'cell clear strip'.

Next, we estimate the entropy. First recall that the free energy $E[n](t)$ (1.4) is decreasing, i.e,

$$\begin{aligned} E[n_0] \geqslant E[n] = &\left(1-\frac{K}{8\pi}\right)\int n\log n dx + \frac{1}{8\pi}\left(K\int n\log n dx\right.\\ &\left.+ 2\iint_{\mathbb{R}^2\times\mathbb{R}^2} n(x)n(y)\log|x-y|dxdy\right) - \int Gndx\\ =: &\left(1-\frac{K}{8\pi}\right)S[n] + T_1 - T_2. \end{aligned} \tag{3.28}$$

To obtain the entropy bound, we need to estimate T_1 from below for some $K < 8\pi$ and estimate T_2 from above. We start by estimating T_1. Similar to [BCM2008], we apply the Logarithmic

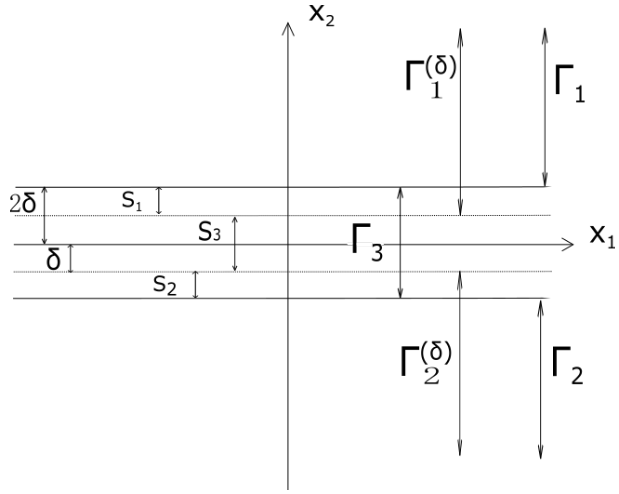

FIGURE 1. Regions $\Gamma_1, \Gamma_2, \Gamma_3$

Hardy-Littlewood-Sobolev inequality in the three regions $\Gamma_2^{(\delta)}, \Gamma_2^{(\delta)}, \Gamma_3$ and obtain:

$$\int_{\Gamma_1^{(\delta)}} n(x)dx \int_{\Gamma_1^{(\delta)}} n\log n dx + 2\iint_{\Gamma_1^{(\delta)}\times\Gamma_1^{(\delta)}} n(x)n(y)\log|x-y|dxdy \geqslant C,$$
$$\int_{\Gamma_2^{(\delta)}} n(x)dx \int_{\Gamma_2^{(\delta)}} n\log n dx + 2\iint_{\Gamma_2^{(\delta)}\times\Gamma_2^{(\delta)}} n(x)n(y)\log|x-y|dxdy \geqslant C,$$
$$\int_{\Gamma_3} n(x)dx \int_{\Gamma_3} n\log n dx + 2\iint_{\Gamma_3\times\Gamma_3} n(x)n(y)\log|x-y|dxdy \geqslant C.$$

Combining the above inequalities yields

$$\begin{aligned}
-C \leqslant& K\int_{\mathbb{R}^2} n\log^+ n dx + 2\iint_{\mathbb{R}^2\times\mathbb{R}^2} n(x)n(y)\log|x-y|dxdy \\
&- 4\iint_{((\Gamma_1^{(\delta)})^c\times\Gamma_1)\cup(\Gamma_2\times(S_1\cup S_3))} n(x)n(y)\log|x-y|dxdy \\
&+ 2\iint_{(S_1\times S_1)\cup(S_2\times S_2)} n(x)n(y)\log|x-y|dxdy \\
=:& I_1 + I_2 - I_3 + I_4.
\end{aligned} \tag{3.29}$$

Here the number K is defined as

$$K := \max\left\{\int_{\Gamma_1^{(\delta)}} ndx + \int_{\Gamma_3} ndx, \int_{\Gamma_2^{(\delta)}} ndx + \int_{\Gamma_3} ndx\right\}. \tag{3.30}$$

Combining the definition (3.30) and (3.27) yields

$$K \leqslant \left(\frac{1}{2} + \frac{(1+\eta)^2}{2R^2}\right)M.$$

By the moderate mass constraint (3.3), we have $K < 8\pi$. Next applying the fact that $|x-y| \geqslant \delta$ for all (x,y) in the integral domain of I_3, we estimate the I_3 and I_4 terms in (3.29) as follows

$$\begin{aligned} I_3 \geqslant & 4M^2 \log \delta, \\ I_4 \leqslant & 2 \iint_{(S_1\times S_1)\cup(S_2\times S_2)} n(x)n(y)\log^+|x-y|dxdy \\ \leqslant & C \iint_{(S_1\times S_1)\cup(S_2\times S_2)} n(x)n(y)(1+|x|^2+|y|^2)dxdy \leqslant C(M^2+2M\int |x|^2 n(x)dx). \end{aligned} \tag{3.31}$$

Combining (3.31) with (3.29) yields

$$\begin{aligned} K\int_{\mathbb{R}^2} n\log^+ n dx + 2\iint_{\mathbb{R}^2\times\mathbb{R}^2} n(x)n(y)\log|x-y|dxdy \\ \geqslant 4M^2\log\delta - C(1+M^2+M\int n|x|^2dx). \end{aligned} \tag{3.32}$$

Recall the well-known upper bound on the negative part of the entropy:

Lemma 3.4. *([BCM2008]) For f positive function, the following estimate holds*

$$\int_{\mathbb{R}^2} f\log^- f dx \leqslant \frac{1}{2}\int_{\mathbb{R}^2}|x|^2 f dx + \log(2\pi)\int_{\mathbb{R}^2} f dx + \frac{1}{e} \leqslant C(1+M+\int |x|^2 n dx). \tag{3.33}$$

Proof. For the proof of this lemma, we refer the interested readers to the papers [BDP2006],[BCM2008]. □

Combining (3.33) and (3.32) yields the lower bound of T_1 in (3.28)

$$T_1 \geqslant -C(M+1)\left(1+M+\int n|x|^2dx\right)+4M^2\log\delta \tag{3.34}$$

for $0 < K < 8\pi$.

For the T_2 term in (3.28), it is bounded above by $\left|\int Hndx\right| \leqslant A\int |x|^2ndx$. Therefore it is enough to show that the second moment is bounded for any finite time. The time evolution of the second moment can be estimated as follows:

$$\frac{d}{dt}\int n|x|^2dx \leqslant 4AM + 4A\int Hn(x)dx \leqslant 4AM + \frac{A^2}{2}\int |x|^2ndx.$$

Gronwall inequality yields that the second moment is bounded for all finite time:

$$\int n|x|^2dx \leqslant C(A,T) < \infty, \quad \forall T < \infty. \tag{3.35}$$

Therefore $T_2 \leqslant C(A,T)$. Combining this with (3.28) and (3.34), and recalling that $K < 8\pi$ yield

$$S[n](T) \leqslant \frac{1}{(1-\frac{K}{8\pi})}\left(E[n_0]+C(M,A,T)-\frac{1}{2\pi}M^2\log\delta\right), \quad \forall T<\infty. \tag{3.36}$$

As a result, we see from (3.36) that the entropy $S[n^\epsilon]$ is uniformly bounded independent of ϵ for any finite time interval $[0,T]$, $\quad T < \infty$. Now by the Proposition 2.1, 2.2, we have that the free energy solution exists on any time interval $[0,T]$, $\quad \forall T < \infty$. □

3.4. **Step 3 — proof of the main theorem.** In the proof of Theorem 3.1, we see that the cell population is separated by a 'cell clear zone' near the x_1 axis. Since total mass in the "cell clear zone" is small, we can heuristically treat the total cell population as a union of two subgroups with subcritical mass ($< 8\pi$). However, since we lack sufficiently good control over the total number of cells near the x_1 axis, we cannot use this idea to prove the optimal result as stated in Theorem 1.1. The idea of proving Theorem 1.1 is that instead of considering the total cell population as the union of two subgroups separated by one fixed 'cell clear zone', we treat it as the union of three subgroups with subcritical mass, namely, the cells in the upper half plane, the lower half plane and the neighborhood of the x_1 axis, respectively. These three subgroups of cells are separated by two 'cell clear zones' varying in time.

The main difficulty in the proof is setting up the three new regions such that:

1. mass inside each region is smaller than 8π;
2. the total mass of cells near their boundaries is well-controlled.

Once the construction is completed, the remaining steps will be similar to step 2.

Proof of Theorem 1.1. We start by constructing the three regions. First we note that the Lemma 3.1 implies that there exists $\delta > 0$ such that the following estimate is satisfied for a fixed $R > 1$ and η chosen small enough:

$$\int_{|x_2|\leqslant 2\delta} ndx \leqslant \frac{(1+\eta)^2}{R^2}\int_{\mathbb{R}^2} ndx \leqslant \frac{1}{2}M, \quad \forall t > 0. \tag{3.37}$$

Now the region $L = \{(x_1, x_2)||x_2| \leqslant 2\delta\}$ have total mass less than $\frac{1}{2}M = M_+ < 8\pi$ for all time.

Secondly, we subdivide the region L into J pieces:

$$\begin{aligned} L = &\cup_1^J L^i, \\ L^i := &\{(x_1, x_2)|\frac{2\delta}{J}(i) > |x_2| \geqslant \frac{2\delta}{J}(i-1)\}. \end{aligned}$$

Here $J = J(M) \geqslant 10$, to be determined later, depends on M. By the pigeon hole principle, there is at least three strips L^i such that

$$\int_{L^i} n(x)dx \leqslant \frac{2}{J}M_+.$$

Suppose there are only two strips with mass smaller than $\frac{2}{J}M_+$, then total mass in L will be bigger than $(J-2)\frac{2}{J}M_+ > M_+$, a contradiction. Now we pick from these three strips the one which is neither L^1 nor L^J. As a result, this strip L^i does not touch the x_1 axis nor the boundary of L. We denote this i by i^*. The L^{i^*} is the 'cell clear zone'. Notice that here $i^* = i^*(n,t)$ depends on time.

Finally, we use this i^* to define the regions. First we define the three regions, each of which has total mass smaller than 8π:

$$\Gamma_1 = \left\{x_2 \geqslant \frac{2\delta}{J}i^*\right\}, \quad \Gamma_2 = \left\{x_2 \leqslant -\frac{2\delta}{J}i^*\right\}, \quad \Gamma_3 = \left\{|x_2| \leqslant \frac{2\delta}{J}(i^*-1)\right\}. \tag{3.38}$$

Next we set

$$\rho = \frac{2\delta}{3J}, \tag{3.39}$$

and define the ρ neighborhood of the above three regions:

$$\Gamma_1^{(\rho)} = \{x_2 > \frac{2\delta}{J}(i^* - \frac{1}{3})\} \quad \Gamma_2^{(\rho)} = \left\{x_2 < -\frac{2\delta}{J}(i^* - \frac{1}{3})\right\}, \quad \Gamma_3^{(\rho)} = \{|x_2| < \frac{2\delta}{J}(i^* - \frac{2}{3})\}. \tag{3.40}$$

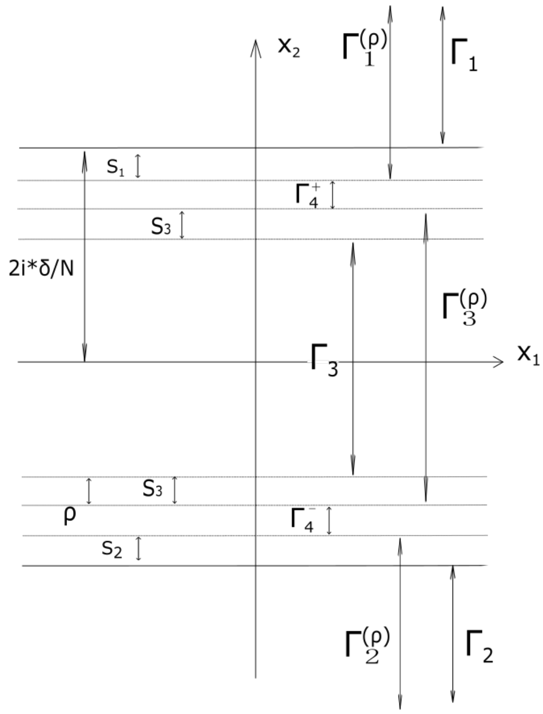

FIGURE 2. Regions $\Gamma_1, \Gamma_2, \Gamma_3$ in the proof of the main theorem

Now we define the complement Γ_4 of the above three regions $\Gamma_i^{(\rho)}, i = 1, 2, 3$:

$$\Gamma_4 = \left\{ \frac{2\delta}{J}\left(i^* - \frac{2}{3}\right) \leqslant |x_2| \leqslant \frac{2\delta}{J}\left(i^* - \frac{1}{3}\right) \right\} = \Gamma_{4+} \cup \Gamma_{4-}, \tag{3.41}$$

$$\Gamma_{4\pm} = \Gamma_4 \cap \mathbb{R}^2_\pm. \tag{3.42}$$

Now we define the complement $\Gamma_4^{(\rho)}$ of $\cup_{i=1}^3 \Gamma_i$ and decompose it into subdomains:

$$\Gamma_4^{(\rho)} = \big(\cup_{i=1}^3 \Gamma_i\big)^c = \big(\Gamma_{4+}^{(\rho)}\big) \cup \big(\Gamma_{4-}^{(\rho)}\big), \quad \big(\Gamma_{4\pm}^{(\rho)}\big) = \Gamma_4^{(\rho)} \cap \mathbb{R}^2_\pm, \tag{3.43}$$

$$\Gamma_4^{(\rho)} = \Gamma_4 \cup S_1 \cup S_2 \cup S_3, \tag{3.44}$$

$$S_1 = \Gamma_1^{(\rho)} \backslash \Gamma_1, \quad S_2 = \Gamma_2^{(\rho)} \backslash \Gamma_2, \quad S_3 = \Gamma_3^{(\rho)} \backslash \Gamma_3. \tag{3.45}$$

Remark 3.4. *It is important to notice that the regions we are constructing is changing with respect to the given time t. Therefore, by doing the argument below, we can only show that the entropy is bounded at time t, but since t is an arbitrary finite time, we have the bound on entropy for $\forall t \in [0, T], \forall T < \infty$.*

We start estimating the entropy. By the free energy dissipation, we obtain

$$\begin{aligned} E[n_0] \geqslant & \left(1 - \frac{K}{8\pi}\right) \int n \log n dx + \frac{1}{8\pi}\left(K \int n \log n dx \right. \\ & \left. + 2 \iint_{\mathbb{R}^2 \times \mathbb{R}^2} n(x) n(y) \log |x - y| dx dy \right) - \int H n dx \\ = & \left(1 - \frac{K}{8\pi}\right) S[n(T)] + T_1 - T_2. \end{aligned} \tag{3.46}$$

To derive entropy bound, we need the estimate T_1 form below for $K < 8\pi$ and estimate T_2 from above. We start by estimating T_1. Combining the definition of L^{i^*} and (3.37) yields

$$\int_{\Gamma_1^{(\rho)}} ndx \leqslant M_+ = \frac{1}{2}M < 8\pi, \tag{3.47}$$

$$\int_{\Gamma_2^{(\rho)}} ndx \leqslant M_+ = \frac{1}{2}M < 8\pi, \tag{3.48}$$

$$\int_{\Gamma_3^{(\rho)}} ndx \leqslant M_+ = \frac{1}{2}M < 8\pi, \tag{3.49}$$

$$\int_{\Gamma_4^{(\rho)}} ndx \leqslant \int_{L^{i^*}} ndx \leqslant \frac{2}{J}(M_+). \tag{3.50}$$

Now by the log-Hardy-Littlewood-Sobolev inequality (3.23), we have that

$$\int_{\Gamma_i^{(\rho)}} n(x)dx \int_{\Gamma_i^{(\rho)}} n(x)\log n(x)dx + 2\iint_{\Gamma_i^{(\rho)}\times\Gamma_i^{(\rho)}} n(x)n(y)\log|x-y|dxdy \geqslant -C, \quad i = 1,2,3, \tag{3.51}$$

$$\int_{\Gamma_{4\pm}^{(\rho)}} n(x)dx \int_{\Gamma_{4\pm}^{(\rho)}} n(x)\log n(x)dx + 2\iint_{\Gamma_{4\pm}^{(\rho)}\times\Gamma_{4\pm}^{(\rho)}} n(x)n(y)\log|x-y|dxdy \geqslant -C. \tag{3.52}$$

Same as in subsection 3.3, we use these estimates to reconstruction the entropy and the potential on the whole $\mathbb{R}^2$ as follows:

$$\begin{aligned}
-C \leqslant& K\int_{\mathbb{R}^2} n(x)\log^+ n(x)dx + 2\iint_{\mathbb{R}^2\times\mathbb{R}^2} n(x)n(y)\log|x-y|dxdy \\
&- 2\iint_R n(x)n(y)\log|x-y|dxdy \\
&+ 2\iint_{(S_1\times S_1)\cup(S_2\times S_2)\cup(S_3^+\times S_3^+)\cup(S_3^-\times S_3^-)} n(x)n(y)\log|x-y|dxdy \\
=:& I_1 + I_2 - I_3 + I_4.
\end{aligned} \tag{3.53}$$

The region R[2] and the integral domain of I_4 is indicated in Figure 3. The K in (3.53) can be estimated using (3.50) as follows

$$K := M_+ + \int_{\Gamma_4^{(\rho)}} ndx \leqslant \left(1 + \frac{2}{J}\right)M_+. \tag{3.54}$$

By the assumption $M_+ < 8\pi$, we can make J big such that $K < 8\pi$. This is where we choose the $J = J(M)$. Applying the fact that $|x-y| \geqslant \frac{2\delta}{3J}$, $\forall(x,y) \in R$, the I_3 and I_4 terms in (3.53) can be

[2] Region R is the union of the following nine regions:

$1)\Gamma_1 \times (\Gamma_1^{(\rho)})^c$, $2)S_1 \times (\Gamma_1 \cup (\Gamma_4^{(\rho)})^+)^c$, $3)\Gamma_4^+ \times ((\Gamma_4^{(\rho)})^+)^c$, $4)S_3^+ \times (\Gamma_3^{(\rho)} \cup (\Gamma_4^{(\rho)})^+)^c$, $5)\Gamma_3 \times (\Gamma_3^{(\rho)})^c$, $6)S_3^- \times (\Gamma_3^{(\rho)} \cup (\Gamma_4^{(\rho)})^-)^c$, $7)\Gamma_4^- \times ((\Gamma_4^{(\rho)})^-)^c$, $8)S_2 \times (\Gamma_2^{(\rho)} \cup (\Gamma_4^{(\rho)})^-)^c$, $9)\Gamma_2 \times (\Gamma_2^{(\rho)})^c$.

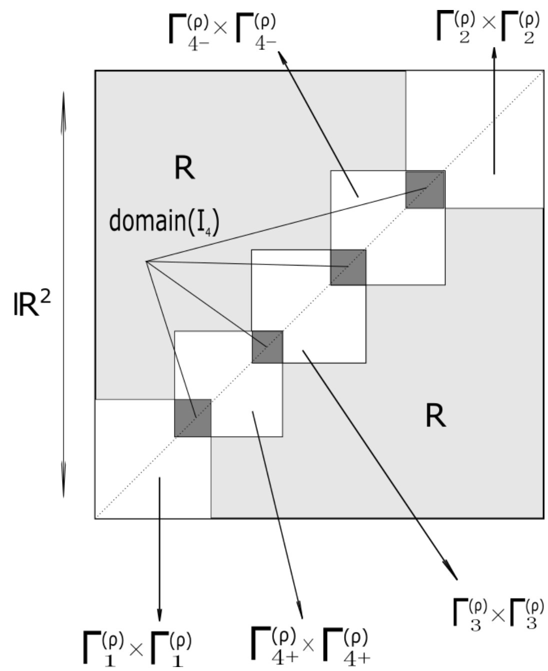

FIGURE 3. Region R in $\mathbb{R}^2\times\mathbb{R}^2$

estimated as follows:

$$\begin{aligned}
I_3 \geqslant & CM^2\log\frac{2\delta}{3J},\\
I_4 \leqslant & 2\iint_{(S_1\times S_1)\cup(S_2\times S_2)\cup(S_3^+\times S_3^+)\cup(S_3^-\times S_3^-)} n(x)n(y)\log^+|x-y|dxdy\\
\leqslant & C\iint_{(S_1\times S_1)\cup(S_2\times S_2)\cup(S_3^+\times S_3^+)\cup(S_3^-\times S_3^-)} n(x)n(y)(1+|x|^2+|y|^2)dxdy\\
\leqslant & C(M^2+M\int|x|^2ndx).
\end{aligned}\tag{3.55}$$

Combining (3.53), (3.54) and (3.55) yields

$$\begin{aligned}
K\int n(x)\log^+ n(x)dx+2\iint_{\mathbb{R}^2\times\mathbb{R}^2} n(x)n(y)\log|x-y|dxdy&\\
\geqslant CM^2\log\frac{2\delta}{3J}-C(1+M^2+M\int|x|^2ndx).&
\end{aligned}$$

Moreover, applying (3.33) yields

$$T_1\geqslant -C(M+1)(1+M+\int|x|^2ndx)+CM^2\log\frac{2\delta}{3J}.\tag{3.56}$$

Combining (3.56), (3.35), (3.46) yields

$$S[n](T)\leqslant\frac{1}{(1-\frac{K}{8\pi})}\left(E[n_0]+C(M,A,T)-CM^2\log\frac{2\delta}{3J}\right)<\infty,\forall T<\infty.$$

Once the entropy is bounded for any finite time, the existence is guaranteed by Proposition 2.1 and Proposition 2.2. □

Appendix A.

In the appendix, we prove the two local existence theorems stated in section 2.2. The proof follows the same line as the analysis in [BCM2008].

A.1. **Proof of proposition 2.1.**

Proof. For any fixed positive ϵ, following the argument as in section 2.5 of [BDP2006], we obtain the global solution in $L^2([0,T],H^1)\cap C([0,T],L^2)$ for the regularized Keller-Segel system with advection (2.3).

The goal is to use the Aubin-Lions Lemma to show that the solutions $\{n^\epsilon\}_{\epsilon\geqslant 0}$ is precompact in certain topology. Same as in the paper [BCM2008], we divide the the proof in steps.

Step 1. A priori estimates on n^ϵ and c^ϵ. In this step, we derive several estimates which we will need later.

First we estimate the second moment

$$V:=\int_{\mathbb{R}^2}|x|^2n^\epsilon dx.$$

The time evolution of V can be estimated using the regularised equation (2.3) and the fact that the gradient of the regularized kernel K^ϵ is bounded $|\nabla K^\epsilon(z)|\leqslant\dfrac{1}{2\pi|z|}$:

$$\begin{aligned}\frac{d}{dt}V=&4M+\iint_{\mathbb{R}^2\times\mathbb{R}^2}n^\epsilon(x,t)n^\epsilon(y,t)(x-y)\nabla K^\epsilon(x-y)dxdy+2\int x\cdot\mathbf{b}n^\epsilon(x)dx\\ \leqslant&4M+2C\int|x|^2n^\epsilon dx.\end{aligned}$$

By Gronwall, we have that

$$V(t)\leqslant\frac{4M}{2C}e^{2CT}+V_0\leqslant C_V(T)<\infty,\quad\forall 0\leqslant t\leqslant T,\tag{A.1}$$

from which we obtain that $(1+|x|^2)n^\epsilon\in L^\infty([0,T],L^1)$ uniformly in ϵ.

Next, we estimate $|n^\epsilon\log n^\epsilon|_{L^\infty_t([0,T];L^1)}$ and $|\int n^\epsilon c^\epsilon dx|_{L^\infty_t}$. Combining the assumption of the proposition 2.1 and (3.33) yields

$$\int|n^\epsilon\log n^\epsilon|dx\leqslant\int n^\epsilon(\log n^\epsilon+|x|^2)dx+2\log(2\pi)M+\frac{2}{e}.\tag{A.2}$$

Therefore, we proved that $n^\epsilon\log n^\epsilon\in L^\infty([0,T],L^1)$ uniformly in ϵ. Recalling the boundedness of the second moment (A.1) and the representation of c^ϵ as $c^\epsilon=K^\epsilon*n^\epsilon$, we deduce the following estimate using the Young's inequality,

$$|c^\epsilon|(x)=\left|\int K^\epsilon(x-y)n^\epsilon(y)dy\right|\leqslant C(M,V)+C(M)\log(1+|x|)\tag{A.3}$$

uniformly in ϵ. Combining this with the mass conservation of n^ϵ and the second moment control (A.1), we deduce that

$$\left|\int n^\epsilon c^\epsilon dx\right|_{L^\infty([0,T])}\leqslant C,\tag{A.4}$$

where C is independent of ϵ.

Next we derive the main a priori estimate, namely, the $L^2([0,T]\times\mathbb{R}^2)$ estimate of $\sqrt{n^\epsilon}\nabla c^\epsilon$. First we calculate the time evolution of $\int n^\epsilon c^\epsilon dx$:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int n^\epsilon c^\epsilon dx &= \int n^\epsilon_t c^\epsilon dx\\ &= \int (\Delta n^\epsilon - \nabla\cdot(\nabla c^\epsilon n^\epsilon) - b\cdot\nabla n^\epsilon)c^\epsilon dx\\ &= \int n^\epsilon \Delta c^\epsilon dx + \int n^\epsilon|\nabla c^\epsilon|^2 dx + \int \mathbf{b}n^\epsilon\cdot\nabla c^\epsilon dx.\end{aligned}$$

Integrating this in time yields:

$$\begin{aligned}&\int_0^T\int n^\epsilon|\nabla c^\epsilon|^2 dxdt\\ &\quad= \frac{1}{2}\int n^\epsilon c^\epsilon dx(T) - \frac{1}{2}\int n^\epsilon c^\epsilon dx(0) - \int_0^T\int n^\epsilon\Delta c^\epsilon dxdt - \int_0^T\int n^\epsilon\mathbf{b}\cdot\nabla c^\epsilon dxdt \qquad (A.5)\end{aligned}$$

Now we estimate the right hand side of (A.5). We see from (A.4) that the first two terms on the right hand side is bounded. Next we estimate the third term on the right hand side of (A.5), which requires information derived from the entropy bound. By the property that $\nabla\cdot\mathbf{b}=0$, we formally calculate the time evolution of $S[n](t)$ as follows,

$$\frac{d}{dt}S[n](t) = -4\int|\nabla\sqrt{n}|^2dx + \int n(t)^2dx. \qquad (A.6)$$

The interested reader is referred to [BDP2006] for more details. We need to estimate the second term in (A.6). Before doing this, note that for $K>1$,

$$\int_{n\geqslant K} ndx \leqslant \frac{1}{\log(K)}\int n_+\log n dx \leqslant \frac{C}{\log(K)} =: \eta(K) \qquad (A.7)$$

can be made arbitrarily small. Now we can use the Gagliardo-Nirenberg-Sobolev inequality together with (A.7) to estimate the second term in (A.6) as follows:

$$\begin{aligned}\int n^2dx \leqslant& MK + \int_{n\geqslant K} n^2 dx \leqslant MK + \left(\int_{n\geqslant K} ndx\right)^{1/2}|n|_3^{3/2}\\ \leqslant& MK + \eta(K)^{1/2}CM^{1/2}|\nabla\sqrt{n}|_2^2.\end{aligned}$$

Combining this with (A.6) and (A.7), we have

$$\frac{d}{dt}S[n](t) = -(4-\eta(K)^{1/2}CM^{1/2})\int|\nabla\sqrt{n}|^2dx + MK. \qquad (A.8)$$

The factor $-(4-\eta(K)^{1/2}CM^{1/2})$ can be made non-positive for K large enough and therefore we have that

$$\int_0^T\int|\nabla\sqrt{n}|^2dxdt \leqslant \frac{S[n](0)-S[n](T)+MKT}{(4-2\eta(K)^{1/2}M^{1/2}C)}.$$

It follows that $\nabla\sqrt{n}$ is bounded in $L^2([0,T]\times\mathbb{R}^2)$. The derivation for $|\nabla\sqrt{n^\epsilon}|_{L^2([0,T];L^2)}\leqslant C$ is similar but more technical, and the interested readers are referred to [BDP2006] for more details. As a consequence of the $L^2([0,T]\times\mathbb{R}^2)$ estimate on $\nabla\sqrt{n^\epsilon}$ and of the computation

$$\frac{d}{dt}S[n^\epsilon](t) = -4\int|\nabla\sqrt{n^\epsilon}|^2dx + \int n^\epsilon(-\Delta c^\epsilon)dx,$$

we have the estimate

$$\int_0^T\int n^\epsilon(-\Delta c^\epsilon)dxdt \leqslant C. \qquad (A.9)$$

This completes the treatment of the third term on the right hand side of (A.5). Next we estimate the last term $\int_0^T \int \mathbf{b}n^\epsilon \cdot \nabla c^\epsilon dxdt$ in (A.5). First we calculate the time evolution of $\int Gn^\epsilon dx$:

$$\begin{aligned}\frac{d}{dt}\int Gn^\epsilon dx &= \int G\nabla\cdot(\nabla n^\epsilon - \nabla c^\epsilon n^\epsilon - \mathbf{b}n^\epsilon)dx \\ &= \int \Delta G n^\epsilon dx + \int \nabla G\cdot\nabla c^\epsilon n^\epsilon dx + \int |\mathbf{b}|^2 n^\epsilon dx \\ &= \int \mathbf{b}\cdot\nabla c^\epsilon n^\epsilon dx + \int |\mathbf{b}|^2 n^\epsilon dx.\end{aligned}$$

Now integrating in time, we obtain

$$\left|\int_0^T \int \mathbf{b}\cdot\nabla c^\epsilon n^\epsilon dx\right| \leqslant \left|\int Gn^\epsilon dx(0)\right| + \left|\int Gn^\epsilon dx(T)\right| + \left|\int_0^T \int |b|^2 n^\epsilon dxdt\right|. \tag{A.10}$$

From the assumption $|\mathbf{b}| \lesssim |x|$, we have that the right hand side of (A.10) can be bounded in terms of the second moment V:

$$\left|\int Gn^\epsilon dx\right| + \left|\int |\mathbf{b}|^2 n^\epsilon dx\right| \leqslant C\int |x|^2 n^\epsilon dx. \tag{A.11}$$

Since the second moment V is bounded (A.1), we have that

$$\left|\int_0^T \int \mathbf{b}\cdot\nabla c^\epsilon n^\epsilon dx\right| \leqslant C. \tag{A.12}$$

Applying estimates (A.4), (A.9) and (A.12) to (A.5), we obtain

$$\int_0^T \int n^\epsilon |\nabla c^\epsilon|^2 dxdt \leqslant C < \infty. \tag{A.13}$$

This concludes our first step.

Step 2- Passing to the limit. As in [BCM2008], the following Aubin-Lions compactness lemma is applied:

Lemma A.1 (Aubin-Lions lemma)**.** [BCM2008] *Take $T > 0$ and $1 < p < \infty$. Assume that $(f_n)_{n\in\mathbb{N}}$ is a bounded sequence of functions in $L^p([0,T];H)$ where H is a Banach space. If $(f_n)_n$ is also bounded in $L^p([0,T];V)$ where V is compactly imbedded in H and $\partial f_n/\partial_t \in L^p([0,T];W)$ uniformly with respect to $n \in \mathbb{N}$ where H is imbedded in W, then $(f_n)_{n\in\mathbb{N}}$ is relatively compact in $L^p([0,T];H)$.*

Our goal now is to find the appropriate spaces V, H, W for n^ϵ. We subdivide the proof into steps, each step determines one space in the lemma.

Space H: Bound on $|n^\epsilon|_{L^2([0,T],L^2)}$: We can estimate the $|n^\epsilon|_2^2$ by applying the following decomposition trick:

$$n^\epsilon = (n^\epsilon - K)_+ + \min\{n^\epsilon, K\}. \tag{A.14}$$

The second part in (A.14) is bounded in $L^p, 1 \leqslant p \leqslant \infty$. The first part is bounded in L^1 by

$$|(n^\epsilon - K)_+|_1 \leqslant \frac{|n^\epsilon \log n^\epsilon|_1}{\log K} = \eta(K), \tag{A.15}$$

which can be made arbitrary small. Now we estimate the time evolution of $\int (n^\epsilon - K)_+^p dx, 1 < p < \infty$ as follows:

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int (n^\epsilon - K)_+^p dx \\
&\quad = \int (n^\epsilon - K)_+^{p-1}(\Delta n^\epsilon - \nabla\cdot(\nabla c^\epsilon n^\epsilon) - \mathbf{b}\cdot\nabla n^\epsilon)dx \\
&\quad = -\frac{4(p-1)}{p^2}\int |\nabla(n^\epsilon - K)_+^{p/2}|^2 dx - \int (n^\epsilon - K)_+^{p-1}\nabla\cdot(\nabla c^\epsilon n^\epsilon)dx - \int (n^\epsilon - K)_+^{p-1}\mathbf{b}\cdot\nabla(n^\epsilon - K)_+ dx \\
&\quad \leqslant -\frac{4(p-1)}{p^2}\int |\nabla(n^\epsilon - K)_+^{p/2}|^2 dx - \frac{p-1}{p}\int (n^\epsilon - K)_+^p \Delta c^\epsilon dx - K\int (n^\epsilon - K)_+^{p-1}\Delta c^\epsilon dx \\
&\quad =: -\frac{4(p-1)}{p^2}\int |\nabla(n^\epsilon - K)_+^{p/2}|^2 dx + T_1 + T_2. \qquad (A.16)
\end{aligned}$$

Note that since $\nabla\cdot\mathbf{b} \equiv 0$, the term involving $\mathbf{b}$ vanishes. For the T_1 term in (A.16), using the facts that $\Delta c^\epsilon = \Delta K^\epsilon * n^\epsilon$ and $|\Delta K^\epsilon|_1$ is uniformly bounded, we can estimate it as follows:

$$\begin{aligned}
T_1 \leqslant& \int (n^\epsilon - K)_+^p |\Delta K^\epsilon| * (n^\epsilon - K)_+ dx + K\int (n^\epsilon - K)_+^p dx |\Delta K^\epsilon|_1 \\
\leqslant& |(n^\epsilon - K)_+|_{p+1}^p \big||\Delta K^\epsilon| * (n^\epsilon - K)_+\big|_{p+1} + CK\int (n^\epsilon - K)_+^p dx \\
\leqslant& C|(n^\epsilon - K)_+|_{p+1}^{p+1} + CK|(n^\epsilon - K)_+|_p^p. \qquad (A.17)
\end{aligned}$$

Similarly, we can estimate the T_2 term in (A.16) as follows:

$$T_2 \leqslant CK|(n^\epsilon - K)_+|_p^p + CK^2|(n^\epsilon - K)_+|_{p-1}^{p-1}. \qquad (A.18)$$

Combining (A.16), (A.17) and (A.18), we obtain

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int (n^\epsilon - K)_+^p dx \leqslant& -\frac{4(p-1)}{p^2}\int |\nabla(n^\epsilon - K)_+^{p/2}|^2 dx \\
&+ C|(n^\epsilon - K)_+|_{p+1}^{p+1} + CK|(n^\epsilon - K)_+|_p^p + CK^2|(n^\epsilon - K)_+|_{p-1}^{p-1}. \qquad (A.19)
\end{aligned}$$

For the highest order term $C|(n^\epsilon - K)_+|_{p+1}^{p+1}$ in (A.19), we use the following Gagliardo-Nirenberg-Sobolev inequality:

$$\int_{\mathbb{R}^2} f^{p+1}dx \leqslant C\int_{\mathbb{R}^2} |\nabla(f^{p/2})|^2 dx \int_{\mathbb{R}^2} f dx, \quad f \geqslant 0 \qquad (A.20)$$

together with (A.15) to estimate it as follows

$$|(n^\epsilon - K)_+|_{p+1}^{p+1} \leqslant C|\nabla((n^\epsilon - K)_+^{p/2})|_2^2 |(n^\epsilon - K)_+|_1 \leqslant C\eta(K)|\nabla((n^\epsilon - K)_+^{p/2})|_2^2. \qquad (A.21)$$

We can take K big such that it is absorbed by the negative dissipation term in (A.19). Now applying Hölder's inequality, Young's inequality and Gronwall inequality to (A.19), we have that

$$|(n^\epsilon - K)_+|_{L^p}(t) \leqslant C(T) < \infty, \quad t \in [0,T], p \in (1,\infty).$$

Applying standard argument, see e.g., [BDP2006] proof of Proposition 3.3, we obtain the estimate

$$|n^\epsilon|_{L^\infty([0,T];L^p)} \leqslant C(T), \quad p \in (1,\infty). \qquad (A.22)$$

In particular, we set $p = 2$ and obtain that

$$|n^\epsilon|_{L^2([0,T];L^2)} \leqslant C(T). \qquad (A.23)$$

We conclude this step by setting $H := L^2(\mathbb{R}^2)$.

Space V: Bound on $|\nabla n^\epsilon|_{L^2([0,T]\times\mathbb{R}^2)}$: First we calculate the time evolution of the quantity $|n^\epsilon|_2^2$:

$$\begin{aligned}\frac{d}{dt}\int |n^\epsilon|^2 dx =& -2\int |\nabla n^\epsilon|^2 dx + 2\int \nabla n^\epsilon \cdot \nabla c^\epsilon n^\epsilon dx - \int \nabla \cdot (\mathbf{b}(n^\epsilon)^2) dx\\ =& -2\int |\nabla n^\epsilon|^2 dx + 2\int \nabla n^\epsilon \cdot \nabla c^\epsilon n^\epsilon dx.\end{aligned}$$

Now integrating in time, we obtain the estimate

$$\begin{aligned}&\int |n^\epsilon|^2 dx(T) - \int |n^\epsilon|^2 dx(0)\\ &\quad= -2\int_0^T\int |\nabla n^\epsilon|^2 dxdt + 2\int_0^T\int n^\epsilon(\nabla n^\epsilon \cdot \nabla c^\epsilon) dxdt\\ &\quad\leqslant -2\int_0^T\int |\nabla n^\epsilon|^2 dxdt + 2\left(\int_0^T\int |n^\epsilon \nabla c^\epsilon|^2 dxdt\right)^{1/2}\left(\int_0^T\int |\nabla n^\epsilon|^2 dxdt\right)^{1/2} \quad (A.24)\end{aligned}$$

The terms on the left hand side of (A.24) are bounded due to (A.22). For the last term on the right hand side, we can estimate it as follows. The Hardy-Littlewood-Sobolev inequality yields

$$|\nabla c^\epsilon|_4 \leqslant C|n^\epsilon|_{4/3}, \quad (A.25)$$

which implies

$$|n^\epsilon \nabla c^\epsilon|_2 \leqslant |n^\epsilon|_4 |\nabla c^\epsilon|_4 \leqslant C|n^\epsilon|_4 |n^\epsilon|_{4/3}.$$

Combining this and the L^p bound (A.22) yields the boundedness of $n^\epsilon \nabla c^\epsilon$ in $L^\infty([0,T],L^2)$. Applying this fact and (A.22) in (A.24) and set $X = (\int_0^T\int |\nabla n^\epsilon|^2 dxdt)^{1/2}$, we obtain

$$X^2 - 2|n^\epsilon \nabla c^\epsilon|_{L^2((0,T)\times\mathbb{R}^2)} X \leqslant \int |n^\epsilon|^2 dx(T) - \int |n^\epsilon|^2 dx(0) \leqslant C.$$

As a result,

$$|\nabla n^\epsilon|_{L^2([0,T]\times\mathbb{R}^2)} \leqslant C. \quad (A.26)$$

Same as in [BCM2008], we set $V := H^1(\mathbb{R}^2)\cap\{n||x|n^2 \in L^1\}$, which is shown to be compactly imbedded in H there. Thanks to the bound (A.23), (A.26) and the (A.1), we have that $n^\epsilon \in L^2([0,T];V)$.

Space W: Bound for the $\partial_t n^\epsilon$: In order to estimate the $L^2([0,T];H^{-1})$ norm of the function $\partial_t n^\epsilon$, we first need to get a bound on the fourth moment of the solution $V_4 := \int n(x_1^4 + x_2^4)dx$. The time evolution of V_4 can be formally estimated as follows:

$$\begin{aligned}\frac{d}{dt}V_4 =& \int (\Delta n - \nabla\cdot(\nabla c n) - \mathbf{b}\nabla n)(x_1^4 + x_2^4) dx\\ =& 12\int n|x|^2 dx - \frac{1}{4\pi}\iint \frac{4(x_1^3 - y_1^3)(x_1 - y_1) + 4(x_2^3 - y_2^3)(x_2 - y_2)}{|x-y|^2} n(x)n(y) dxdy + \int \mathbf{b}n\cdot(4x_1^3, 4x_2^3) dx\\ \leqslant& C(M)\int n|x|^2 dx + C\int n(x_1^4 + x_2^4) dx. \quad (A.27)\end{aligned}$$

Combining the second moment estimate (A.1) and the Gronwall inequality, we have that

$$\int n(x,t)|x|^4 dx \leqslant C, \;\; t\in[0,T]. \quad (A.28)$$

One can adapt this calculation to the regularized solution n^ϵ without much difficulty. We leave the details to the interested reader.

Now we can estimate the $L^2([0,T];H^{-1})$ norm of the $\partial_t n^\epsilon$. Combining the L^p bound on n^ϵ (A.22), the bound on ∇c (A.25) and the fourth moment control (A.28) and testing the equation (2.3) with $f \in L^2([0,T],H^1(\mathbb{R}^2))$, we have

$$\begin{aligned}\langle \partial_t n^\epsilon, f\rangle_{L^2([0,T]\times\mathbb{R}^2)} \leqslant & |\nabla n^\epsilon|_{L^2([0,T];L^2)}|f|_{L^2([0,T];H^1)} + |\nabla c^\epsilon n^\epsilon|_{L^2([0,T];L^2)}|f|_{L^2([0,T];H^1)} \\ & + |\mathbf{b}n^\epsilon|_{L^2([0,T];L^2)}|f|_{L^2([0,T];H^1)} \\ \leqslant & |\nabla n^\epsilon|_{L^2([0,T];L^2)}|f|_{L^2([0,T];H^1)} + T^{1/2}|\nabla c^\epsilon|_{L^\infty([0,T];L^4)}|n^\epsilon|_{L^\infty([0,T];L^4)}|f|_{L^2([0,T];H^1)} \\ & + CT^{1/2}\sup_t V_4^{1/4}|n^\epsilon|^{3/4}_{L^\infty([0,T];L^3)}|f|_{L^2([0,T];H^1)} \\ \leqslant & C|f|_{L^2([0,T];H^1)}.\end{aligned}$$

As a result, we have that $\partial_t n^\epsilon$ is uniformly bounded in $L^2([0,T];H^{-1})$.

Combining the results from all the steps above and the Aubin-Lions lemma, we have that $(n^\epsilon)_\epsilon$ is precompact in $L^2([0,T];L^2)$. We denote n as the limit of one converging subsequence $(n^{\epsilon_k})_{\epsilon_k}$. Moreover, combining (A.25) and the $L^{4/3}([0,T]\times\mathbb{R}^2)$ bound on n^ϵ which can be derived from (A.22), we have that $n^{\epsilon_k}\nabla c^{\epsilon_k}$ converge to $n\nabla c$ in distribution sense.

Step 3- Free energy estimates. By the convexity of the functional $n \to \int_{\mathbb{R}^2}|\nabla\sqrt{n}|^2dx$, the fact that $n^{\epsilon_k}\nabla c^{\epsilon_k}$ converge to $n\nabla c$ in distribution sense and weak semi-continuity, we have

$$\begin{aligned}\iint_{[0,T]\times\mathbb{R}^2}|\nabla\sqrt{n}|^2dxdt &\leqslant \liminf_{k\to\infty}\iint_{[0,T]\times\mathbb{R}^2}|\nabla\sqrt{n^{\epsilon_k}}|^2dxdt,\\ \iint_{[0,T]\times\mathbb{R}^2}n|\nabla c|^2dxdt &\leqslant \liminf_{k\to\infty}\iint_{[0,T]\times\mathbb{R}^2}n^{\epsilon_k}|\nabla c^{\epsilon_k}|^2dxdt.\end{aligned}$$

Moreover, it can be checked that $S[n^\epsilon](t) \to S[n](t)$ for almost every t, whose proof is similar to the one used in [BDP2006] Lemma 4.6.

Next we show the free energy estimate (1.5) using the strong convergence of $\{n^{\epsilon_k}\}$ in $L^2([0,T]\times\mathbb{R}^2)$. The key is to show the following entropy dissipation term is lower semi-continuous for the sequence (n^{ϵ_k}):

$$\begin{aligned}&\int_0^T\int n^{\epsilon_k}|\nabla\log n^{\epsilon_k} - \nabla c^{\epsilon_k} - \mathbf{b}|^2dxdt\\ &\quad =4\iint_{[0,T]\times\mathbb{R}^2}|\nabla\sqrt{n^{\epsilon_k}}|^2dxdt + \iint_{[0,T]\times\mathbb{R}^2}n^{\epsilon_k}|\nabla c^{\epsilon_k}|^2dxdt + \iint_{[0,T]\times\mathbb{R}^2}n^{\epsilon_k}|\mathbf{b}|^2dxdt\\ &\quad\quad - 2\iint_{[0,T]\times\mathbb{R}^2}(n^{\epsilon_k})^2dxdt - 2\iint_{[0,T]\times\mathbb{R}^2}n^{\epsilon_k}\mathbf{b}\cdot\nabla c^{\epsilon_k}dxdt\\ &\quad =: T_1 + T_2 + T_3 + T_4 + T_5.\end{aligned}\tag{A.29}$$

For the sake of simplicity, later we use n^ϵ to denote n^{ϵ_k}.

First, we estimate the T_3 term in (A.29). By the Fatou Lemma, we have the following inequality:

$$\iint_{[0,T]\times\mathbb{R}^2}n|\mathbf{b}|^2dxdt \leqslant \liminf_{\epsilon_k\to 0}\iint_{[0,T]\times\mathbb{R}^2}n^{\epsilon_k}|\mathbf{b}|^2dxdt.\tag{A.30}$$

This finises the treatment of T_3.

Next, we show that the term T_5 in (A.29) actually converges as $\epsilon_k \to 0$. We decompose the difference between T_5 and its formal limit into two parts

$$\begin{aligned}
&\Big|\int_0^T\int_{\mathbb{R}^2}\mathbf{b}\cdot\nabla cn - \mathbf{b}\cdot\nabla c^\epsilon n^\epsilon dxdt\Big| \\
&\qquad\leqslant \int_0^T\int_{\mathbb{R}^2}|\nabla c\cdot\mathbf{b}n - \nabla c^\epsilon\cdot\mathbf{b}n + n\mathbf{b}\cdot\nabla c^\epsilon - \mathbf{b}\cdot\nabla c^\epsilon n^\epsilon|dxdt \\
&\qquad\leqslant \int_0^T\int_{\mathbb{R}^2}|\nabla c - \nabla c^\epsilon||\mathbf{b}n|dxdt + \int_0^T\int_{\mathbb{R}^2}|\mathbf{b}n - \mathbf{b}n^\epsilon||\nabla c^\epsilon|dxdt \\
&\qquad=: T_{51} + T_{52}.
\end{aligned} \tag{A.31}$$

For the first term T_{51} in (A.31), applying Hölder and Hardy-Littlewood-Sobolev inequality, we can estimate it as follows

$$\begin{aligned}
T_{51} \leqslant& \Big(\int_0^T|\nabla(c-c^\epsilon)|_4^4dt\Big)^{1/4}\Big(\int_0^T\int_{\mathbb{R}^2}|b\sqrt{n}|^2dxdt\Big)^{1/2}\Big(\int_0^T\int_{\mathbb{R}^2}\sqrt{n}^4dxdt\Big)^{1/4} \\
\leqslant& C(\int_0^T|n-n^\epsilon|_{4/3}^4dt)^{1/4}(\int_0^T\int_{\mathbb{R}^2}|x|^2ndxdt)^{1/2}(\int_0^T\int_{\mathbb{R}^2}n^2dxdt)^{1/4} \\
\leqslant& C\Big(\int_0^T|n-n^\epsilon|_2^2\,|n-n^\epsilon|_1^2dt\Big)^{1/4}\Big(\int_0^T\int_{\mathbb{R}^2}|x|^2ndxdt\Big)^{1/2}\Big(\int_0^T\int_{\mathbb{R}^2}n^2dxdt\Big)^{1/4} \\
\leqslant& C\Big(\int_0^T|n-n^\epsilon|_2^2dt\Big)^{1/4}\sup_t|n-n^\epsilon|_1^{1/2}\Big(\int_0^T\int_{\mathbb{R}^2}|x|^2ndxdt\Big)^{1/2}\Big(\int_0^T\int_{\mathbb{R}^2}n^2dxdt\Big)^{1/4}.
\end{aligned}$$

From the $L^2([0,T]\times\mathbb{R}^2)$ strong convergence of $(n^\epsilon)_\epsilon$, we have that the first factor goes to zero. Other factors are bounded due to (A.1), Fatou's Lemma, L^1 bound on n^ϵ, n and $n \in L^2([0,T]\times\mathbb{R}^2)$. As a result, T_{51} converges to zero. Next we estimate the T_{52} term in (A.31). Applying the fact that $|\sqrt{|c|}-\sqrt{|a|}| \leqslant \sqrt{|c-a|}$, the Hölder and Hardy-Littlewood-Sobolev inequality, we have

$$\begin{aligned}
T_{52} =& \int_0^T\int_{\mathbb{R}^2}|\mathbf{b}(\sqrt{n}^2 - \sqrt{n^\epsilon}^2)||\nabla c^\epsilon|dxdt \\
\leqslant& \int_0^T\int_{\mathbb{R}^2}|\mathbf{b}\sqrt{n}(\sqrt{n}-\sqrt{n^\epsilon})|\nabla c^\epsilon|dxdt + \int_0^T\int_{\mathbb{R}^2}|\mathbf{b}\sqrt{n^\epsilon}(\sqrt{n}-\sqrt{n^\epsilon})|\nabla c^\epsilon|dxdt \\
\leqslant& |\nabla c^\epsilon|_{L^4([0,T]\times\mathbb{R}^2)}|b\sqrt{n}|_{L^2([0,T]\times\mathbb{R}^2)}\Big(\int_0^T\int_{\mathbb{R}^2}\sqrt{|n-n^\epsilon|}^4dxdt\Big)^{1/4} \\
&+ |\nabla c^\epsilon|_{L^4([0,T]\times\mathbb{R}^2)}|b\sqrt{n^\epsilon}|_{L^2([0,T]\times\mathbb{R}^2)}(\int_0^T\int_{\mathbb{R}^2}\sqrt{|n-n^\epsilon|}^4dxdt)^{1/4} \\
\leqslant& C\sup_t|n^\epsilon|_{L^1(\mathbb{R}^2)}{}^{1/2}|n^\epsilon|_{L^2([0,T]\times\mathbb{R}^2)}^{1/2}\Big(\int_0^T\int(n^\epsilon+n)|x|^2dxdt\Big)^{1/2}|n-n^\epsilon|_{L^2([0,T]\times\mathbb{R}^2)}^{1/2}
\end{aligned}$$

By the same reasoning as in T_{51}, we have that this term goes to zero. This finishes the treatment for the term.

Now for the T_1, T_2, T_4 terms in (A.29), we can handle them in the same way as in [BDP2006]. Combining all the estimates above we have that

$$\int_0^T\int n|\nabla\log n - \nabla c - \mathbf{b}|^2dxdt \leqslant \liminf_{\epsilon_k\to 0}\int_0^T\int n^{\epsilon_k}|\nabla\log n^{\epsilon_k} - \nabla c^{\epsilon_k} - \mathbf{b}|^2dxdt \tag{A.32}$$

The remaining part of the proof is the same as the one in [BDP2006]. □

A.2. **Proof of proposition 2.2.** The proof of the proposition follows along the lines of [BCM2008].

Acknowledgment. Research was supported by NSF grants DMS16-13911, RNMS11-07444 (KI-Net), ONR grant N00014-1512094 and the Sloan research fellowship FG-2015-66019. SH thanks the Ann G. Wylie Dissertation Fellowship and Jacob Bedrossian and Scott Smith for many fruitful discussions. We thank ETH Institute for Theoretical Studies (ETH-ITS) for the support and hospitality.

References

[BCZ2015] J. Bedrossian and M. Coti Zelati, Enhanced dissipation, hypoellipticity, and anomalous small noise inviscid limits in shear flows, ArXiv:1510.08098, 2015.

[BH2016] J. Bedrossian, S. He, Suppression of blow-up in Patlak-Keller-Segel via shear flows, arXiv:1609.02866 2016.

[BK2010] , J. Bedrossian and I. Kim, Global existence and finite time blow-up for critical Patlak-Keller-Segel models with inhomogeneous diffusion, SIMA 45(3), 934–964, 2013.

[BM2014] J. Bedrossian and N. Masmoudi, Existence, Uniqueness and Lipschitz Dependence for Patlak-Keller-Segel and Navier-Stokes in $\mathbb{R}^2$ with Measure-valued initial data, Arch. Rat. Mech. Anal, 214(3), 717–801, 2014.

[Bil1995] P. Biler, The Cauchy problem and self-similar solutions for a nonlinear parabolic equation, Studia Math., 114(2):181192, 1995.

[BCC2008] A. Blanchet, V. Calvez and J. A. Carrillo, *Convergence of the mass-transport steepest descent scheme for the subcritical Patlak-Keller-Segel model*, SIAM J. Numer. Anal., 46 (2008), 691-721.

[BDP2006] A. Blanchet and J. Dolbeault and B. Perthame, Two-Dimensional Keller-Segel Model: Optimal Critical Mass and Qualitative Properties of the Solutions, EJDE 44, 1–32, 2006.

[BCM2008] A. Blanchet, J. A. Carrillo and N. Masmoudi, Infinite Time Aggregation for the Critical Parlak-Keller-Segel model in $\mathbb{R}^2$, Comm. Pure Appl. Math., **61** 1449-1481, 2008.

[CL92] E. Carlen and M. Loss, *Competing symmetries, the logarithmic HLS inequality and Onofri's inequality on* $\mathbb{S}^n$, Geom. Func. Anal.(1992) 2: 90. https://doi.org/10.1007/BF01895706, 90-104.

[CC2006] V. Calvez and J.A. Carrillo, *Volume effects in the Keller-Segel model: energy estimates preventing blow-up*, JMPA, (2006), volume 86, pages 155-175.

[CC2008] V. Calvez and L. Corrias, The parabolic-parabolic Keller-Segel model in $\mathbb{R}^2$, Commun. Math. Sci., 6(2) 417-447, 2008.

[CCLW2012] J. A. Carrillo, L. Chen, J.-G. Liu and J. Wang, *A Note on the Subcritical Two Dimensional Keller-Segel System*, Acta Appl Math, V. 119, Issue 1, (2012), 43-55.

[CR2010] J.A. Carrillo and J. Rosado, *Uniqueness of bounded solutions to Aggregation equations by optimal transport methods*, Proc. 5th Euro. Congress of Math. Amsterdam, (2008), European Mathematical Society.

[Del1991] J.M. Delort, *Existence de nappes de tourbillon en dimension deux.* J. Am. Math. Soc. 4, 386553 (1991)

[DS2009] J. Dolbeault and C. Schmeiser, *The two-dimensional Keller-Segel model after blow-up*, Discrete and Continuous Dynamical Systems, 25 (2009), pp. 109-121.

[DLM2010] R.-J. Duan, A. Lorz and P. Markowich, *Global Solutions to the Coupled ChemotaxisFluid Equations*, Comm. Partial Differential Equations, Vol. 35, (2010), page 1635-1673.

[EM2016] G. Egaa and S. Mischler, *Uniqueness and long time asymptotic for the Keller-Segel equation: the parabolic-elliptic case*, Arch. Ration. Mech. Anal. 220 (2016), no. 3, 1159-1194.

[FLM2010] M. Di Francesco, A. Lorz and P. Markowich, *Chemotaxisfluid coupled model for swimming bacteria with nonlinear diffusion: global existence and asymptotic behavior*, Discrete Contin. Dyn. Syst. Ser. A, Vol. 28, (2010), page 1437-1453.

[Hor2003] D. Hortsmann, From 1970 until present: the Keller-Segel model in chemotaxis and its consequences, I, Jahresber. Deutsch. Math.-Verein, 105(3), 103–165, 2003.

[JL1992] W. Jäger and S. Luckhaus, *On Explosions of Solutions to a System of Partial Differential Equations Modelling Chemotaxis*, Trans. Am. Math. Soc., (1992), volume 329, pages 819-824.

[KS1970] E. F. Keller and L. A. Segel, *Initiation of slide mold aggregation viewed as an instability*, J. Theor. Biol., 26 (1970).

[KJCY2017] S. Khan, J. Johnson, E. Cartee, and Y. Yao, *Global Regularity of Chemotaxis Equations with Advection*, to appear in Involve.

[KX2015] A. Kiselev and X. Xu, Suppression of chemotactic explosion by mixing, Archive for Rational Mechanics and Analysis, 1-36, 2015.

[LL2011] J.-G. Liu and A. Lorz, *A coupled chemotaxis-fluid model: global existence*, Annales de lInstitut Henri Poincare. Analyse Non Lineaire, Vol. 28, (2011), page 643-652.

[L2010] A. Lorz, *Coupled chemotaxis fluid model*, Math. Models Methods Appl. Sci., Vol. 20, (2010), page 987-1004.

[L2012] A. Lorz, *A coupled Keller-Segel-Stokes model: global existence for small initial data and blow-up delay*, Communications in Mathematical Sciences, Vol. 10, (2012), page 555-574.

[Pat1953] C. S. Patlak, Random walk with persistence and external bias, Bull. Math. Biophys., 15 (1953), pp. 311338.

[Po2002] F. Poupaud, *Diagonal defect measures, adhesion dynamics and Euler equations*, Meth. Appl. Anal. 9 (2002), pp. 533-561.

[TS2001] Takasi Senbaa, Takashi Suzuki, *Weak Solutions to a Parabolic-Elliptic System of Chemotaxis* J. Functional. Analysis 47 (2001), 17-51.

Department of Mathematics and Center for Scientific Computation and Mathematical Modeling (CSCAMM), University of Maryland, College Park

E-mail address: simhe@cscamm.umd.edu

Department of Mathematics, Center for Scientific Computation and Mathematical Modeling (CSCAMM), and Institute for Physical Sciences & Technology (IPST), University of Maryland, College Park

E-mail address: tadmor@cscamm.umd.edu