Feedback boundary control of linear hyperbolic systems with relaxation

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***Abstract*—We consider boundary stabilization for one–dimensional systems of linear hyperbolic partial differential equations with relaxation structure. Such equations appear in many applications. By combining weighted Lyapunov functions, the structure is used to derive new stabilization results. The result is illustrated with an application to boundary stabilization of water flows in open canals.**

***Index Terms*—Stabilisation, Hyperbolic relaxation systems, Lyapunov methods, Feedback boundary control**

I. INTRODUCTION

We are interested in boundary stabilization of general hyperbolic PDEs (partial differential equations). Our particular focus is on the influence of the source term on the design of (dissipative) feedback laws. The control of hyperbolic PDEs has recently gained interest in the mathematical and engineering community due to the wide range of possible applications. Most of the development of the design of suitable boundary feedback control was driven by the St. Venant equations [1]–[6]. Other contributions cover the case of gas dynamics [7], traffic flow [8] or supply chains [9].

In this paper, we are concerned with a class of hyperbolic PDEs appearing as (intermediate) mathematical models between the Boltzmann equation and hyperbolic conservation laws. They describe various irreversible processes including chemical reactive flows, radiation hydrodynamics, inviscid gas dynamics with relaxation, nonlinear optics, viscoelasticity fluid flows, and many more [10], [11]. The fundamental properties of these physically relevant models have been successfully extracted in [10], [12], [13]. They will be exploited in the following to investigate exponential stability. The exponential stability will be proven by extending the recently proposed class of Lyapunov functions [14], [15]. We also refer to [15]–[22] for related investigations using this particular class of Lyapunov functions.

The focus of this paper is the investigation of exponential stability in the presence of physically relevant source terms. For such problems, a general result using a smallness assumption on the source terms is given in [14, Theorem 13.12] or [20]. However, this assumption is typically not fulfilled by the previously mentioned mathematical models. In the linear case a weaker condition is proposed in the recent paper [1, Condition C2, Theorem 1]. As mentioned in [1, Remark 2] it is not straightforward to check whether or not this condition is true. Here we pursue a different approach. We use a modified Lyapunov function exploiting the relaxation structure.

To the best of our knowledge, this seems the first place where explicitly the structure is used to prove exponential stability. As in [1] we consider the linear cases with linear boundary conditions. However, we do not require the source term to be marginally diagonally stable as in [1, Theorem 2]. Finally, we apply the result to the Saint Venant Exner model. This is the same example as discussed in [1, Section 4]. With the new Lyapunov function we could also improve the result presented therein.

II. MOTIVATION AND RELAXATION STRUCTURE

Motivated by [1] and [10], we consider a one–dimensional linear system

$$u_t + a\, u_x + b\, q_x = 0,\; q_t + c\, u_x + d\, q_x = -e\, q \qquad (1)$$

for $x \in [0,1]$ and $t \geq 0$. Here $u : [0,\infty) \times [0,1] \to \mathbb{R}^{n-r}$, $q : [0,\infty) \times [0,1] \to \mathbb{R}^r$ and $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{n \times n},\ e \in \mathbb{R}^{r \times r}$. Unlike that in [1], system (1) is *not* in its characteristic form, rather than in its standard form [13].

About this system, we make the following two assumptions.

(A1) There exists a symmetric positive definite matrix $A_0 \in \mathbb{R}^{n \times n}$ such that

$$A_0 \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ is symmetric and } A_0 = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}$$

with $X_1 \in \mathbb{R}^{(n-r)\times(n-r)}$ and $X_2 \in \mathbb{R}^{r \times r}$.

(A2) The matrix

$$X_2\, e + e^t\, X_2 \text{ is positive definite.}$$

Remark 1: Assumptions (A1) and (A2) are exactly the structural stability conditions proposed in [12] for general system $U_t + AU_x = QU$: There exists an invertible matrix $\bar{P} \in \mathbb{R}^{n\times n}$ and an invertible matrix $S \in \mathbb{R}^{r \times r}$ such that $\bar{P} Q \bar{P}^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix}$; there exists a symmetric positive definite matrix $\bar{A}_0$ such that $\bar{A}_0 A$ is symmetric; and

$$\bar{A}_0 Q + Q^t \bar{A}_0 \leq -\bar{P}^t \begin{pmatrix} 0 & 0 \\ 0 & Id_{r\times r} \end{pmatrix} \bar{P}.$$

As shown in [12] and [10], these conditions are fulfilled by many classical models from mathematical physics. They

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ensure existence of the zero-relaxation limit for initial-value problems of general multi-dimensional nonlinear systems.

Assumption (A1) implies that the system (1) is hyperbolic. Thus, we can diagonalize the coefficient matrix A with a transformation matrix $T \in \mathbb{R}^{n\times n}$ such that

$$T^{-1}AT = \Lambda, \Lambda := \begin{pmatrix} \Lambda_+ & 0 \\ 0 & \Lambda_- \end{pmatrix}, \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = T^{-1}U, \quad (2)$$

where $\Lambda_\pm$ are diagonal and contain the positive and negative eigenvalues of A, respectively. As in [1], we assume that the system (1) has no vanishing eigenvalues.

Further, we let $\xi_+ \in \mathbb{R}^m$ and $\xi_- \in \mathbb{R}^{n-m}$. Boundary conditions are specified as

$$\xi_+(t,0) = K_{00}\xi_+(t,1) \text{ and } \xi_-(t,1) = K_{11}\xi_-(t,0). \quad (3)$$

In addition, equation (1) is accompanied by suitable initial data

$$u(x,0) = u_0(x) \text{ and } q(x,0) = q_0(x). \quad (4)$$

Remark 2: More general conditions of the type

$$\begin{pmatrix} \xi_+(t,0) \\ \xi_-(t,1) \end{pmatrix} = \begin{pmatrix} K_{00} & K_{01} \\ K_{10} & K_{11} \end{pmatrix} \begin{pmatrix} \xi_+(t,1) \\ \xi_-(t,0) \end{pmatrix}$$

have been considered in [1], [15]. However, our focus is the treatment of the relaxation term and therefore only consider the simplified setting of equation (3).

Assumptions (A1) and (A2) guarantee exponential decay in q. The goal is to prescribe a feedback boundary control yielding also exponential decay in the conservative variable u. It is known that for $(u_0, q_0) \in L^2((0,1);\mathbb{R}^n)$ the problem (1) together with (3) and (4) has a unique weak solution $(u,q)(t,\cdot) \in L^2((0,1);\mathbb{R}^n)$ [23, Sec 2.1].

Definition 1: The system (1) together with (3) and (4) is exponentially stable, if there exists $\nu > 0$ and $C > 0$, such that for every $(u_0, q_0) \in L^2((0,1);\mathbb{R}^n)$, the weak solution to the Cauchy problem (1) together with (3) and (4) satisfies

$$\|(u,q)(t,\cdot)\|_{L^2((0,1);\mathbb{R}^n)} \le C\exp(-\nu t)\|(u_0,q_0)\|_{L^2((0,1);\mathbb{R}^n)}.$$

In [1, Theorem 2] the authors prove exponential stability under the assumption that the source term $M := T^{-1}\begin{pmatrix} 0 & 0 \\ 0 & -e \end{pmatrix} T$ is diagonally marginally stable, i.e., there exists a diagonal positive definite matrix P, such that $M^TP + PM$ is negative semi–definite. Unfortunately, it seems *a priori* not clear if such a matrix P exists. Further, its construction might be difficult. Here we exploit the physically relevant assumptions (A1) and (A2) to obtain exponential stability without any further requirements.

We will use the following notation: $\lambda_{min}(A)$ and $\lambda_{max}(A)$ denote the smallest and largest eigenvalue of a matrix A, respectively. To simplify the notation we set $q(t) := q(\cdot,t) \in L^2((0,1);\mathbb{R}^n)$ and we denote by $\|q(t)\|_A^2 = \int_0^1 q^T(t,x)Aq(t,x)dx$ for a positive definite matrix A. We drop the subindex if the usual L^2-scalar product is used. Clearly, $\lambda_{min}(A)\|q(t)\|^2 \le \|q(t)\|_A^2 \le \lambda_{max}(A)\|q(t)\|^2$.

III. A MODIFIED LYAPUNOV FUNCTION FOR EXPONENTIAL DECAY

We state the main result on exponential stability using a Lyapunov function given by equation (5) below.

Theorem 3.1: Suppose the system (1) fulfills the assumptions (A1) and (A2). Then there exist K_{00} and K_{11} such that the system (1) together with (3) and (4) is exponentially stable.

The key for the proof of Theorem 3.1 is the choice of an appropriate Lyapunov function. Here we choose

$$\begin{aligned} \mathcal{L}(t) &= \int_0^1 U^t\left(\alpha A_0 + \mu(x)\right)U dx \quad (5) \\ &= \alpha\|(u,q)(t)\|_{A_0}^2 + \|(u,q)(t)\|_\mu^2 \end{aligned}$$

for some $\alpha > 0$ and a family of matrices $\mu(x) \in \mathbb{R}^{n\times n}$ given by

$$\mu(x) := T^{-t}\exp(-\Lambda x)T^{-1} \quad (6)$$

for $x \in [0,1]$ and T and Λ given by equation (2). We denote by $\exp(-\Lambda x)$ the diagonal matrix with entries $\exp(-\Lambda_{ii}x)$ for $i = 1,\dots,n$. Note that $\mu(x)$ is symmetric, positive definite with uniformly bounded eigenvalues. Further, μ is componentwise differentiable. If we denote by μ_x its componentwise derivative we obtain $\mu_x(x)A = T^{-t}DT^{-1}$ where D is a diagonal matrix with entries $D_{ii} = -\Lambda_{ii}^2\exp(-\Lambda_{ii}x) < 0$. Therefore, $\mu_x(x)A$ is negative definite with uniformly bounded eigenvalues.

Remark 3: The result of Theorem 3.1 remains true in the following situation: Assume there exists a family $\mu(\cdot) \in \mathbb{R}^{n\times n}$ parametrized by $x \in [0,1]$ of positive definite symmetric matrix with uniformly bounded eigenvalues. Further, assume that $\mu(x)$ is componentwise differentiable and the matrix $\mu_x(x)A$ is negative definite for all $x \in [0,1]$ with uniformly bounded eigenvalues. Further assume $\mu(x)A$ is symmetric. In the general case it is however not clear how the matrix μ can be constructed.

As in [1, Theorem 2] the results can be extended to more general boundary conditions stated in Remark 2. We refer to [1, Section 3] for the precise requirement on the boundary feedback matrix in this case.

Clearly, we have $\mathcal{L}(t) \ge 0$ for all t and $\mathcal{L}(t) = 0$ implies $U(t,\cdot) = 0$. Theorem 3.1 is establish using the following preliminary results. The first lemma exposes a relation between the transformation T in (2) and the symmetrizer A_0 in Assumption (A1).

Lemma 3.2: Assume (A1) holds true and system (1) has no vanishing eigenvalues. Let T be given by equation (2). Then there exist symmetric positive definite matrices $\tilde{X}_1 \in \mathbb{R}^{m\times m}$, $\tilde{X}_2 \in \mathbb{R}^{(n-m)\times(n-m)}$ such that

$$T^tA_0T = \begin{pmatrix} \tilde{X}_1 & 0 \\ 0 & \tilde{X}_2 \end{pmatrix}.$$

This lemma can be proved by observing that $T^{-1}AT = \Lambda = T^tA^tT^{-t}$ where Λ is given by equation (2). Since $A_0A = A^tA_0$, we have $T^tA_0T\Lambda = \Lambda T^tA_0T$. Namely, the diagonal matrix Λ commutes with the symmetric matrix T^tA_0T. Therefore, the latter is of block-diagonal with $\tilde{X}_1$ and $\tilde{X}_2$ of proper dimensions. Since A_0 is positive definite, so are $\tilde{X}_1$ and $\tilde{X}_2$.

The next lemma shows that we can obtain decay in $\|q\|$ in the Lyapunov function. This result can be used to estimate mixed terms including u and q.

Lemma 3.3 (Energy estimate): For equation (1) satisfying the assumptions (A1) and (A2), there exists $C_q > 0$ such that

$$\partial_t \|(u,q)(t)\|_{A_0}^2 = \partial_t \|q(t)\|_{X_2}^2 + \partial_t \|u\|_{X_1}^2 \le -C_q \|q\|_{X_2}^2 - BC_1,$$

where

$$BC_1 = U^t(t,1)A_0AU(t,1) - U^t(t,0)A_0AU(t,0).$$

Indeed, we multiply the system (1) with U^tA_0 to obtain

$$(U^tA_0U)_t + (U^tA_0AU)_x = -q^t\left(X_2e + e^tX_2\right)q.$$

By integrating this over $x \in [0,1]$, it immediately yields the assertion for

$$C_q = \frac{\lambda_{min}(X_2e + e^tX_2)}{\lambda_{max}(X_2)} > 0.$$

The third lemma introduces the boundary conditions yielding also decay in u.

Lemma 3.4 (Exponential decay): Suppose system (1) is hyperbolic. Then there exists $C_u > 0$ and $C_{qu} \in \mathbb{R}$ such that

$$\partial_t \|(u,q)(t)\|_\mu^2 + BC_2 \le -C_u \|u\|_{X_1}^2 + C_{qu} \|q\|_{X_2}^2,$$

where

$$BC_2 = U^t(t,1)\mu(1)AU(t,1) - U^t(t,0)\mu(0)AU(t,0).$$

Actually, we multiply the system (1) with $U^t\mu(x)$ to obtain

$$\begin{aligned}&(U^t\mu(x)U)_t + (U^t\mu(x)AU)_x = U\mu_x(x)AU\\ &-U^t\begin{pmatrix}0 & \mu_{1,2}(x)e\\ e^t\mu_{2,1}(x) & e^t\mu_{2,2}(x) + \mu_{2,2}(x)e\end{pmatrix}U,\end{aligned}$$

where we denote by $\mu_{i,j}(x)$ the corresponding submatrix of $\mu(x)$ at position (i,j). Note that $\mu_{2,2}(x) \in \mathbb{R}^{r\times r}$ and $\mu_{2,1}^t(x) = \mu_{1,2}(x) \in \mathbb{R}^{(n-r)\times r}$. Integrating the last equality over $x \in [0,1]$ yields

$$\partial_t \|(u,q)(t)\|_\mu^2 + BC_2 \le -\tilde{\lambda}\|(u,q)(t)\|^2 + \int_0^1 S(t,x)dx,$$

where $\tilde{\lambda} := -\max_{x\in[0,1]}\{\lambda_{max}(\mu_x(x)A)\} > 0$. Furthermore, S is estimated as follows

$$\begin{aligned}S(t,x) &= -q^te^t\mu_{2,1}(x)u - u^t\mu_{1,2}(x)eq\\ -q^t&\left(e^t\mu_{2,2}(x) + \mu_{2,2}(x)e\right)q \le C_{12}|q||u| + C_{22}|q|^2,\end{aligned}$$

where $|u|$ denotes the 2-vector norm

$$C_{12} = \max_{x\in[0,1]}\left(|e^t\mu_{2,1}(x)|_\infty + |\mu_{1,2}(x)e|_\infty\right)$$

with $|A|_\infty$ being the infinity norm of the matrix A, and the non-negative constant C_{22} is given by

$$C_{22} = \max_{x\in[0,1]} \lambda_{\max}(-e^t\mu_{2,2}(x) - \mu_{2,2}(x)e).$$

Integration of S yields then for any $\delta > 0$

$$\begin{aligned}\int_0^1 S(t,x)dx &\le C_{12}\int_0^1 |u||q|dx + C_{22}\|q\|^2\\ &\le \frac{C_{12}}{2}\left(\delta\|u\|^2 + \frac{1}{\delta}\|q\|^2\right) + C_{22}\|q\|^2\\ &\le \frac{C_{12}\delta}{2}\|u\|^2 + \left(\frac{C_{12}}{2\delta} + C_{22}\right)\|q\|^2.\end{aligned}$$

Then we arrive at

$$\begin{aligned}&\partial_t \|(u,q)(t)\|_\mu^2 + BC_2 \le\\ &\frac{C_{12}\delta - 2\tilde{\lambda}}{2}\|u\|^2 + \left(\frac{C_{12}}{2\delta} + C_{22} - \tilde{\lambda}\right)\|q\|^2.\end{aligned}$$

By taking $\delta = \frac{\tilde{\lambda}}{C_{12}} > 0$, the assertion follows with

$$C_u = \frac{\tilde{\lambda}}{2\lambda_{max}(X_1)},\ C_{qu} = \frac{\frac{C_{12}^2}{2\tilde{\lambda}} + \max\{0, C_{22} - \tilde{\lambda}\}}{\lambda_{min}(X_2)}. \quad (7)$$

With these preparations, we now turn to prove the main result.

Proof of the main result. According to Lemma 3.3 and 3.4, we have

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) &= \partial_t \|(u,q)(t)\|_\mu^2 + \alpha\partial_t \|(u,q)(t)\|_{A_0}^2\\ &\le -C_u\|u\|_{X_1}^2 + (C_{qu} - \alpha C_q)\|q\|_{X_2}^2,\end{aligned}$$

provided that

$$\begin{aligned}&BC_2 + \alpha BC_1 =\\ &U^t(t,1)\mu(1)AU(t,1) - U^t(t,0)\mu(0)AU(t,0) +\\ &\alpha\left(U^t(t,1)A_0AU(t,1) - U^t(t,0)A_0AU(t,0)\right) \ge 0. \quad (8)\end{aligned}$$

Choose α positive and $\alpha > \frac{C_{qu}}{C_q}$. It follows that

$$\frac{d}{dt}\mathcal{L}(t) \le -\min\{C_u, \alpha C_q - C_{qu}\}\frac{1}{\alpha + C}\mathcal{L}(t).$$

with constant $C \ge \frac{\|(u,q)\|_\mu}{\|(u,q)\|_{A_0}}$. Thus we obtain exponential decay in the sense of Definition 1 of $\mathcal{L}(t)$ at rate

$$\nu = \min\{C_u, \alpha C_q - C_{qu}\}\frac{1}{\alpha + C}. \quad (9)$$

It remains to discuss boundary conditions (3) such that inequality (8) holds true. Choose T, Λ as in equation (2) and set $\xi(t,x) = (\xi_+, \xi_-)(t,x)$. Due to Lemma 3.2, we have

$$\begin{aligned}&U^t(t,1)\mu(1)AU(t,1) + \alpha U^t(t,1)A_0AU(t,1)\\ &\quad = \xi^t(t,1)\left(T^t\mu(1)T + \alpha T^tA_0T\right)\Lambda\xi(t,1)\\ &= \xi^t(t,1)\left(T^t\mu(1)T\Lambda + \alpha\begin{pmatrix}\tilde{X}_1\Lambda_+ & 0\\ 0 & \tilde{X}_2\Lambda_-\end{pmatrix}\right)\xi(t,1).\end{aligned}$$

Recall the definition of μ in equation (6). We obtain

$$T^t\mu(1)T = \begin{pmatrix}\exp(-\Lambda_+) & 0\\ 0 & \exp(-\Lambda_-)\end{pmatrix},\ T^t\mu(0)T = Id.$$

In the case of general μ fulfilling the assumptions stated in Remark 3 we obtain at least that $T^t\mu(\cdot)T$ is block-diagonal with positive definite symmetric entries. In summary, we obtain

$$\begin{aligned} &U^t(t,1)\mu(1)AU(t,1) - U^t(t,0)\mu(0)AU(t,0) + \\ &\alpha\left(U^t(t,1)A_0AU(t,1) - U^t(t,0)A_0AU(t,0)\right) = \\ &\xi(t,1)^t \begin{pmatrix} (e^{-\Lambda_+} + \alpha\tilde{X}_1)\Lambda_+ & 0 \\ 0 & (e^{-\Lambda_-} + \alpha\tilde{X}_2)\Lambda_- \end{pmatrix} \xi(t,1) \\ &-\xi(t,0)^t \begin{pmatrix} (Id + \alpha\tilde{X}_1)\Lambda_+ & 0 \\ 0 & (Id + \alpha\tilde{X}_2)\Lambda_- \end{pmatrix} \xi(t,0) \\ &= \xi_+(t,1)^t\mathbf{K}_{00}\xi_+(t,1) + \xi_-(t,0)^t\mathbf{K}_{11}\xi_-(t,0). \end{aligned}$$

In the last step we have applied the boundary condition (3). This yields the following expression for $\mathbf{K}_{ii}$:

$$\begin{aligned} \mathbf{K}_{00} &= (e^{-\Lambda_+} + \alpha\tilde{X}_1)\Lambda_+ - K_{00}^t\left(Id_{m\times m} + \alpha\tilde{X}_1\right)\Lambda_+K_{00}, \\ \mathbf{K}_{11} &= -\left(Id_{(n-m)\times(n-m)} + \alpha\tilde{X}_2\right)\Lambda_- + \\ &\quad K_{11}^t(e^{-\Lambda_-} + \alpha\tilde{X}_2)\Lambda_-K_{11}. \end{aligned}$$

It suffices to have that $\mathbf{K}_{00}$ and $\mathbf{K}_{11}$ are both symmetric non-negative definite. To this end, we choose $K_{00} = \kappa_{00}Id_{m\times m}$ and $K_{11} = \kappa_{11}Id_{(n-m)\times(n-m)}$. Then we have

$$\begin{aligned} \mathbf{K}_{00} &= \exp(-\Lambda_+)\Lambda_+ - \kappa_{00}^2\Lambda_+ + \alpha(1-\kappa_{00}^2)\tilde{X}_1\Lambda_+, \\ \mathbf{K}_{11} &= \exp(-\Lambda_-)\Lambda_-\kappa_{11}^2 - \Lambda_- + \alpha(\kappa_{11}^2 - 1)\tilde{X}_2\Lambda_-. \end{aligned}$$

Note that $\tilde{X}_1\Lambda_+$ and $\tilde{X}_2\Lambda_-$ are symmetric, for $\begin{pmatrix} \tilde{X}_1\Lambda_+ & 0 \\ 0 & \tilde{X}_2\Lambda_- \end{pmatrix} = T^tA_0AT$. Thus, $\mathbf{K}_{00}$ and $\mathbf{K}_{11}$ are both symmetric non-negative definite if κ_{00}^2 and κ_{11}^2 are sufficiently small, for example,

$$\kappa_{00}^2 \le \exp(-\max_i\{\Lambda_{+,ii}\}) \quad \text{and} \quad \kappa_{11}^2 \le \exp(\min_i\{\Lambda_{-,ii}\}).$$

In this way, we have shown that $\mathcal{L}(t)$ is a Lyapunov function and there exist feedback matrices K_{00}, K_{11} such that (u,q) enjoys exponential decay at rate ν given by equation (9). This finishes the proof.

IV. Example of the Saint–Venant–Exner model

The Saint–Venant–Exner model describes hydraulic systems in open canals with moving bathymetry. A control problem is derived by linearizing the shallow water system at a subcritical flow, see [1]. The states are described with the water height $H(t,x)$, the velocity $V(t,x)$ and the bathymetry $B(t,x)$. We denote by $x \in [0,1]$ the position in the canal and by $t \ge 0$ time. Denote by g the gravitational constant, S_b is the constant bottom slope of the open canal, $C_f > 0$ is friction and $a > 0$ is a parameter including porosity and viscosity effects. A steady state $(H^*, V^*, B^*) \neq (0,0,0)$ of the Saint–Venant–Exner model fulfills

$$gS_bH^* = C_fV^*. \tag{10}$$

Denote by $h(t,x) = H(t,x) - H^*$, $v(t,x) = V(t,x) - V^*$ and $b(t,x) = B(t,x) - B^*$ the deviation of the steady state. By control mechanism the deviation (h,b,u) should be driven to zero for $t \to \infty$. The deviation fulfills the linear system

$$\partial_t \begin{pmatrix} h \\ b \\ u \end{pmatrix} + \mathcal{A}\partial_x \begin{pmatrix} h \\ b \\ u \end{pmatrix} = \mathcal{Q} \begin{pmatrix} h \\ b \\ u \end{pmatrix} \tag{11}$$

with

$$\mathcal{A} = \begin{pmatrix} V^* & 0 & H^* \\ 0 & 0 & a(V^*)^2 \\ g & g & V^* \end{pmatrix},$$

$$\mathcal{Q} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{(gS_b)^2}{C_f} & 0 & -2gS_b \end{pmatrix}.$$

Note that system (11) is not of the form (1).

In order to apply Theorem 3.1, we firstly show that the linearized system (11) satisfies the structural stability condition in [12] (also see Remark 1). Namely, there exist an invertible matrix $\bar{P}$ and a symmetric positive-definite matrix $\bar{A}_0$ such that

$$\bar{P}\mathcal{Q}\bar{P}^{-1} = \begin{pmatrix} 0_{2\times2} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \end{pmatrix} & s \end{pmatrix}$$

with s a non-zero real number, $\bar{A}_0\mathcal{A} = \mathcal{A}^t\bar{A}_0$, and

$$\bar{A}_0Q + Q^t\bar{A}_0 \le -\bar{P}^t \begin{pmatrix} 0_{2\times2} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \end{pmatrix} & \delta \end{pmatrix} \bar{P}.$$

Here $\delta > 0$ is a positive constant chosen below and depending on the constants C_f, S_b and g.

Provided that such $\bar{P}$ and $\bar{A}_0$ have been found, it is easy to see that $U = \bar{P}(h,b,u)^t \in \mathbb{R}^3$ fulfills an equation of type (1) with $n = 3$ and $r = 1$. Further, the assumptions (A1) and (A2) hold true with $A_0 = P^{-t}\bar{A}_0P^{-1}$. Hence, the linearized Saint–Venant–Exner model is exponentially stable in the sense of Definition 1.

For $\bar{P}$, we simply take

$$\bar{P} = \begin{pmatrix} Id_{2\times2} & \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\ \begin{matrix} \xi_2^{-1}\xi_1 & 0 \end{matrix} & 1 \end{pmatrix}, \; \xi_1 = \frac{(gS_b)^2}{C_f}, \; \xi_2 = -2gS_b.$$

Indeed, for this choice of $\bar{P}$ we obtain $s = \xi_2 < 0$ since

$$\bar{P}\mathcal{Q} = \mathcal{Q} = \begin{pmatrix} 0_{2\times2} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \end{pmatrix} & s \end{pmatrix} \bar{P}.$$

Thanks to Lemma 2.2 in [12], the symmetrizer $\bar{A}_0$ above has to be of the form

$$\bar{A}_0 = \bar{P}^t \begin{pmatrix} \alpha & \beta & 0 \\ \beta & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \bar{P} \tag{12}$$

with α, γ and β specified below. Since $\begin{pmatrix} \alpha & \beta & 0 \\ \beta & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathcal{Q} = \mathcal{Q}$, we have

$$\bar{A}_0\mathcal{Q} + \mathcal{Q}^t\bar{A}_0 = \bar{P}^t\mathcal{Q} + \mathcal{Q}^t\bar{P} = 2\begin{pmatrix} \xi_2^{-1}\xi_1^2 & 0 & \xi_1 \\ 0 & 0 & 0 \\ \xi_1 & 0 & \xi_2 \end{pmatrix},$$

$$\bar{P}^t \begin{pmatrix} 0_{2\times2} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \end{pmatrix} & \delta \end{pmatrix} \bar{P} = \delta \begin{pmatrix} \xi_2^{-2}\xi_1^2 & 0 & \xi_2^{-1}\xi_1 \\ 0 & 0 & 0 \\ \xi_2^{-1}\xi_1 & 0 & 1 \end{pmatrix}.$$

Furthermore,

$$\bar{A}_0\mathcal{Q} + \mathcal{Q}^t\bar{A}_0 + \delta\bar{P}^t \begin{pmatrix} 0_{2\times 2} & \begin{pmatrix}0\\0\end{pmatrix} \\ \begin{pmatrix}0 & 0\end{pmatrix} & 1 \end{pmatrix} \bar{P} = (\delta + 2\xi_2)\begin{pmatrix} \xi_2^{-2}\xi_1^2 & 0 & \xi_2^{-1}\xi_1 \\ 0 & 0 & 0 \\ \xi_2^{-1}\xi_1 & 0 & 1 \end{pmatrix}.$$

The resulting matrix is negative semi–definite provided that $\delta = -2\xi_2$.

Finally, we turn to choose α, β and γ. From equation (12) it follows that

$$\bar{A}_0 = \begin{pmatrix} \alpha + \frac{r^2}{4C_f^2} & \beta & -\frac{r}{2C_f} \\ \beta & \gamma & 0 \\ -\frac{r}{2C_f} & 0 & 1 \end{pmatrix}.$$

Thanks to $\bar{A}_0\mathcal{A} = \mathcal{A}^t\bar{A}_0$, we obtain the following relations for α, β and γ :

$$\beta = -\frac{g}{2H^*},$$
$$a(V^*)^2\beta - g + (H^*)\left(\alpha + \frac{(V^*)^2}{4(H^*)^2}\right) = 0,$$
$$a(V^*)^2\gamma - g + (H^*)\beta = 0.$$

For $H^*, V^* \neq 0$, these uniquely determine α, β and γ as

$$\begin{aligned} 4(H^*)^2\alpha &= 2ga(V^*)^2 + 4gH^* - (V^*)^2, \\ 2H^*\beta &= -g, \\ 2a(V^*)^2\gamma &= 3g. \end{aligned}$$

It remains to verify that the symmetric matrix $\bar{A}_0$ defined in equation (12) is positive definite. Clearly, $\gamma > 0$. Then we use the equilibrium relation (10) to compute

$$\det\bar{A}_0 = \alpha\gamma - \beta^2,$$
$$8(H^*)^2a(V^*)^2\det\bar{A}_0 = g\left(12gH^* - 3(V^*)^2 + 4a(V^*)^2g\right) = g\left(\frac{12C_f}{S_b}V^* - 3(V^*)^2 + 4a(V^*)^2g\right).$$

This leads to analyse if

$$\frac{12C_f}{S_b}V^* + (4ag - 3)(V^*)^2 > 0. \quad (13)$$

If

$$a \geq \frac{3}{4g} \quad \text{and} \quad V^* > 0, \quad (14)$$

then the condition (13) holds true. Otherwise, if

$$0 < V^* < \frac{12C_f}{S_b(3 - 4ag)},$$

then the condition also holds true.

Physically, those conditions mean that either the porosity and viscosity is sufficiently large or the velocity of the equilibrium state is sufficiently small. In contrast to [1] we do *not* require V^* to be bounded from below by the (positive) second eigenvalue of $\mathcal{A}$. Further in [24] the porosity and viscosity effects of the bed are modeled by equation [24, Eq. 3] and by equation [25, Equation 3.6] as

$$a = 3\,\frac{1}{1-\sigma},\ 0 \leq \sigma < 1.$$

In this case, clearly $a \geq \frac{3}{4g}$ and equation (14) is fulfilled. We summarize our findings in the following

Corollary: Consider the linearized Saint–Venant–Exner model given by equation (11). Assume that the steady state fulfills $gS_bH^* = C_fV^*$ where g denotes the gravitational constant, $C_f > 0$ friction, $S_b > 0$ the constant bottom slope and $a = \frac{3}{1-\sigma}$ for some $0 \leq \sigma < 1$.

Then there exists a feedback boundary control such that the linearized Saint–Venant–Exner is exponentially stable.

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