An Asymptotic-Preserving Stochastic Galerkin Method for the Semiconductor Boltzmann Equation with Random Inputs and Diffusive Scalings*

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Abstract

In this paper, we develop a generalized polynomial chaos approach based stochastic Galerkin (gPC-SG) method for the linear semi-conductor Boltzmann equation with random inputs and diffusive scalings. The random inputs are due to uncertainties in the collision kernel or initial data. We study the regularity of the solution in the random space, and prove the spectral accuracy of the gPC-SG method. We then use the asymptotic-preserving framework for the deterministic counterpart developed in [8] to come up with the stochastic asymptotic-preserving gPC-SG method for the problem under study which is efficient in the diffusive regime. Numerical experiments are conducted to validate the accuracy and asymptotic properties of the method.

Key words. semi-conductor Boltzmann equation, uncertainty quantification, diffusion limit, asymptotic preserving, random inputs, generalized polynomial chaos, spectral accuracy

1 Introduction

We consider the linear semiconductor Boltzmann equation with random inputs. Here the random inputs arise in the collision kernel or initial data due to modeling or measurement errors, which are typical for kinetic equations that are often derived via mean-field limits from particle systems [2, 15]. In recent years there have been significant interests in uncertainty quantification for physical models that contain uncertain coefficients, but few works have been concentrated on kinetic equations which are of practical importance in mesoscopic modeling of physical, biological to social sciences. Here we mention the stochastic Galerkin method employed for neutron transport equation with random scattering coefficients [1] and more recently, for the nonlinear Boltzmann equation [5].

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Another challenge in numerical approximations of kinetic and transport equations arise from varying magnitude of the Knudsen number, which is the dimensionless mean free path measuring the ratio between the particle mean free path and a typical length scale. When the Knudsen number is small the equation becomes numerically stiff thus demand prohibitive, Knudsen number dependent mesh sizes and time steps. To overcome this difficulty, asymptotic-preserving (AP) schemes, which mimic the asymptotic transition from the kinetic equations to the hydrodynamic or diffusion limit, combined with efficient time integrators, have proved to be very efficient to handle small or multiple scales in kinetic or hyperbolic problems, see [6, 7]. For AP schemes for linear (deterministic) kinetic equations similar to the equation under study, see [12, 9, 8, 11]. For linear transport equations with diffusive scales and random inputs, stochastic asymptotic-preserving (s-AP) schemes were recently introduced in [10]. A s-AP scheme allows the use of mesh sizes, time steps and the number of terms in the orthogonal polynomial expansions *independent of* the Knudsen number, yet can still capture the solution of the limiting, macroscopic equations.

In this paper, we aim to develop a s-AP scheme for the linear semiconductor Boltzmann equation with random inputs and diffusive scalings. Our method is based on the generalized polynomial chaos approach in the stochastic Galerkin (hereafter referred to as the gPC-SG) framework [4, 13, 18]. The advantage of the gPC-SG method over the classical Monte-Carlo method is that the former enjoys a spectral accuracy in the random space–provided sufficient regularity of the solution– while the latter converges with only half-th order accuracy.

It was realized in [10] that for random transport equations with diffusive scalings, upon adopting the gPC-SG formulation, the underlying system is a vector analogy of its deterministic counterpart, thus the deterministic AP machineries can be easily employed to give rise to s-AP schemes. It is also the case here. Once we use the standard gPC-SG approximation, the resulting system, which is deterministic, has the same form as the original Boltzmann equation except that it is vectorized. Indeed we will use the deterministic AP scheme developed in [8] for the gPC-SG system which induces a s-AP scheme, in the sense that its solution will approach, in the zero Knudsen number limit, to that of the corresponding drift diffusion equation with random inputs. Here the physical equation is different from those studied in [8], arising in different areas. In addition to the development of the gPC-SG scheme, we also study the regularity of the solution, proving that the regularity of the initial data in the random space is preserved at later time. Further, we also establish the spectral convergence of the method in the random space.

The paper is organized as follows. In section 2, we introduce the semiconductor Boltzmann equation with random inputs, and show its drift-diffusion limit. The regularity of the distribution function in the random space will also be studied. Section 3 describes the gPC-SG method for the Boltzmann equation. Section 4 focuses on the spectral convergence analysis in the random space, which also motivates the need for an AP method when the Knudsen number is small. In section 5, we adopt a fully-discretized AP scheme under the diffusive scalings. A formal proof of the stochastic AP property is also given. In Section 6, extensive numerical examples are presented to show the validity, spectral convergence and asymptotic property of the proposed scheme.

2 The Semiconductor Boltzmann equation with random inputs

2.1 The model

We are interested in the linear Boltzmann equation for semiconductor devices [15] under the diffusive scaling with random inputs:

$$\begin{cases} \epsilon\partial_t f + \mathbf{v}\cdot\nabla_{\mathbf{x}} f + \dfrac{q}{m}\nabla_{\mathbf{x}}\phi(t,\mathbf{x},\mathbf{z})\cdot\nabla_{\mathbf{v}} f = \dfrac{1}{\epsilon}\mathcal{Q}(f)(t,\mathbf{x},\mathbf{v},\mathbf{z}) + \epsilon G(t,\mathbf{x},\mathbf{v},\mathbf{z}), \\ \qquad\qquad\qquad\qquad\qquad\qquad t>0, \mathbf{x}\in\Omega\subseteq\mathbb{R}^N, \mathbf{v}\in\mathbb{R}^d, \mathbf{z}\in I_{\mathbf{z}}, \\ f(0,\mathbf{x},\mathbf{v},\mathbf{z}) = f_I(\mathbf{x},\mathbf{v},\mathbf{z}), \\ f(t,\mathbf{x},\mathbf{v},\mathbf{z}) = g(t,\mathbf{x},\mathbf{v},\mathbf{z}), \quad \mathbf{x}\in\partial\Omega, \quad \mathbf{v}\cdot\mathbf{n}\le 0. \end{cases} \tag{2.1}$$

Here $f(t,\mathbf{x},\mathbf{v},\mathbf{z})$ is the probability density distribution for particles at $\mathbf{x}\in\Omega$, with velocity $\mathbf{v}\in\mathbb{R}^d$. $\mathbf{n}$ is the unit outer normal vector to the boundary $\partial\Omega$ of the spatial domain, ϵ is the Knudsen number, $\phi(t,\mathbf{x},\mathbf{z})$ is the electric potential and $\mathbf{E}(t,\mathbf{x},\mathbf{z}) = -\nabla_{\mathbf{x}}\phi(t,\mathbf{x},\mathbf{z})$ is the electric field. Here we consider the electric field given a priori for analytical study, though it is usually coupled with a Poisson equation [15] (the scheme can be easily extended to include the Poisson equation, and will be tested in section 6). The constants q and m are respectively the elementary charge and the effective mass of the electron. In this paper, we set $q = m = 1$.

The anisotropic collision operator $\mathcal{Q}$ describes a linear approximation of the electron-phonon interaction. It is given by

$$\mathcal{Q}(f)(\mathbf{v},\mathbf{z}) = \int_{\mathbb{R}^d} \sigma(\mathbf{v},\mathbf{w},\mathbf{z})\left\{M(\mathbf{v})f(\mathbf{w},\mathbf{z}) - M(\mathbf{w})f(\mathbf{v},\mathbf{z})\right\} d\mathbf{w}, \tag{2.2}$$

where M is the normalized Maxwellian at temperature θ,

$$M(\mathbf{v}) = \frac{1}{(2\pi\theta)^{d/2}}\exp^{-\frac{|\mathbf{v}|^2}{2\theta}}.$$

We assume the anisotropic scattering kernel σ to be symmetric and bounded,

$$\exists\ \sigma_0,\sigma_1>0, \quad \sigma_0\le\sigma(\mathbf{v},\mathbf{w},\mathbf{z}) = \sigma(\mathbf{w},\mathbf{v},\mathbf{z})\le\sigma_1, \tag{2.3}$$

and the collision frequency

$$\lambda(\mathbf{v},\mathbf{z}) = \int_{\mathbb{R}^d}\sigma(\mathbf{v},\mathbf{w},\mathbf{z})M(\mathbf{w})d\mathbf{w}\le\lambda_0. \tag{2.4}$$

The uncertainties may come from the collision kernel, the electric potential, initial data or boundary data. The random variable $\mathbf{z}$ is an n-dimensional random vector with support $I_{\mathbf{z}}$ characterizing the random inputs of the system, so essentially all functions in (2.1) depend on $\mathbf{z}$.

We now introduce some spaces, inner product and norms that will be used throughout this paper.

$$H = L^2(I_{\mathbf{z}};\pi(\mathbf{z})d\mathbf{z}), \quad <f,g>_H = \int_{I_{\mathbf{z}}} fg\pi(\mathbf{z})d\mathbf{z}, \quad \|f\|_H = \left(\int_{I_{\mathbf{z}}} f^2\pi(\mathbf{z})d\mathbf{z}\right)^{1/2} \tag{2.5}$$

$$\|f(t,\cdot,\cdot,\cdot)\|^2_{\Gamma(t)} := \int_{\mathbb{R}^d}\int_\Omega ||f(t,\mathbf{x},\mathbf{v},\mathbf{z})||^2_H\ e^{-2\phi(\mathbf{x},t)}/M(\mathbf{v})d\mathbf{x}d\mathbf{v}, \quad t\ge 0. \tag{2.6}$$

An important property of the collision operator $\mathcal{Q}$ is the symmetry property [16]

$$\int_{\mathbb{R}^d} \mathcal{Q}(f)(\mathbf{v})g(\mathbf{v})/M(\mathbf{v})d\mathbf{v} = -\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \sigma(\mathbf{v},\mathbf{w},\mathbf{z})M(\mathbf{v})M(\mathbf{w}) \cdot \left(\frac{f(\mathbf{v})}{M(\mathbf{v})} - \frac{f(\mathbf{w})}{M(\mathbf{w})}\right)\left(\frac{g(\mathbf{v})}{M(\mathbf{v})} - \frac{g(\mathbf{w})}{M(\mathbf{w})}\right) d\mathbf{w}d\mathbf{v},$$

from which one can deduce

$$\begin{aligned}&\int_{\mathbb{R}^d} \mathcal{Q}(f)(\mathbf{v})f(\mathbf{v})/M(\mathbf{v})d\mathbf{v}\\ &= -\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \sigma(\mathbf{v},\mathbf{w},\mathbf{z})M(\mathbf{v})M(\mathbf{w})\left(\frac{f(\mathbf{v})}{M(\mathbf{v})} - \frac{f(\mathbf{w})}{M(\mathbf{w})}\right)^2 d\mathbf{w}d\mathbf{v} \le 0,\end{aligned} \tag{2.7}$$

where the positivity $\sigma(\mathbf{v},\mathbf{w},\mathbf{z}) \ge \sigma_0(\mathbf{z}) > 0$ is used.

For each $\mathbf{z}$, (2.1) is a deterministic equation. As $\epsilon \to 0$, $\mathcal{Q}(f) = 0$, then $f(\mathbf{x},\mathbf{v},t) = \rho(\mathbf{x},t)M(\mathbf{v})$, where ρ satisfies the drift-diffusion equation [16, 15]

$$\partial_t \rho = \nabla_{\mathbf{x}}(D(\nabla_{\mathbf{x}}\rho + 2\rho\mathbf{E}(\mathbf{x}))), \tag{2.8}$$

where the diffusion matrix D is defined by

$$D = \int_{\mathbb{R}^d} \frac{\mathbf{v}\otimes\mathbf{v}M(\mathbf{v})}{\lambda} d\mathbf{v}.$$

The limit $\epsilon \to 0$ is known as the drift-diffusion limit.

2.2 Regularity of the solution in z

We first study the regularity of f in the random variable $\mathbf{z}$. To this aim, we assume σ depends on $\mathbf{z}$ linearly. This is mostly the case when one approximates a random field, for example by the Karhunen-Loeve expansion [14],

$$\sigma(\mathbf{v},\mathbf{w},\mathbf{z}) \approx \bar{\sigma}_0(\mathbf{v},\mathbf{w}) + \sum_{i=1}^{n} \tilde{\sigma}_i(\mathbf{v},\mathbf{w})z_i, \tag{2.9}$$

with $z_1, z_2, \cdots, z_n$ independent random variables with probability density function $\pi(\mathbf{z})$.

In this section, we assume $I_{\mathbf{z}}$ to be a bounded domain. Several distributions, such as the uniform or Beta distributions. Then the boundedness of σ in $\mathbf{z}$, as a linear function of $\mathbf{z}$, can be assumed. For simplicity we also assume $\phi = \phi(\mathbf{x},t)$ independent of $\mathbf{z}$. The l-th order formal differentiation of the Boltzmann equation with respect to $\mathbf{z}$ gives

$$\epsilon^2\partial_t(\partial_{\mathbf{z}}^l f) + \epsilon\mathbf{v}\cdot\nabla_{\mathbf{x}}(\partial_{\mathbf{z}}^l f) + \epsilon\nabla_{\mathbf{x}}\phi\cdot\nabla_{\mathbf{v}}(\partial_{\mathbf{z}}^l f) = \partial_{\mathbf{z}}^l Q(f), \tag{2.10}$$

with

$$\begin{aligned}\partial_{\mathbf{z}}^l Q(f) &= M(\mathbf{v})\int_{\mathbb{R}^d} \partial_{\mathbf{z}}^l\left[\sigma(\mathbf{v},\mathbf{w},\mathbf{z})f(\mathbf{w},\mathbf{z})\right] d\mathbf{w} - \partial_{\mathbf{z}}^l\left[\lambda(\mathbf{v},\mathbf{z})f(\mathbf{v},\mathbf{z})\right]\\ &= M(\mathbf{v})\int_{\mathbb{R}^d} \sigma(\mathbf{v},\mathbf{w},\mathbf{z})\partial_{\mathbf{z}}^l f(\mathbf{w},\mathbf{z})d\mathbf{w} + lM(\mathbf{v})\int_{\mathbb{R}^d} \partial_{\mathbf{z}}\sigma(\mathbf{v},\mathbf{w},\mathbf{z})\partial_{\mathbf{z}}^{l-1} f(\mathbf{w},\mathbf{z})d\mathbf{w}\\ &\quad - \lambda(\mathbf{v},\mathbf{z})\lambda_{\mathbf{z}}^l f(\mathbf{v},\mathbf{z}) - l\partial_{\mathbf{z}}\lambda(\mathbf{v},\mathbf{z})\partial_{\mathbf{z}}^{l-1} f(\mathbf{v},\mathbf{z}), \quad \text{for } 0 \le l \le m.\end{aligned} \tag{2.11}$$

We have the following theorem about the regularity of the solution in $\mathbf{z}$.

Theorem 2.1 *Assume σ depends on $\mathbf{z}$ linearly, and*

$$\max_{\mathbf{z}} |\sigma| \le \gamma_0, \qquad \max_{\mathbf{z}} |\partial_{\mathbf{z}}\sigma| \le \gamma_1, \qquad \max_{\mathbf{z}} |\partial_{\mathbf{z}}\lambda| \le \gamma_1.$$

We also assume that $\epsilon < \sqrt{2\gamma_1}$ and $m > 0$ is an integer. If for some integer $m \ge 0$, $||\partial_{\mathbf{z}}^l f_I||_{\Gamma(0)} \le \beta$, for all $l = 0, \cdots, m$, then

$$||\partial_{\mathbf{z}}^l f||^2_{\Gamma(t)} \le \beta \frac{1}{1 - \frac{\epsilon^2}{2\gamma_1}} \; e^{\frac{2\gamma_1 l}{\epsilon^2} t}, \quad l = 0, \cdots, m. \tag{2.12}$$

Proof. Multiplying $(\partial_{\mathbf{z}}^l f) e^{-2\phi(\mathbf{x},t)} \pi(\mathbf{z})/M(\mathbf{v})$ to both sides of (2.10) and integrating on $\Omega \times \mathbb{R}^d \times I_{\mathbf{z}}$, one gets

$$\begin{aligned}
\frac{\epsilon^2}{2}\partial_t \left|\left|\partial_{\mathbf{z}}^l f\right|\right|^2_{\Gamma(t)} &= \int_{I_{\mathbf{z}}}\int_{\Omega}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \sigma(\mathbf{v},\mathbf{w},\mathbf{z})\partial_{\mathbf{z}}^l f(\mathbf{w},\mathbf{z})\partial_{\mathbf{z}}^l f(\mathbf{v},\mathbf{z}) e^{-2\phi} d\mathbf{w} d\mathbf{v} d\mathbf{x}\, \pi(\mathbf{z}) d\mathbf{z} \\
&\quad + l\int_{I_{\mathbf{z}}}\int_{\Omega}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \partial_{\mathbf{z}}\sigma(\mathbf{v},\mathbf{w},\mathbf{z})\partial_{\mathbf{z}}^{l-1} f(\mathbf{w},\mathbf{z})\partial_{\mathbf{z}}^l f(\mathbf{v},\mathbf{z}) e^{-2\phi} d\mathbf{w} d\mathbf{v} d\mathbf{x}\, \pi(\mathbf{z}) d\mathbf{z} \\
&\quad - \int_{I_{\mathbf{z}}}\int_{\Omega}\int_{\mathbb{R}^d} \lambda(\mathbf{v},\mathbf{z}) \left(\partial_{\mathbf{z}}^l f(\mathbf{v},\mathbf{z})\right)^2 e^{-2\phi}/M(\mathbf{v}) d\mathbf{v} d\mathbf{x}\, \pi(\mathbf{z}) d\mathbf{z} \\
&\quad - l\int_{I_{\mathbf{z}}}\int_{\Omega}\int_{\mathbb{R}^d} \partial_{\mathbf{z}}\lambda(\mathbf{v},\mathbf{z})\partial_{\mathbf{z}}^{l-1} f(\mathbf{v},\mathbf{z})\partial_{\mathbf{z}}^l f(\mathbf{v},\mathbf{z}) e^{-2\phi}/M(\mathbf{v}) d\mathbf{v} d\mathbf{x}\, \pi(\mathbf{z}) d\mathbf{z} \\
&\le \int_{I_{\mathbf{z}}}\int_{\Omega}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left(\sqrt{\sigma(\mathbf{v},\mathbf{w},\mathbf{z})}\sqrt{M(\mathbf{v})}\partial_{\mathbf{z}}^l f(\mathbf{w},\mathbf{z})/\sqrt{M(\mathbf{w})}\right) \\
&\quad \cdot \left(\sqrt{\sigma(\mathbf{v},\mathbf{w},\mathbf{z})}\sqrt{M(\mathbf{w})}\partial_{\mathbf{z}}^l f(\mathbf{v},\mathbf{z})/\sqrt{M(\mathbf{v})}\right) d\mathbf{w} d\mathbf{v} e^{-2\phi} d\mathbf{x}\, \pi(\mathbf{z}) d\mathbf{z} \\
&\quad + l\gamma_1 \int_{I_{\mathbf{z}}}\int_{\Omega} \left|\int_{\mathbb{R}^d} \partial_{\mathbf{z}}^{l-1} f(\mathbf{w},\mathbf{z}) d\mathbf{w}\right| \cdot \left|\int_{\mathbb{R}^d} \partial_{\mathbf{z}}^l f(\mathbf{v},\mathbf{z}) d\mathbf{v}\right| e^{-2\phi} d\mathbf{x}\, \pi(\mathbf{z}) d\mathbf{z} \\
&\quad - \int_{I_{\mathbf{z}}}\int_{\Omega}\int_{\mathbb{R}^d} \lambda(\mathbf{v},\mathbf{z}) \left(\partial_{\mathbf{z}}^l f(\mathbf{v},\mathbf{z})\right)^2 e^{-2\phi}/M(\mathbf{v}) d\mathbf{v} d\mathbf{x}\, \pi(\mathbf{z}) d\mathbf{z} \\
&\quad + l\gamma_1 \int_{I_{\mathbf{z}}}\int_{\Omega}\int_{\mathbb{R}^d} \left|\partial_{\mathbf{z}}^{l-1} f(\mathbf{v},\mathbf{z})\right| \cdot \left|\partial_{\mathbf{z}}^l f(\mathbf{v},\mathbf{z})\right| e^{-2\phi}/M(\mathbf{v}) d\mathbf{v} d\mathbf{x}\, \pi(\mathbf{z}) d\mathbf{z} \\
&\le \left(\int_{I_{\mathbf{z}}}\int_{\Omega}\int_{\mathbb{R}^d} \lambda(\mathbf{w},\mathbf{z}) \left(\partial_{\mathbf{z}}^l f(\mathbf{w},\mathbf{z})\right)^2 /M(\mathbf{w}) d\mathbf{w} e^{-2\phi} d\mathbf{x}\, \pi(\mathbf{z}) d\mathbf{z}\right)^{\frac{1}{2}} \\
&\quad \cdot \left(\int_{I_{\mathbf{z}}}\int_{\Omega}\int_{\mathbb{R}^d} \lambda(\mathbf{v},\mathbf{z}) \left(\partial_{\mathbf{z}}^l f(\mathbf{v},\mathbf{z})\right)^2 /M(\mathbf{v}) d\mathbf{v} e^{-2\phi} d\mathbf{x}\, \pi(\mathbf{z}) d\mathbf{z}\right)^{\frac{1}{2}} \\
&\quad + l\gamma_1 \left(\int_{I_{\mathbf{z}}}\int_{\Omega} \left(\int_{\mathbb{R}^d} \partial_{\mathbf{z}}^{l-1} f(\mathbf{w},\mathbf{z}) d\mathbf{w}\right)^2 e^{-2\phi} d\mathbf{x}\, \pi(\mathbf{z}) d\mathbf{z}\right)^{\frac{1}{2}} \\
&\quad \cdot \left(\int_{I_{\mathbf{z}}}\int_{\Omega} \left(\int_{\mathbb{R}^d} \partial_{\mathbf{z}}^l f(\mathbf{v},\mathbf{z}) d\mathbf{v}\right)^2 e^{-2\phi} d\mathbf{x}\, \pi(\mathbf{z}) d\mathbf{z}\right)^{\frac{1}{2}} \\
&\quad - \int_{I_{\mathbf{z}}}\int_{\Omega}\int_{\mathbb{R}^d} \lambda(\mathbf{v},\mathbf{z}) \left(\partial_{\mathbf{z}}^l f(\mathbf{v},\mathbf{z})\right)^2 e^{-2\phi}/M(\mathbf{v}) d\mathbf{v} d\mathbf{x}\, \pi(\mathbf{z}) d\mathbf{z} \\
&\quad + l\gamma_1 \left(\int_{I_{\mathbf{z}}}\int_{\Omega}\int_{\mathbb{R}^d} (\partial_{\mathbf{z}}^{l-1} f(\mathbf{v},\mathbf{z}))^2 /M(\mathbf{v}) d\mathbf{v} e^{-2\phi} d\mathbf{x}\, \pi(\mathbf{z}) d\mathbf{z}\right)^{\frac{1}{2}}
\end{aligned}$$

$$\cdot \left(\int_{I_{\mathbf{z}}} \int_{\Omega} \int_{\mathbb{R}^d} (\partial_{\mathbf{z}}^l f(\mathbf{v},\mathbf{z}))^2 / M(\mathbf{v}) d\mathbf{v} e^{-2\phi} d\mathbf{x}\, \pi(\mathbf{z}) d\mathbf{z} \right)^{\frac{1}{2}}$$

$$= l\gamma_1 ||\partial_{\mathbf{z}}^{l-1} f||_{\Gamma(t)} \cdot ||\partial_{\mathbf{z}}^{l} f||_{\Gamma(t)} + l\gamma_1 ||\partial_{\mathbf{z}}^{l-1} f||_{\Gamma(t)} \cdot ||\partial_{\mathbf{z}}^{l} f||_{\Gamma(t)}$$

$$\leq l\gamma_1 (||\partial_{\mathbf{z}}^{l-1} f||^2_{\Gamma(t)} + ||\partial_{\mathbf{z}}^{l} f||^2_{\Gamma(t)}).$$

Here in the first inequality we used the assumption $\max_{\mathbf{z}} |\partial_{\mathbf{z}} \sigma| \leq \gamma_1$, and $\max_{\mathbf{z}} |\partial_{\mathbf{z}} \lambda| \leq \gamma_1$. In the second inequality, we used the Cauchy-Schwartz inequality. In the second equality, the first and third term cancels. The last equality is a result of $2ab \leq a^2 + b^2$ for any a, b. Therefore,

$$\epsilon^2 \partial_t ||\partial_{\mathbf{z}}^{l} f||^2_{\Gamma(t)} \leq 2\gamma_1 l (||\partial_{\mathbf{z}}^{l} f||^2_{\Gamma(t)} + ||\partial_{\mathbf{z}}^{l-1} f||^2_{\Gamma(t)}). \tag{2.13}$$

We now use mathematical induction to prove (2.12). It clearly holds for $l = 0$. Assume that

$$||\partial_{\mathbf{z}}^{l} f||^2_{\Gamma(t)} \leq \beta \ \frac{1}{1 - \frac{\epsilon^2}{2\gamma_1}} \ e^{\frac{2\gamma_1 l}{\epsilon^2} t}.$$

Then by Gronwall's inequality for (2.13),

$$\begin{aligned}
||\partial_{\mathbf{z}}^{l+1} f||^2_{\Gamma(t)} &\leq e^{\frac{2\gamma_1 (l+1)}{\epsilon^2} t} \ ||\partial_{\mathbf{z}}^{l+1} f_I||^2_{\Gamma(t)} + \int_0^t e^{\frac{2\gamma_1 (l+1)}{\epsilon^2} (t-s)} \ ||\partial_{\mathbf{z}}^{l} f||^2_{\Gamma(s)} \ ds \\
&\leq \beta \ e^{\frac{2\gamma_1 (l+1)}{\epsilon^2} t} + \beta \ \frac{1}{1 - \frac{\epsilon^2}{2\gamma_1}} \ e^{\frac{2\gamma_1 (l+1)}{\epsilon^2} t} \ \int_0^t e^{-\frac{2\gamma_1}{\epsilon^2} s} \ ds \\
&= \beta \ e^{\frac{2\gamma_1 (l+1)}{\epsilon^2} t} + \beta \ \frac{1}{1 - \frac{\epsilon^2}{2\gamma_1}} \ e^{\frac{2\gamma_1 (l+1)}{\epsilon^2} t} \cdot \frac{\epsilon^2}{2\gamma_1} \left(1 - e^{-\frac{2\gamma_1}{\epsilon^2} t} \right) \\
&\leq \beta \ \left(1 + \frac{\frac{\epsilon^2}{2\gamma_1}}{1 - \frac{\epsilon^2}{2\gamma_1}} \right) \ e^{\frac{2\gamma_1 (l+1)}{\epsilon^2} t} = \beta \ \frac{1}{1 - \frac{\epsilon^2}{2\gamma_1}} \ e^{\frac{2\gamma_1 (l+1)}{\epsilon^2} t}.
\end{aligned}$$

Therefore,

$$||\partial_{\mathbf{z}}^{l} f||^2_{\Gamma(t)} \leq \beta \frac{1}{1 - \frac{\epsilon^2}{2\gamma_1}} \ e^{\frac{2\gamma_1 l}{\epsilon^2} t}, \quad l = 0, \cdots, m. \qquad \square$$

This Theorem shows that the regularity of the initial data is preserved by the solution at later times.

3 A gPC stochastic Galerkin method

By the stochastic Galerkin method, one seeks for an orthogonal polynomial expansion for the solution of problem (2.1). That is, for random variable $z \in I_{\mathbf{z}}$,

$$f(t,\mathbf{x},\mathbf{v},\mathbf{z}) \approx f_K(t,\mathbf{x},\mathbf{v},\mathbf{z}) = \sum_{|\mathbf{k}|=0}^{K} \alpha_{\mathbf{k}}(t,\mathbf{x},\mathbf{v}) \psi_{\mathbf{k}}(\mathbf{z}) = \boldsymbol{\alpha} \cdot \boldsymbol{\psi}, \quad M = \binom{n+K}{n}. \tag{3.1}$$

Here $\mathbf{k} = (k_1, \ldots, k_n)$ is a multi-index with $|\mathbf{k}| = k_1 + \cdots + k_n$, and the coefficient vectors are given by

$$\boldsymbol{\alpha} = (\alpha_0, \alpha_2, \cdots, \alpha_K), \qquad \boldsymbol{\psi} = (\psi_0, \psi_1, \cdots, \psi_K).$$

$\{\Psi_{\mathbf{k}}(\mathbf{z})\}$ are the orthonormal basis functions that form $\mathbb{P}_K^n$ (the set of n-variate orthonormal polynomials of degree $K \geq 1$) and satisfy

$$\int_{I_{\mathbf{z}}} \psi_{\mathbf{k}}(\mathbf{z}) \psi_{\mathbf{l}}(\mathbf{z}) \pi(\mathbf{z}) d\mathbf{z} = \delta_{\mathbf{kl}}, \qquad 0 \leq |\mathbf{k}|, |\mathbf{l}| \leq K = \dim(\mathbb{P}_K^n),$$

where $\delta_{\mathbf{kl}}$ the Kronecker Delta function. The commonly used pairs of $\{\psi_{\mathbf{k}}(\mathbf{z})\}$ and $\pi(\mathbf{z})$ include Hermite-Gaussian, Legendre-uniform, Laguerre-Gamma, etc. If the random dimension $n > 1$, one can re-order the multi-dimensional polynomials $\{\psi_{\mathbf{k}}(\mathbf{z})\}$ of $\mathbf{z}$ into a single index k. One can refer to Section 5.2 of [18], where the graded lexicographic ordering is introduced.

The initial values of each component of $\boldsymbol{\alpha}$ are the gPC coefficients of the initial datum for f which is denoted by $f_I(\mathbf{x}, \mathbf{v}, \mathbf{z})$,

$$\alpha_{\mathbf{k}}(0, \mathbf{x}, \mathbf{v}) =< f_I(\mathbf{x}, \mathbf{v}, \cdot), \psi_{\mathbf{k}} >_H, \quad |\mathbf{k}| = 0, \cdots, K. \tag{3.2}$$

We apply f_K into (2.1) and take an inner product with $\psi_{\mathbf{k}}(\mathbf{z})$ $(0 \le |\mathbf{k}| \le K)$ to get

$$\begin{aligned}
&\epsilon \frac{\partial \boldsymbol{\alpha}}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} \boldsymbol{\alpha} + \nabla_{\mathbf{x}} \phi \cdot \nabla_{\mathbf{v}} \boldsymbol{\alpha} \\
=&< \mathcal{Q}(f_K), \boldsymbol{\psi} >_H = \frac{1}{\epsilon} M(\mathbf{v}) \int_{\mathbb{R}^d} B(\mathbf{v}, \mathbf{w}) \boldsymbol{\alpha}(\mathbf{w}) d\mathbf{w} - \frac{1}{\epsilon} F(\mathbf{v}) \boldsymbol{\alpha} + \epsilon \tilde{G}(\mathbf{v}) \\
=& \frac{1}{\epsilon} \mathbf{Q}(\boldsymbol{\alpha}) + \epsilon \tilde{G},
\end{aligned} \tag{3.3}$$

where we denote

$$\mathbf{Q}(\boldsymbol{\alpha}) = M(\mathbf{v}) \int_{\mathbb{R}^d} B(\mathbf{v}, \mathbf{w}) \boldsymbol{\alpha}(\mathbf{w}) d\mathbf{w} - F(\mathbf{v}) \boldsymbol{\alpha}, \tag{3.4}$$

and matrices $B(\mathbf{v}, \mathbf{w}) = (B_{\mathbf{ij}})_{K \times K}$, $F(\mathbf{v}) = (F_{\mathbf{ij}})_{K \times K}$ as well as the vector $\tilde{G} = (\tilde{G}_{\mathbf{k}})_{K \times 1}$ are given by

$$\begin{aligned}
B_{\mathbf{ij}}(\mathbf{v}, \mathbf{w}) &= \int_{I_{\mathbf{z}}} \sigma(\mathbf{v}, \mathbf{w}, \mathbf{z}) \psi_{\mathbf{i}}(\mathbf{z}) \psi_{\mathbf{j}}(\mathbf{z}) \pi(\mathbf{z}) d\mathbf{z}, \\
F_{\mathbf{ij}}(\mathbf{v}) &= \int_{I_{\mathbf{z}}} \lambda(\mathbf{v}, \mathbf{z}) \psi_{\mathbf{i}}(\mathbf{z}) \psi_{\mathbf{j}}(\mathbf{z}) \pi(\mathbf{z}) d\mathbf{z}, \\
\tilde{G}_{\mathbf{k}}(t, \mathbf{x}, \mathbf{v}) &= \int_{I_{\mathbf{z}}} G(t, \mathbf{x}, \mathbf{v}, \mathbf{z}) \psi_{\mathbf{k}}(\mathbf{z}) \pi(\mathbf{z}) d\mathbf{z}.
\end{aligned} \tag{3.5}$$

Since $\sigma, \lambda > 0$, both matrices B, F are symmetric and positive definite [19]. Note

$$F(\mathbf{v}) = \int_{\mathbb{R}^d} B(\mathbf{v}, \mathbf{w}) M(\mathbf{w}) d\mathbf{w}. \tag{3.6}$$

Similar to the property that $\ker \mathcal{Q} = \text{span}\{M(\mathbf{v})\}$, we have the following result which will be used when we prove the stochastic AP property of our scheme. The proof is similar to that in [16] for the collision operator $\mathcal{Q}$ defined in (2.2).

Lemma 3.1 $\mathbf{r}(\mathbf{v}) = M(\mathbf{v}) \int_{\mathbb{R}^d} \mathbf{r}(\mathbf{w}) d\mathbf{w}$ *is the unique solution of the system of equations* $\mathbf{Q}(\mathbf{r}) = 0$.

Proof, Use the symmetry of B and F, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} \frac{\mathbf{Q}(\mathbf{r})^T \mathbf{r}(\mathbf{v})}{M(\mathbf{v})} d\mathbf{v} \\
=& \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbf{r}(\mathbf{w})^T B^T(\mathbf{v}, \mathbf{w}) \mathbf{r}(\mathbf{v}) d\mathbf{w} d\mathbf{v} - \int_{\mathbb{R}^d} \frac{\mathbf{r}(\mathbf{v})^T F^T(\mathbf{v}) \mathbf{r}(\mathbf{v})}{M(\mathbf{v})} d\mathbf{v} \\
=& \frac{1}{2} \Big(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\mathbf{r}(\mathbf{w})^T}{M(\mathbf{w})} B(\mathbf{v}, \mathbf{w}) M(\mathbf{v}) M(\mathbf{w}) \frac{\mathbf{r}(\mathbf{v})}{M(\mathbf{v})} d\mathbf{w} d\mathbf{v} \\
&+ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\mathbf{r}(\mathbf{v})^T}{M(\mathbf{v})} B(\mathbf{w}, \mathbf{v}) M(\mathbf{w}) M(\mathbf{v}) \frac{\mathbf{r}(\mathbf{w})}{M(\mathbf{w})} d\mathbf{v} d\mathbf{w}
\end{aligned}$$

$$
\begin{aligned}
&- \int_{\mathbb{R}^d} \frac{\mathbf{r}(\mathbf{v})^T F(\mathbf{v})\mathbf{r}(\mathbf{v})}{M(\mathbf{v})}\, d\mathbf{v} - \int_{\mathbb{R}^d} \frac{\mathbf{r}(\mathbf{w})^T F(\mathbf{w})\mathbf{r}(\mathbf{w})}{M(\mathbf{w})} d\mathbf{w}\Big) \\
&= \frac{1}{2}\Big(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\mathbf{r}(\mathbf{w})^T}{M(\mathbf{w})} B(\mathbf{v},\mathbf{w}) M(\mathbf{v}) M(\mathbf{w}) \frac{\mathbf{r}(\mathbf{v})}{M(\mathbf{v})} d\mathbf{w} d\mathbf{v} \\
&\quad + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\mathbf{r}(\mathbf{v})^T}{M(\mathbf{v})} B(\mathbf{w},\mathbf{v}) M(\mathbf{w}) M(\mathbf{v}) \frac{\mathbf{r}(\mathbf{w})}{M(\mathbf{w})} d\mathbf{v} d\mathbf{w} \\
&\quad - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\mathbf{r}(\mathbf{v})^T}{M(\mathbf{v})} B(\mathbf{v},\mathbf{w}) M(\mathbf{w}) M(\mathbf{v}) \frac{\mathbf{r}(\mathbf{v})}{M(\mathbf{v})} d\mathbf{w} d\mathbf{v} \\
&\quad - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\mathbf{r}(\mathbf{w})^T}{M(\mathbf{w})} B(\mathbf{w},\mathbf{v}) M(\mathbf{v}) M(\mathbf{w}) \frac{\mathbf{r}(\mathbf{w})}{M(\mathbf{w})} d\mathbf{v} d\mathbf{w}\Big) \\
&= -\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left(\frac{\mathbf{r}(\mathbf{w})}{M(\mathbf{w})} - \frac{\mathbf{r}(\mathbf{v})}{M(\mathbf{v})}\right)^T B(\mathbf{v},\mathbf{w}) M(\mathbf{v}) M(\mathbf{w}) \left(\frac{\mathbf{r}(\mathbf{w})}{M(\mathbf{w})} - \frac{\mathbf{r}(\mathbf{v})}{M(\mathbf{v})}\right) d\mathbf{v} d\mathbf{w} \le 0,
\end{aligned}
$$

since B is positive definite. The second equality is due to symmetry of matrices B and F. We use (3.6) in the third equality and use $B(\mathbf{v},\mathbf{w}) = B(\mathbf{w},\mathbf{v})$ in the fourth equality. Clearly if $\mathbf{Q}(\boldsymbol{\alpha}) = 0$, the above inequality becomes equality. Since B is positive definite, the term in the last line implies $\dfrac{\mathbf{r}(\mathbf{w})}{M(\mathbf{w})} = \dfrac{\mathbf{r}(\mathbf{v})}{M(\mathbf{v})}$ for any $\mathbf{v}$ and $\mathbf{w}$. This implies $r(\mathbf{v}) = cM(\mathbf{v})$ for c independent of $\mathbf{v}$. Integrating over $\mathbf{v}$ gives $c = \int_{\mathbb{R}^d} r(\mathbf{v}) d\mathbf{v}$.

4 Spectral convergence analysis

We now prove that the gPC-SG method is convergent and stable under suitably defined energy norm given in (2.6). An error estimate and spectral convergence rate in the probability space will also be presented. The proofs are similar to the convergence analysis of the moment method for linear kinetic model [20].

Let f be the solution to the Boltzmann equation (2.1). We define the projection operator

$$
P_K f = \sum_{|\mathbf{k}|=0}^{K} < f, \psi_{\mathbf{k}} >_H \psi_{\mathbf{k}}.
$$

The error arisen from the gPC-SG can be split into two parts R_K and e_K,

$$
f - f_K = f - P_K f + P_K f - f_K := R_K + e_K, \tag{4.1}
$$

where $R_K = f - P_K f$ is the projection error, and

$$
e_K = P_K f - f_K = \sum_{|\mathbf{k}|=0}^{K} \big(< f, \psi_{\mathbf{k}} >_H - f_{\mathbf{k}}\big)\psi_{\mathbf{k}} = \hat{\mathbf{e}} \cdot \boldsymbol{\psi},
$$

where $\hat{\mathbf{e}} = \big(< f, \psi_0 >_H - f_0, \cdots, < f, \psi_{\mathbf{k}} >_H - f_{\mathbf{k}}\big)$ is the numerical error.

We first give the projection error. By standard error estimate for orthogonal polynomial approximations, and (2.12), for $0 \le t \le T$,

$$
||R_K||_{\Gamma(t)} \le C_1 K^{-m} ||\partial_{\mathbf{z}}^m f||_{\Gamma(t)} \le C_1 \frac{\beta}{1 - \frac{\epsilon^2}{2\gamma_1}} \frac{e^{\frac{2\gamma_1 m T}{\epsilon^2}}}{K^m}, \tag{4.2}
$$

with C_1 independent of m.

It remains to estimate e_K. Define the operator

$$\mathcal{L} = \epsilon^2 \partial_t + \epsilon \mathbf{v} \cdot \nabla_{\mathbf{x}} + \epsilon \nabla_{\mathbf{x}} \phi \cdot \nabla_{\mathbf{v}} - \mathcal{Q}. \tag{4.3}$$

We first prove two results.

Lemma 4.1 $< \mathcal{L}(R_K), \boldsymbol{\psi} >_H = - < \mathcal{Q}(R_K), \boldsymbol{\psi} >_H$.

Proof. Since $R_K = f - P_K f = f - \sum_{|\mathbf{k}|=0}^{K} < f, \psi_{\mathbf{k}} >_H \psi_{\mathbf{k}}$,

$$\begin{aligned} < \partial_t R_K, \boldsymbol{\psi} >_H &= < \partial_t f, \boldsymbol{\psi} >_H - < \sum_{|\mathbf{k}|=0}^{K} \partial_t < f, \psi_{\mathbf{k}} >_H \psi_{\mathbf{k}}, \boldsymbol{\psi} >_H \\ &= < \partial_t f, \boldsymbol{\psi} >_H - \sum_{|\mathbf{k}|=0}^{K} < \partial_t f, \psi_{\mathbf{k}} >_H < \psi_{\mathbf{k}}, \boldsymbol{\psi} >_H \\ &= < \partial_t f, \boldsymbol{\psi} >_H - < \partial_t f, \boldsymbol{\psi} >_H = 0. \end{aligned} \tag{4.4}$$

The second and third terms of $< \mathcal{L}(R_K), \boldsymbol{\psi} >_H$ are also zero by similar arguments. Therefore, $< \mathcal{L}(R_K), \boldsymbol{\psi} >_H = - < \mathcal{Q}(R_K), \boldsymbol{\psi} >_H$.

Lemma 4.2 $||\mathcal{Q}(R_K)||_{\Gamma(t)} \leq C_1 \dfrac{\beta}{1 - \frac{\epsilon^2}{2\gamma_1}} \dfrac{e^{\frac{2\gamma_1 mT}{\epsilon^2}}}{K^m}$, *for* $0 \leq t \leq T$.

Since $Q(R_K)(\mathbf{v}, \mathbf{z}) = M(\mathbf{v}) \int_{\mathbb{R}^d} \sigma(\mathbf{v}, \mathbf{w}, \mathbf{z}) R_K(\mathbf{w}, \mathbf{z}) d\mathbf{w} - \lambda(\mathbf{v}, \mathbf{z}) R_K(\mathbf{v}, \mathbf{z}) := I + II$, then $||Q(R_K)||^2_{\Gamma(t)} \leq 2 \left(||I||^2_{\Gamma(t)} + ||II||^2_{\Gamma(t)} \right)$. It is easy to see that $||II||^2_{\Gamma(t)} \leq \lambda_0^2 ||R_K||^2_{\Gamma(t)}$. We now estimate I.

$$\begin{aligned} \|I\|^2_{\Gamma(t)} &= \left\| M(\mathbf{v}) \int_{\mathbb{R}^d} \sigma(\mathbf{v}, \mathbf{w}, \mathbf{z}) R_K(\mathbf{w}, \mathbf{z}) d\mathbf{w} \right\|^2_{\Gamma(t)} \\ &= \int_{I_{\mathbf{z}}} \int_{\mathbb{R}^d} M(\mathbf{v}) \int_{\Omega} \left(\int_{\mathbb{R}^d} \sigma(\mathbf{v}, \mathbf{w}, \mathbf{z}) R_K(\mathbf{w}, \mathbf{z}) d\mathbf{w} \right)^2 e^{-2\phi} d\mathbf{x} d\mathbf{v} \pi(\mathbf{z}) d\mathbf{z} \\ &\leq \int_{I_{\mathbf{z}}} \int_{\mathbb{R}^d} M(\mathbf{v}) \int_{\Omega} \left(\int_{\mathbb{R}^d} \sigma^2(\mathbf{v}, \mathbf{w}, \mathbf{z}) M(\mathbf{w}) d\mathbf{w} \right) \left(\int_{\mathbb{R}^d} R_K^2(\mathbf{w}, \mathbf{z}) / M(\mathbf{w}) d\mathbf{w} \right) \\ &\quad \cdot e^{-2\phi} \, d\mathbf{x} d\mathbf{v} \pi(\mathbf{z}) d\mathbf{z} \leq \sigma_1^2 ||R_K||^2_{\Gamma(t)}, \end{aligned}$$

where we used

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \sigma^2(\mathbf{v}, \mathbf{w}, \mathbf{z}) M(\mathbf{v}) M(\mathbf{w}) d\mathbf{w} d\mathbf{v} \leq \sigma_1^2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} M(\mathbf{v}) M(\mathbf{w}) \, d\mathbf{v} d\mathbf{w} = \sigma_1^2.$$

Therefore, $||Q(R_K)||^2_{\Gamma(t)} \leq 2 \left(\sigma_1^2 ||R_K||^2_{\Gamma(t)} + \lambda_0^2 ||R_K||^2_{\Gamma(t)} \right)$, so $||\mathcal{Q}(R_K)||_{\Gamma(t)} \lesssim ||R_K||_{\Gamma(t)} \leq C_1 \dfrac{\beta}{1 - \frac{\epsilon^2}{2\gamma_1}} \dfrac{e^{\frac{2\gamma_1 mT}{\epsilon^2}}}{K^m}$, for $0 \leq t \leq T$.

Since $\mathcal{L}(f) = 0$ and $P_K \mathcal{L}(f_K) = 0$, from (4.1), $< \mathcal{L}(e_K), \psi_{\mathbf{k}} >_H = - < \mathcal{L}(R_K), \psi_{\mathbf{k}} >_H$, for $|\mathbf{k}| = 0, \cdots K$, that is,

$$< \mathcal{L}(e_K), \boldsymbol{\psi} >_H = - < \mathcal{L}(R_K), \boldsymbol{\psi} >_H . \tag{4.5}$$

Taking the scalar product of (4.5) with $e^{-2\phi(\mathbf{x},t)} \cdot \hat{\mathbf{e}}/M(\mathbf{v})$ and integrating on $\Omega \times \mathbb{R}^d$ give

$$\begin{aligned}
&\frac{\epsilon^2}{2}\left(\partial_t \int_{\mathbb{R}^d}\int_\Omega e^{-2\phi}\ e_K^2/M(\mathbf{v})\ d\mathbf{x}d\mathbf{v} + \int_{\mathbb{R}^d}\int_\Omega \partial_t\phi e^{-2\phi}\ e_K^2/M(\mathbf{v})\ d\mathbf{x}d\mathbf{v}\right)\\
&\quad - \int_{\mathbb{R}^d}\int_\Omega < \mathcal{Q}(e_K), \boldsymbol{\psi} >_H \cdot\hat{\mathbf{e}}\ e^{-2\phi}/M(\mathbf{v})\ d\mathbf{x}d\mathbf{v}\\
&= -\int_{\mathbb{R}^d}\int_\Omega < \mathcal{L}(R_K), \boldsymbol{\psi} >_H \cdot\hat{\mathbf{e}}\ e^{-2\phi}/M(\mathbf{v})\ d\mathbf{x}d\mathbf{v}\\
&= \int_{\mathbb{R}^d}\int_\Omega < \mathcal{Q}(R_K), \boldsymbol{\psi} >_H \cdot\hat{\mathbf{e}}\ e^{-2\phi}/M(\mathbf{v})\ d\mathbf{x}d\mathbf{v}\\
&= \int_{\mathbb{R}^d}\int_\Omega < \mathcal{Q}(R_K), e_K >_H e^{-2\phi}/M(\mathbf{v})\ d\mathbf{x}d\mathbf{v}\\
&\le \int_{\mathbb{R}^d}\int_\Omega ||\mathcal{Q}(R_K)||_H \cdot ||e_K||_H\ e^{-2\phi}/M(\mathbf{v})\ d\mathbf{x}d\mathbf{v}\\
&\le \left(\int_{\mathbb{R}^d}\int_\Omega ||\mathcal{Q}(R_K)||_H^2\ e^{-2\phi}/M(\mathbf{v})\ d\mathbf{x}d\mathbf{v}\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^d}\int_\Omega ||e_K||_H^2\ e^{-2\phi}/M(\mathbf{v})\ d\mathbf{x}d\mathbf{v}\right)^{\frac{1}{2}}\\
&= ||\mathcal{Q}(R_K)||_{\Gamma(t)} \cdot ||e_K||_{\Gamma(t)},
\end{aligned}$$

where the second equality uses Lemma 4.1 and the inequality is by the Cauchy-Schwartz inequality.

From (2.7), we get

$$\frac{\epsilon^2}{2}\partial_t\left(||e_K||_{\Gamma(t)}^2\right) \le \frac{\epsilon^2}{2}\hat{\phi}||e_K||_{\Gamma(t)}^2 + ||\mathcal{Q}(R_K)||_{\Gamma(t)} \cdot ||e_K||_{\Gamma(t)},$$

with $\hat{\phi} = \|\partial_t\phi\|_{L_\infty(\Omega;(0,T))}$.

Cancelling one $\|e_K\|_{\Gamma(t)}$ term on both sides from the above inequality and using Lemma 4.2, we have

$$\frac{d}{dt}||e_K||_{\Gamma(t)} \le \frac{\hat{\phi}}{2}||e_K||_{\Gamma(t)} + \frac{C_1}{\epsilon^2}\ \frac{\beta}{1-\frac{\epsilon^2}{2\gamma_1}}\frac{e^{\frac{2\gamma_1 mt}{\epsilon^2}}}{K^m}.$$

Apply Gronwall's Lemma,

$$||e_K||_{\Gamma(t)} \le \exp(\hat{\phi}t/2)||e_K||_{\Gamma(0)} + \frac{C_1}{\epsilon^2}\ \frac{\beta}{1-\frac{\epsilon^2}{2\gamma_1}}\frac{e^{\frac{2\gamma_1 mt}{\epsilon^2}}}{K^m}\int_0^t \exp\left(\hat{\phi}(t-\tau)/2\right)\ d\tau. \tag{4.6}$$

We are now ready to state the main convergence theorem:

Theorem 4.3 *Assume σ depends on $\mathbf{z}$ linearly,*

$$\max_{\mathbf{z}}|\sigma| \le \gamma_0, \qquad \max_{\mathbf{z}}|\partial_{\mathbf{z}}\sigma| \le \gamma_1, \qquad \max_{\mathbf{z}}|\partial_{\mathbf{z}}\lambda| \le \gamma_1.$$

and $\epsilon < \sqrt{2\gamma_1}$. Assume $\hat{\phi}$ bounded, and for some integer $m \ge 0$, $||\partial_{\mathbf{z}}^l f_I||_{\Gamma(0)} \le \beta$, for $l = 0, \cdots, m$. Then

$$||f - f_K||_{\Gamma(t)} \le C\frac{e^{\frac{2\gamma_1 mT}{\epsilon^2}}}{K^m}, \tag{4.7}$$

where $C > 0$ depends on the initial data, $\hat{\phi}$ and T.

Proof. From (4.1), one has

$$\|f - f_K\|_{\Gamma(t)} \leq \|R_K\|_{\Gamma(t)} + \|e_K\|_{\Gamma(t)}\,.$$

Note $e_K(0) = P_k f - f_k|_{t=0} = 0$. Now combining (4.2) with (4.6) gives the desired inequality (4.7).

Remark 4.4 *When $\epsilon \to 0$, the right-hand-side of (4.7) requires $K > e^{\frac{2\gamma_1 mT}{\epsilon^2}}$, so the error is not good for small ϵ unless K is very large. This is exactly the motivation for the development of s-AP scheme which allows the choice of K independent of ϵ.*

5 A fully discrete stochastic AP Scheme

For clarify, we consider $v \in \mathbb{R}$. The space, time and velocity discretization here follow that of the deterministic case in [9], using the even- and odd- parities formulation in velocity, Wild sum approximation to handle the stiffness in time, center and upwind approximations in space, and spectral approximation in velocity.

First, (2.1) can be split into two equations, one for $v > 0$ and one for $-v < 0$,

$$\begin{cases} \epsilon\partial_t f + v\partial_x f - E\partial_v f = \dfrac{1}{\epsilon}\mathcal{Q}(f) + \epsilon G, \\ \epsilon\partial_t f - v\partial_x f + E\partial_v f = \dfrac{1}{\epsilon}\mathcal{Q}(f) + \epsilon G. \end{cases} \tag{5.1}$$

Introduce the even and odd parities r and j for $v > 0$,

$$\begin{aligned} r(t,x,v,\mathbf{z}) &= \frac{1}{2}\left(f(t,x,v,\mathbf{z}) + f(t,x,-v,\mathbf{z})\right), \\ j(t,x,v,\mathbf{z}) &= \frac{1}{2\epsilon}(f(t,x,v,\mathbf{z}) - f(t,x,-v,\mathbf{z})), \end{aligned} \tag{5.2}$$

adding and subtracting the two equations in (5.1), one has

$$\begin{cases} \partial_t r + v\partial_x j - E\partial_v j = \dfrac{1}{\epsilon^2}\mathcal{Q}(r) + G, \\ \partial_t j + \dfrac{1}{\epsilon^2}(v\partial_x r - E\partial_v r) = -\dfrac{1}{\epsilon^2}\lambda j. \end{cases} \tag{5.3}$$

From now on we will only consider the case of $v > 0$.

By the gPC-SG approach, one inserts the approximate solutions r_K, j_K,

$$r_K(t,x,v,\mathbf{z}) = \sum_{|\mathbf{k}|=0}^{K} \hat{r}_{\mathbf{k}}(t,x,v)\psi_{\mathbf{k}}(\mathbf{z}), \quad j_K(t,x,v,\mathbf{z}) = \sum_{|\mathbf{k}|=0}^{K} \hat{j}_{\mathbf{k}}(t,x,v)\psi_{\mathbf{k}}(\mathbf{z}),$$

into (5.3) and enforces the residual to be orthogonal to the polynomial space spanned by $\boldsymbol{\psi}(\mathbf{z}) = (\psi_0(\mathbf{z}), \psi_1(\mathbf{z}), \psi_2(\mathbf{z}), \cdots, \psi_K(\mathbf{z}))^T$. Let

$$\hat{\mathbf{r}}(t,x,v) = (\hat{r}_0(t,x,v), \cdots, \hat{r}_K(t,x,v))^T, \qquad \hat{\mathbf{j}}(t,x,v) = (\hat{j}_0(t,x,v), \cdots, \hat{j}_K(t,x,v))^T.$$

A system of equations for vectors $\hat{\mathbf{r}}$ and $\hat{\mathbf{j}}$ can then be obtained as

$$\begin{cases} \partial_t \hat{\mathbf{r}} + v\partial_x \hat{\mathbf{j}} - E\partial_v \hat{\mathbf{j}} = \dfrac{1}{\epsilon^2}\mathbf{Q}(\hat{\mathbf{r}}) + \displaystyle\int_{I_{\mathbf{z}}} G(t,x,v,\mathbf{z})\boldsymbol{\psi}(\mathbf{z})\pi(\mathbf{z})d\mathbf{z}, \\ \partial_t \hat{\mathbf{j}} + \dfrac{1}{\epsilon^2}(v\partial_x \hat{\mathbf{r}} - E\partial_v \hat{\mathbf{r}}) = -\dfrac{1}{\epsilon^2}F(v)\hat{\mathbf{j}}, \end{cases} \tag{5.4}$$

where matrices B, F are defined in (3.6). As was done in [8], we rewrite (5.4) into a diffusive relaxation system,

$$\begin{cases} \partial_t \hat{\mathbf{r}} + v\partial_x \hat{\mathbf{j}} - E\partial_v \hat{\mathbf{j}} = \dfrac{1}{\epsilon^2}\mathbf{Q}(\hat{\mathbf{r}}) + \displaystyle\int_{I_{\mathbf{z}}} G(t,x,v,\mathbf{z})\boldsymbol{\psi}(\mathbf{z})\pi(\mathbf{z})d\mathbf{z}, \\ \partial_t \hat{\mathbf{j}} + \phi\left(v\partial_x \hat{\mathbf{r}} - E\partial_v \hat{\mathbf{r}}\right) = -\dfrac{1}{\epsilon^2}\left[F(v)\hat{\mathbf{j}} + (1-\epsilon^2\phi)(v\partial_x \hat{\mathbf{r}} - E\partial_v \hat{\mathbf{r}})\right], \end{cases} \tag{5.5}$$

where $\phi = \phi(\epsilon)$ is a control parameter satisfying $0 \le \phi \le 1/\epsilon^2$. A simple choice of ϕ is

$$\phi(\epsilon) = \min\left\{1, \frac{1}{\epsilon^2}\right\}.$$

5.1 A Stochastic AP Time-splitting

Here the main difficulty is to handle the stiff collision operator in an efficient way. We employ the conventional time-splitting procedure [9, 8] for the diffusive relaxation system (5.5), which is composed of a relaxation step

$$\begin{cases} \partial_t \hat{\mathbf{r}} = \dfrac{1}{\epsilon^2}\mathbf{Q}(\hat{\mathbf{r}}) + \displaystyle\int_{I_{\mathbf{z}}} G(t,x,v,\mathbf{z})\boldsymbol{\psi}(\mathbf{z})\pi(\mathbf{z})d\mathbf{z}, \\ \partial_t \hat{\mathbf{j}} = -\dfrac{1}{\epsilon^2}\left(F(v)\hat{\mathbf{j}} + (1-\epsilon^2\phi)\mathbf{d}(\hat{\mathbf{r}})\right), \end{cases} \tag{5.6}$$

where the vector $\mathbf{d}(\hat{\mathbf{r}}) = v\partial_x \hat{\mathbf{r}} - E\partial_v \hat{\mathbf{r}}$, followed by a convection step

$$\begin{cases} \partial_t \hat{\mathbf{r}} + v\partial_x \hat{\mathbf{j}} - E\partial_v \hat{\mathbf{j}} = 0, \\ \partial_t \hat{\mathbf{j}} + \phi(v\partial_x \hat{\mathbf{r}} - E\partial_v \hat{\mathbf{r}}) = 0. \end{cases} \tag{5.7}$$

The AP property of the splitting can be shown as follows. Under the diffusive scaling, as $\epsilon \to 0$, (5.6) becomes $\mathbf{Q}(\hat{\mathbf{r}}) = 0$. By **Lemma** 3.1, the solution is given by

$$\hat{r}_{\mathbf{k}}(v) = \hat{\rho}_{\mathbf{k}} M(v), \tag{5.8}$$

where $\hat{\rho}_{\mathbf{k}} = \int_{\mathbb{R}} \hat{\mathbf{r}}_{\mathbf{k}}(w)dw$. The second equation of (5.6) gives

$$\hat{j}_{\mathbf{k}} = -\sum_{\mathbf{l}} (F^{-1})_{\mathbf{kl}}(v\partial_x \hat{r}_{\mathbf{l}} - E\partial_v \hat{r}_{\mathbf{l}}). \tag{5.9}$$

Applying (5.8) and (5.9) to the first equation of (5.7) and integrating over v, one gets

$$\partial_t \hat{\rho}_{\mathbf{k}} = \partial_x \left(T \sum_{\mathbf{l}} (F^{-1})_{\mathbf{kl}}(\partial_x \hat{\rho}_{\mathbf{l}} + 2\hat{\rho}_{\mathbf{l}} E) \right), \tag{5.10}$$

where

$$T = \int_{\mathbb{R}} v^2 M(v)dv = \theta.$$

On the other hand, applying the ansatz $\rho(t,x,\mathbf{z}) = \displaystyle\sum_{|\mathbf{k}|=0}^{K} \hat{\rho}_{\mathbf{k}}(t,x)\psi_{\mathbf{k}}(\mathbf{z})$ and conducting the Galerkin projection for the limiting drift-diffusion equation (2.8), we obtain

$$\partial_t \hat{\rho}_{\mathbf{k}} = \partial_x \left(T \sum_{\mathbf{l}} S_{\mathbf{kl}}(\partial_x \hat{\rho}_{\mathbf{l}} + 2\hat{\rho}_{\mathbf{l}} E) \right), \tag{5.11}$$

where the matrix $S = (S_{\mathbf{kl}})_{K\times K}$ are given by

$$S_{\mathbf{kl}} = \int_{I_{\mathbf{z}}} \frac{1}{\lambda(\mathbf{z})}\psi_{\mathbf{k}}(\mathbf{z})\psi_{\mathbf{l}}(\mathbf{z})\pi(\mathbf{z})d\mathbf{z}.$$

We will demonstrate in the numerical test that (5.10) is a good approximation of (5.11), namely $S = (S_{\mathbf{kl}})_{K\times K} \sim F^{-1} = (F_{\mathbf{kl}})^{-1}_{K\times K}$ with spectral accuracy.

5.2 The fully discretized systems

The relaxation step (5.6) is stiff, thus we adopt the Wild sum approximation as in [3]. Denote $\mathbf{P}(\hat{\mathbf{r}}) = \mathbf{Q}(\hat{\mathbf{r}}) + \lambda_0 \hat{\mathbf{r}}$ and rewrite (5.6) as

$$\partial_t \hat{\mathbf{r}} = \frac{1}{\epsilon^2}(\mathbf{P}(\hat{\mathbf{r}}) - \lambda_0 \hat{\mathbf{r}}) \tag{5.12}$$

$$\partial_t \hat{\mathbf{j}} = -\frac{1}{\epsilon^2}\left(F(v)\hat{\mathbf{j}} + (1-\epsilon^2\phi)\mathbf{d}(\hat{\mathbf{r}})\right). \tag{5.13}$$

We discretize time using a uniform time step $\Delta t = t^{n+1} - t^n$, where n is the temporal index. Denote by the index $(*)$ the intermediate numerical values obtained after one relaxation step from t^n. Introduce the new variables [8]

$$\tau = 1 - e^{-\lambda_0 t/\epsilon^2}, \quad \hat{\mathbf{R}} = \hat{\mathbf{r}} e^{\lambda_0 t/\epsilon^2},$$

(5.12) becomes

$$\partial_\tau \hat{\mathbf{R}} = \frac{1}{\lambda_0(1-\tau)}\mathbf{P}(\hat{\mathbf{R}}).$$

By Taylor expansion,

$$\hat{\mathbf{R}}(v,\tau) = \sum_{k=0}^{\infty} \tau^k \hat{\mathbf{r}}^{(k)}(v),$$

with

$$\hat{\mathbf{r}}^{(k+1)}(v) = \frac{1}{k+1}\sum_{n=0}^{k}\frac{1}{\lambda_0}P(\hat{\mathbf{r}}^{(n)}), \quad k \geq 1.$$

We revert to the old notations and truncate the expansion $\hat{\mathbf{r}}(v,t) = (1-\tau)\sum_{k=0}^{\infty}\tau^k\hat{\mathbf{r}}^{(k)}(v)$ by replacing the higher order terms with the corresponding local equilibrium state $\hat{\mathbf{r}}^{(\infty)} = \hat{\boldsymbol{\rho}}^* M = \hat{\boldsymbol{\rho}}^n M$, which leads to

$$\hat{\mathbf{r}}^* = (1-\tau)\sum_{k=0}^{m}\tau^k\hat{\mathbf{r}}^{(k)} + \tau^{m+1}\hat{\mathbf{r}}^{(\infty)}.$$

Plug in (5.2), the first-order time relaxed (**TR**) scheme for $\hat{\mathbf{r}}$ is given by

$$\hat{\mathbf{r}}^* = (1-\tau)\hat{\mathbf{r}}^n + \tau(1-\tau)\frac{\mathbf{P}(\hat{\mathbf{r}}^n)}{\lambda_0} + \tau^2\hat{\boldsymbol{\rho}}^n M,$$

with $\hat{\boldsymbol{\rho}}^n = \int_{\mathbb{R}} \hat{\mathbf{r}}^n(v)dv$.

To update the values for $\hat{\mathbf{j}}$, we solve (5.13) by the backward Euler method to get $\hat{\mathbf{j}}^\star$,

$$\hat{\mathbf{j}}^\star = \left(I + F(v)\frac{\Delta t}{\epsilon^2}\right)^{-1}\left[\hat{\mathbf{j}}^n - \frac{\Delta t(1-\epsilon^2\phi)}{\epsilon^2}\mathbf{d}(\hat{\mathbf{r}}^\star)\right],$$

where the matrix $I + F(v)\frac{\Delta t}{\epsilon^2}$ is invertible, thanks to the positive definite F.

For notation clarity, we describe the spatial discretization in one space dimension. Consider the spatial domain $\Omega = [x_L, x_R]$ which is partitioned into N grid cells with a uniform mesh size $\Delta x = 1/N$. Define the left boundary x_L as $x_{1/2}$, right boundary x_R as $x_{N+1/2}$, choose the spatial grid points $x_{j-1/2} = x_{1/2} + (j-1)\Delta x$, for $j = 1, \cdots, N+1$. The j-th interior cell is $[x_{j-1/2}, x_{j+1/2}]$, for $j = 1, \cdots, N$, with the cell average at time level t^n given by

$$\hat{\mathbf{U}}_j^n = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \hat{\mathbf{U}}(t^n, x, v, \mathbf{z})dx.$$

In the relaxation step, the second order center difference is used for the spatial derivative. In the transport step, since it is not stiff, we solve it by using an explicit second-order upwind scheme with slope limiters, which need to be implemented on the diagonal form on the Riemann invariants:

$$\hat{\mathbf{U}} = (\hat{\mathbf{r}} + \phi^{-\frac{1}{2}}\hat{\mathbf{j}})/2, \qquad \hat{\mathbf{V}} = (\hat{\mathbf{r}} - \phi^{-\frac{1}{2}}\hat{\mathbf{j}})/2, \tag{5.14}$$

which solve

$$\begin{cases} \partial_t \hat{\mathbf{U}} + \sqrt{\phi}(v\partial_x \hat{\mathbf{U}} - E\partial_v \hat{\mathbf{U}}) = 0, \\ \partial_t \hat{\mathbf{V}} - \sqrt{\phi}(v\partial_x \hat{\mathbf{V}} - E\partial_v \hat{\mathbf{V}}) = 0, \end{cases}$$

The scheme is given by, for $j = 1, \cdots, N$,

$$\begin{cases} \hat{\mathbf{U}}_j^{n+1} = \hat{\mathbf{U}}_j^* - \lambda \left[\hat{\mathbf{U}}_j^* - \hat{\mathbf{U}}_{j-1}^* + \dfrac{\Delta x}{4}(\boldsymbol{\sigma}_j^+ - \boldsymbol{\sigma}_{j-1}^+) \right] + (\partial_v \hat{\mathbf{U}}_j^*) E_i \Delta t, \\ \hat{\mathbf{V}}_j^{n+1} = \hat{\mathbf{V}}_j^* + \lambda \left[\hat{\mathbf{V}}_{j+1}^* - \hat{\mathbf{V}}_j^* - \dfrac{\Delta x}{4}(\boldsymbol{\sigma}_{j+1}^- - \boldsymbol{\sigma}_j^-) \right] - (\partial_v \hat{\mathbf{V}}_j^*) E_i \Delta t, \end{cases}$$

For $\lambda < 0$, the upwind stencil has to be changed. Here $\boldsymbol{\sigma}_i^{\pm}$ is the slope of $\hat{\mathbf{r}} \pm \phi^{-\frac{1}{2}}\hat{\mathbf{j}}$ on the i-th cell at $(*)$-th time step, with l-th component denoted by $(\sigma_i^{\pm})^{(l)}$ and given by

$$(\sigma_i^{\pm})^{(l)} = \frac{1}{\Delta x}\left[\pm r_{i\pm1}^{(l)} - \phi^{-\frac{1}{2}} j_i^{(l)} \mp r_i^{(l)} + \phi^{-\frac{1}{2}} j_{i\pm1}^{(l)} \right] \psi((\theta_i^{\pm})^{(l)}), \qquad (0 \le m \le M)$$

where $r^{(l)}$, $j^{(l)}$ are the l-th component of $\hat{\mathbf{r}}$ and $\hat{\mathbf{j}}$ respectively. $(\theta_i^{\pm})^{(l)}$ is defined by

$$(\theta_i^{\pm})^{(l)} = \left(\frac{r_i^{(l)} \pm \phi^{-\frac{1}{2}} j_i^{(l)} - r_{i-1}^{(l)} \mp \phi^{-\frac{1}{2}} j_{i-1}^{(l)}}{r_{i+1}^{(l)} \pm \phi^{-\frac{1}{2}} j_{i+1}^{(l)} - r_i^{(l)} \mp \phi^{-\frac{1}{2}} j_i^{(m)}} \right)^{\pm}.$$

A simple minmod slope limiter function is chosen,

$$\psi(\theta) = \max\{0, \min\{1, \theta\}\}.$$

one can then update the values for $\hat{\mathbf{r}}_j^{n+1}$ and $\hat{\mathbf{j}}_j^{n+1}$ by using (5.14).

Now combine the AP-property of the time splitting shown in section 5.1, with the AP property of the deterministic scheme given in [8], easily imply that the fully discrete time and space approximations are s-AP, in the sense that its $\epsilon \to 0$ limit, with $\Delta t, \Delta x$ fixed, approaches to the fully-discrete gPC-SG approximation of the drift-diffusion equation (2.8). We omit the details.

The velocity discretization is performed using spectral approximation based on the Hermite polynomials, which is equivalent to the moment method. We refer the reader to [17, 8].

5.3 Boundary conditions

Consider the inflow boundary conditions $f(t,x,v)|_{x_L} = F_L(v,\mathbf{z})$, $f(t,x,-v)|_{x_R} = F_R(v,\mathbf{z})$ for $v > 0$. Then

$$f_k(t,x,v)|_{x_L} = (\vec{F}_1(v))_k, \qquad f_k(t,x,-v)|_{x_R} = (\vec{F}_2(v))_k, \quad (0 \le l \le M)$$

where

$$(\vec{F}_1(v))_k = \int_{I_{\mathbf{z}}} F_L(v,\mathbf{z})\psi_k(\mathbf{z})\pi(\mathbf{z})d\mathbf{z}, \qquad (\vec{F}_2(v))_k = \int_{I_{\mathbf{z}}} F_R(v,\mathbf{z})\psi_k(\mathbf{z})\pi(\mathbf{z})d\mathbf{z}.$$

By gPC Galerkin approach (for $v > 0$ only),

$$\hat{r}_l + \epsilon \hat{j}_l|_{x_L} = (\vec{F}_1(v))_l, \qquad \hat{r}_l - \epsilon \hat{j}_l|_{x_R} = (\vec{F}_2(v))_l, \tag{5.15}$$

where r_l, j_l are the k-th component ($0 \le m \le M$) of $\hat{\mathbf{r}}$ and $\hat{\mathbf{j}}$ respectively.

As $\epsilon \to 0$, (5.6) leads to

$$\hat{\mathbf{j}} = -F^{-1}(v)\mathbf{d}(\hat{\mathbf{r}}), \tag{5.16}$$

which is plugging into (5.15) to form the numerical boundary conditions.

To obtain a second order numerical boundary conditions, we use two ghost cells outside x_L, namely $[x_{-3/2}, x_{-1/2}]$ and $[x_{-1/2}, x_{1/2}]$, with average cell values $\hat{\mathbf{U}}_{-1}$, $\hat{\mathbf{U}}_0$ defined respectively. Two ghost cells outside x_R are $[x_{N+1/2}, x_{N+3/2}]$ and $[x_{N+3/2}, x_{N+5/2}]$ with $\hat{\mathbf{U}}_{N+1}$, $\hat{\mathbf{U}}_{N+2}$ are similarly defined. Using the second-order central difference for spatial derivative of $\hat{\mathbf{r}}$ in (5.15) and (5.16), we define the ghost cells values $\hat{\mathbf{r}}_{-1}$, $\hat{\mathbf{r}}_0$ by the following equations

$$\begin{aligned}
&\frac{\hat{\mathbf{r}}_{-1} + \hat{\mathbf{r}}_2}{2} - \epsilon \Big[F^{-1}(v) \Big(\frac{v}{3\Delta x}(\hat{\mathbf{r}}_2 - \hat{\mathbf{r}}_{-1}) - E(x_{1/2}) \nabla_v \hat{\mathbf{r}}(x_{1/2}) \Big) \Big] = \vec{F}_1, \\
&\frac{\hat{\mathbf{r}}_0 + \hat{\mathbf{r}}_1}{2} - \epsilon \Big[F^{-1}(v) \Big(\frac{v}{\Delta x}(\hat{\mathbf{r}}_1 - \hat{\mathbf{r}}_0) - E(x_{1/2}) \nabla_v \hat{\mathbf{r}}(x_{1/2}) \Big) \Big] = \vec{F}_1,
\end{aligned} \tag{5.17}$$

where $\nabla_v \hat{\mathbf{r}}$ is approximated by

$$\nabla_v \hat{\mathbf{r}}(x_{1/2}) \approx \int_{I_{\mathbf{z}}} (\nabla_v F_L) \boldsymbol{\psi}(\mathbf{z}) \pi(\mathbf{z}) d\mathbf{z}.$$

Notice that the matrix $\frac{1}{2} I + \frac{\epsilon v}{\Delta x} F^{-1}$ is positive definite, so the above equations are solvable. For $\epsilon \ll 1$, (5.17) is an $O(\epsilon^2)$ approximation to (5.15) and (5.16). Boundary values for $\hat{\mathbf{j}}$ can be obtained by using approximation to (5.16)

$$\begin{aligned}
\hat{\mathbf{j}}_{-1} &= -2F^{-1}(v) \Big(\frac{v}{3\Delta x}(\hat{\mathbf{r}}_2 - \hat{\mathbf{r}}_{-1}) - E(x_{1/2}) \nabla_v \hat{\mathbf{r}}(x_{1/2}) \Big) - \hat{\mathbf{j}}_2, \\
\hat{\mathbf{j}}_0 &= -2F^{-1}(v) \Big(\frac{v}{\Delta x}(\hat{\mathbf{r}}_1 - \hat{\mathbf{r}}_0) - E(x_{1/2}) \nabla_v \hat{\mathbf{r}}(x_{1/2}) \Big) - \hat{\mathbf{j}}_1.
\end{aligned}$$

The boundary condition at x_R is treated similarly.

5.4 The stochastic collocation method

For numerical comparison, the stochastic collocation (SC) method will be employed. Let $\{\mathbf{z}^{(j)}\}_{j=1}^{N_s} \subset I_{\mathbf{z}}$ be the set of collocation nodes, N_s the number of samples. For each fixed individual sample $\mathbf{z}^{(j)}$, $j = 1, \ldots, N_s$, one applies the AP scheme to the deterministic equations as in [8], obtains the solution ensemble $u_j(t, x, v) = u(t, x, v, \mathbf{z}^{(j)})$, then adopts an interpolation approach to construct a gPC approximation, such as

$$u(t, x, v, \mathbf{z}) = \sum_{j=1}^{N_s} u_j(t, x, v) l_j(\mathbf{z}),$$

where $l_j(\mathbf{z})$ depends on the construction method. The Lagrange interpolation method is used here by choosing $l_j(\mathbf{z}^{(i)}) = \delta_{ij}$. This is straightforward and brings no coding difficulty. An overview of such stochastic collocation method can be found in [18]. In contrary to the collocation method which is not guaranteed to be AP, especially in high dimensional random space, the proposed gPC-SG method delivers a stochastic AP property in the whole random domain for any ϵ, regardless of the order of polynomial expansions.

6 Numerical Examples

In this section, several numerical tests are shown to illustrate the validity and effectiveness of our stochastic AP scheme. Randomness will be risen from collision kernel, initial data or boundary data. For simplicity, we will always assume the random variable $\mathbf{z}$ obeys a uniform distribution, defined on $[-1,1]^n$ with n up to 2, so the Legendre polynomial gPC basis are used.

Often one is only interested in the solution statistics, such as the mean and standard deviation of the macroscopic physical quantities. The macroscopic quantities ρ, $\mathbf{u}$ that stands for density and bulk velocity are defined by

$$\rho = \int_{\mathbb{R}} f(v)dv, \qquad \mathbf{u} = \frac{1}{\rho}\int_{\mathbb{R}} f(v)vdv.$$

Given the gPC coefficients $f_{\mathbf{k}}$ of f, the statistical mean, variance and standard deviation are

$$[f] \approx f_0, \qquad \mathrm{Var}[f] \approx \sum_{|\mathbf{k}|=1}^{M} f_{\mathbf{k}}^2, \qquad \mathcal{S}[f] = \sqrt{\sum_{|\mathbf{k}|=1}^{M} f_{\mathbf{k}}^2}.$$

To measure the difference in the solutions, we use the difference in mean and in standard deviation, with L^2 norm in x,

$$\mathcal{E}_{\mathrm{mean}}(t) = \left|\left|E[u^h] - E[u]\right|\right|_{L^2},$$
$$\mathcal{E}_{\mathrm{std}}(t) = \left|\left|\sigma[u^h] - \sigma[u]\right|\right|_{L^2},$$

where u^h and u are respectively numerical solutions of gPC-SG and the reference solution obtained by the high-order collocation method.

For the collocation method, with samples $\{\mathbf{z}^{(j)}\}$ and corresponding weights $\{w^{(j)}\}$ chosen from a quadrature rule, the integrals are approximated by

$$\int_{I_{\mathbf{z}}} f(t,x,v,\mathbf{z})\pi(\mathbf{z})d\mathbf{z} \approx \sum_{j=1}^{N_s} f(t,x,v,\mathbf{z}^{(j)})w^{(j)}.$$

6.1 Test I: A random collision kernel

$$x \in [0,1], \quad F_L(v) = M(v), \quad F_R(-v) = M(v), \quad G = 0,$$
$$\phi = \exp\left(-50\exp(1)(1/4-x)^2\right), \quad \epsilon = 0.5 \text{ or } \epsilon = 0.002.$$

The initial distribution is $f(x,v,t=0) = M(v)$. Let $\sigma(v,w,z) = 2+z \geq 0$ and $\sigma(v,w,z) = 2+0.3z_1+0.7z_2$ for the one and two-dimensional tests, respectively.

Lacking the analytic solutions, we use the high-order stochastic collocation method with 16 Legendre-Gauss quadrature points to compute a reference solution. In Fig. 1 and Fig. 2, we show the results by gPC-SG with $K = 4$, both small and large ϵ are chosen. In Fig. 3, we give the results of two dimensional random variables and $K = 4$. We compare our solutions with the reference solutions. The gPC solutions are represented by stars and reference solutions by solid lines with fine meshes, namely $\Delta x = 0.005$, $\Delta t = 2\times 10^{-5}$. One can clearly observe a satisfactory agreement between the solution of our gPC-SG scheme and the reference ones.

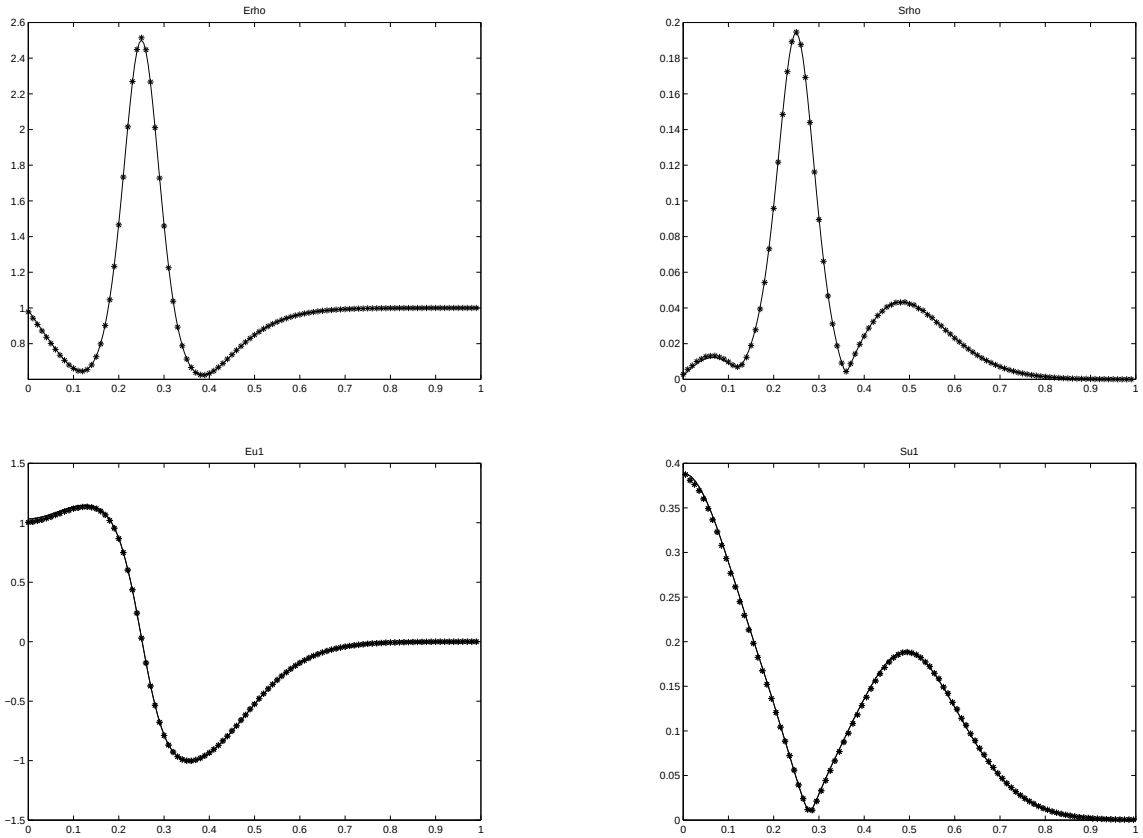


Figure 1: Test I with one-dimensional random variable. First row: mean and standard deviation of density ρ. Second row: mean and standard deviation of momentum density ρu (denoted by u_1), at $t = 0.03$, $\Delta x = 0.01$, $\Delta t = 5 \times 10^{-5}$, $\epsilon = 0.002$. Star: gPC-SG with $K = 4$. Solid line: reference solutions using collocation with $N_z = 16$.

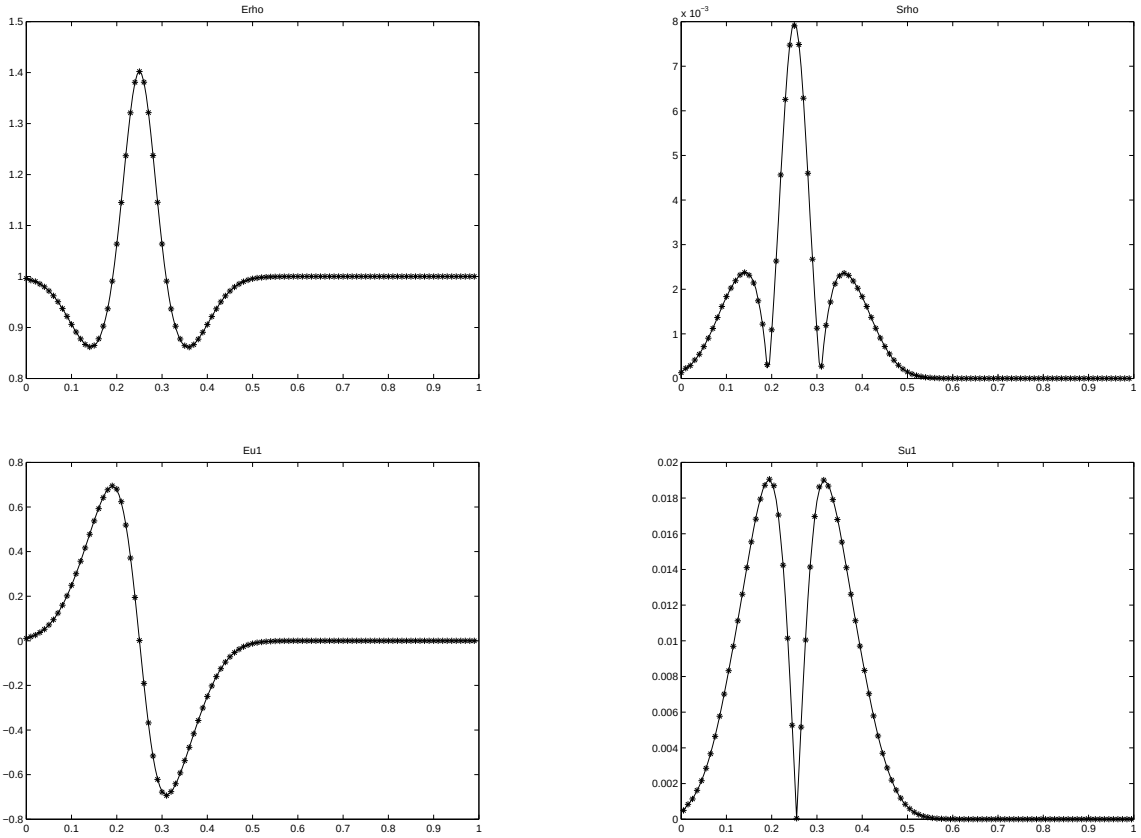


Figure 2: Test I with one-dimensional random variable. First row: mean and standard deviation of ρ. Second row: mean and standard deviation of ρu, at $t = 0.03$, $\Delta x = 0.01$, $\Delta t = 5 \times 10^{-5}$, $\epsilon = 0.5$. Star: gPC-SG with $K = 4$. Solid line: reference solutions using collocation with $N_z = 16$.

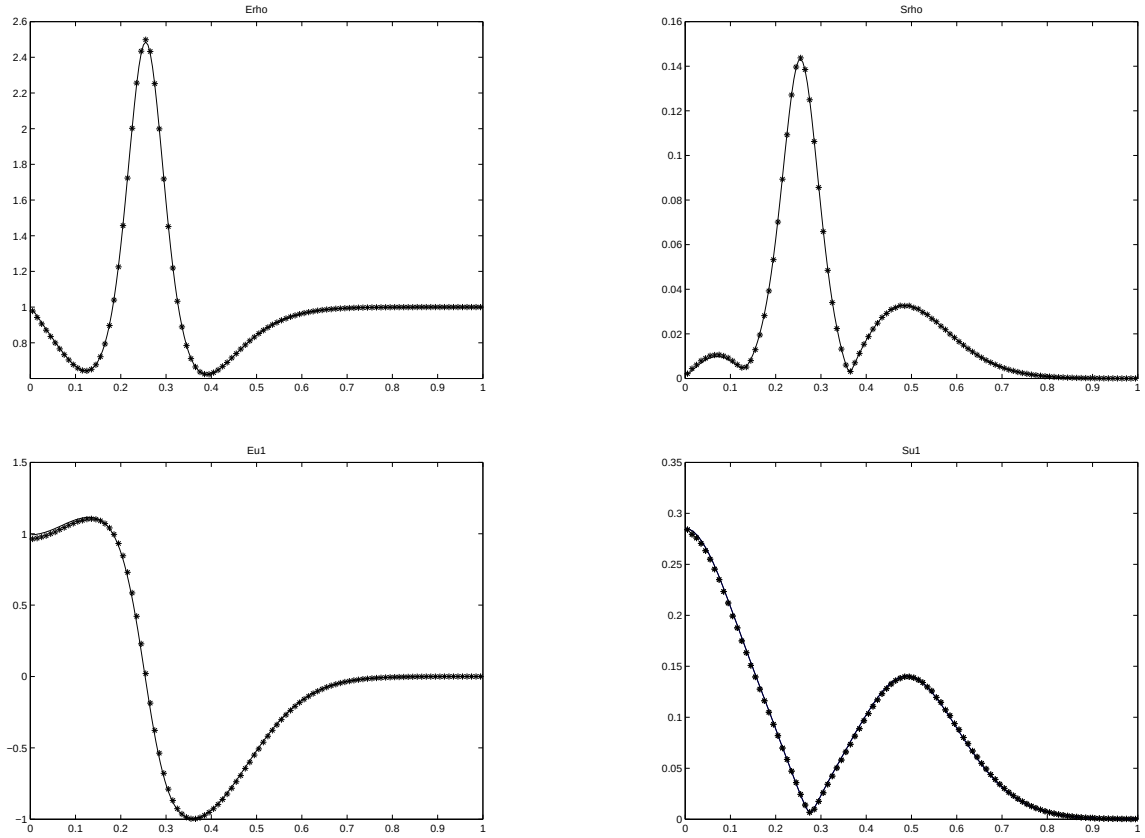


Figure 3: Test I with 2 dimensional random variables. First row: mean and standard deviation of ρ. Second row: mean and standard deviation of ρu, at $t = 0.03$, $\Delta x = 0.01$, $\Delta t = 5 \times 10^{-5}$, $\epsilon = 0.5$. Star: gPC-SG with $K = 4$. Solid line: reference solutions using collocation with $N_z = 16$.

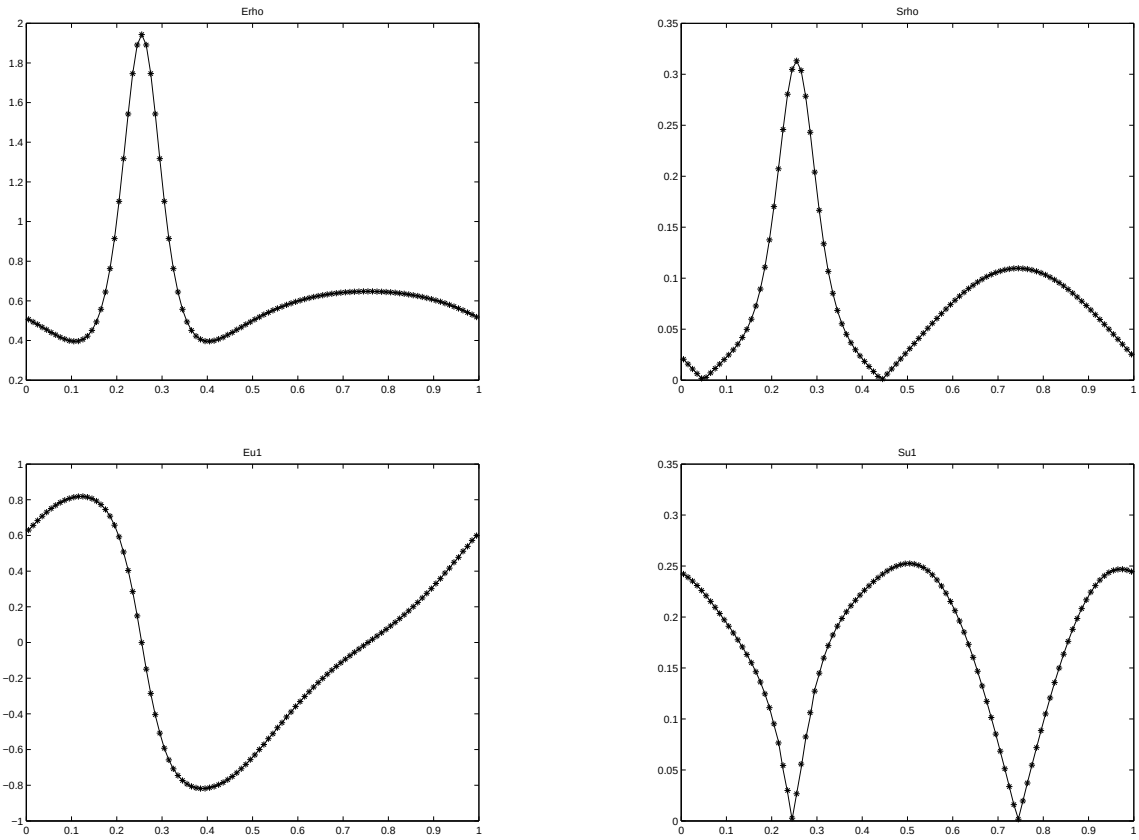


Figure 4: Test II. First row: mean and standard deviation of ρ. Second row: mean and standard deviation of ρu, at $t = 0.03$, $\Delta x = 0.01$, $\Delta t = 5 \times 10^{-5}$, $\epsilon = 0.002$. Star: gPC-SG with $K = 4$. Solid line: reference solutions using collocation with $N_z = 16$.

6.2 Test II: Random initial data

$$x \in [0,1], \quad \phi = \exp\left(-50\exp(1)(1/4-x)^2\right), \quad \sigma(v,w) = 1, \quad G = 0,$$
$$f^0(x,v,z) = \frac{\rho^0(x,z)}{\sqrt{\pi}} e^{-v^2}, \quad \rho^0(x,z) = \frac{2 + z\sin(2\pi x)}{3}, \quad \epsilon = 0.002.$$

ρ^0 is chosen to mimic the Karhunen-Loeve expansion. Periodic boundary condition is assumed in x. We show the 4-th order gPC solutions and reference solutions in Fig. 4. The two solutions are in good agreement.

6.3 Test III: Random boundary data

$$x \in [0,1], \quad F_L(v) = (1+0.5z)M(v), \quad F_R(-v) = (1+0.5z)M(v),$$
$$\sigma(\mathbf{v},\mathbf{w}) = 1, \quad G = 0, \quad \phi = \exp\left(-50\exp(1)(1/4-x)^2\right), \quad \epsilon = 0.002.$$

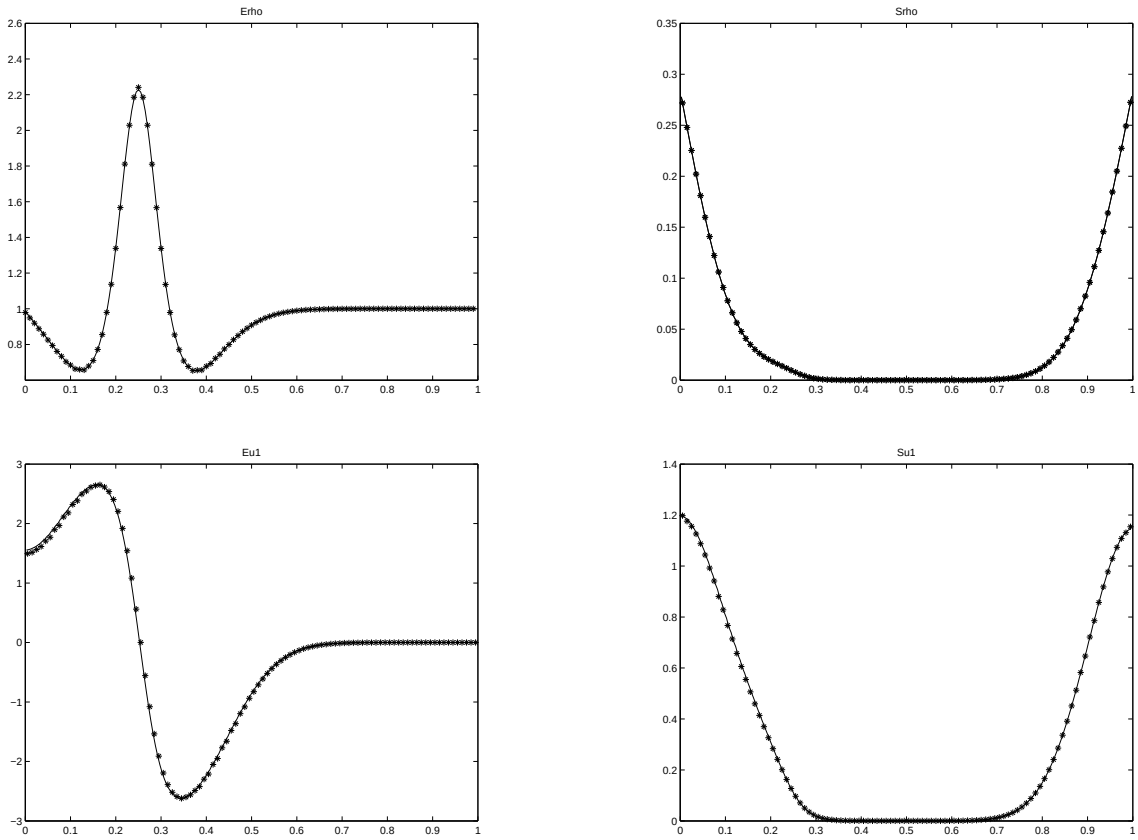


Figure 5: Test III. First row: mean and standard deviation of ρ. Second row: mean and standard deviation of ρu, at $t = 0.01$, $\Delta x = 0.01$, $\Delta t = 5 \times 10^{-5}$, $\epsilon = 0.002$. Star: gPC-SG with $K = 4$. Solid line: reference solutions using collocation with $N_z = 16$.

In this numerical test, we assume the inflow boundary conditions involved with random inputs, that is, a random fluctuation around the equilibrium at both boundaries. The 4-th order gPC solutions and reference solutions are shown in Fig. 5.

6.4 Test IV: The Boltzmann-Poisson equation with random parameters

$$x \in [0,1], \quad F_L(v) = M(v), \quad F_R(-v) = M(v),$$
$$f(x,v,t=0) = M(v), \quad G = 0, \quad \epsilon = 0.001.$$

We consider the case where the electric potential is given by the solution of a Poisson equation. The numerical data is chosen from [8].

$$\beta(z)\partial_{xx}\phi = \rho - c(x,z), \qquad \phi(0) = 0, \qquad \phi(1) = V,$$

where $\beta(z)$ is the scaled Debye length, V the applied bias voltage, and $c(x,z)$ is the doping profile. The electric field $E = -\partial_x \phi$ is obtained by central difference approximation. In this

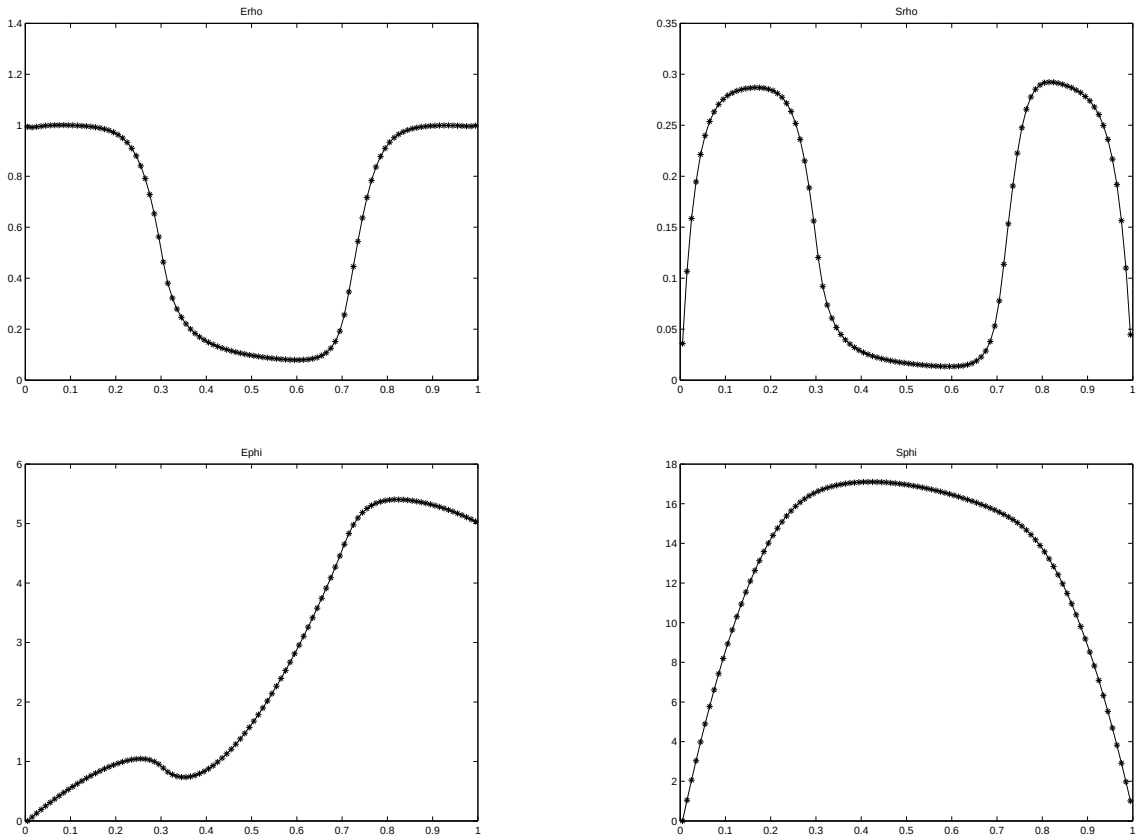


Figure 6: Test IV. First row: mean and standard deviation of ρ. Second row: mean and standard deviation of the potential ϕ, at $t = 0.05$, $\Delta x = 0.01$, $\Delta t = 0.0001$, $\epsilon = 0.001$. Star: gPC-SG with $K = 4$. Solid line: reference solutions using collocation with $N_z = 16$.

numerical test, we involve random perturbations on the parameters $\beta(z)$ and $c(x, z)$, namely $\beta(z) = 0.002(1 + 0.2z)$, and

$$c(x,z) = \left(1 - (1-M)\rho(0,t=0)\left[\tanh(\frac{x-x_1}{s}) - \tanh(\frac{x-x_2}{s})\right]\right)(1+0.5z),$$

with $s = 0.02$, $M = (1 - 0.001)/2$, $x_1 = 0.3$, $x_2 = 0.7$ and $V = 5$. The 4-th order gPC solutions and reference solutions are shown in Fig. 6, and are in good agreement.

6.5 Numerical tests for the AP property

In the following Fig. 7, 8 and 9, we use Test I with one-dimensional random variable, $\sigma(v, w, z) = 2 + z$. Fig. 7 compares the numerical solutions of the gPC-SG system for the limit diffusion equation given in (5.11) and that of the limit equation for the gPC system given in (5.10), both use the gPC approximations with $K = 4$. The analytic solution of two equations are unknown, so we use the forward Euler in time and central difference in space to obtain the approximated solutions. We can see the two sets of solutions in good agreement.

In Fig. 8, we compare mean and standard deviation of ρ between the 4-th order gPC solutions of our proposed scheme and that of the limit diffusion equation (5.10) as reference

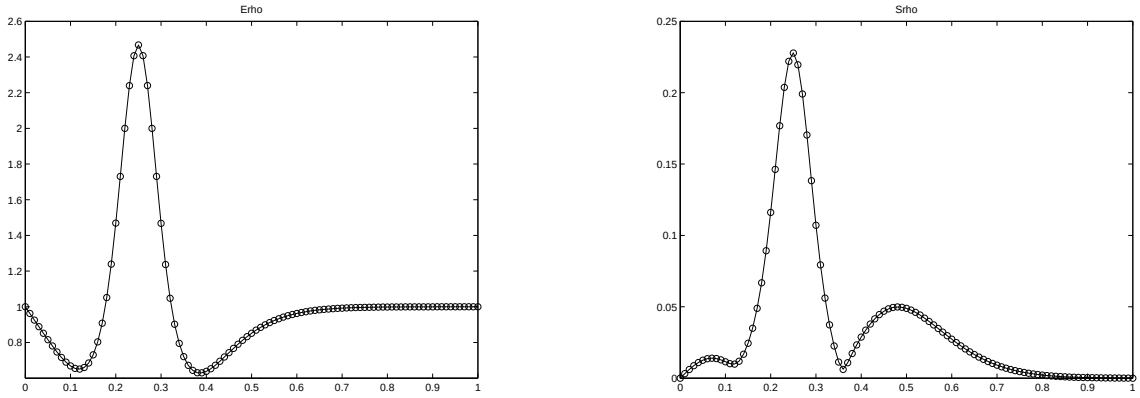


Figure 7: Test I with one-dimensional random variable. Mean and standard deviation of numerical solution ρ of the limiting diffusion equations, at $t = 0.025$. $\Delta x = 0.01$, $\Delta t = 5 \times 10^{-5}$. Solid line: numerical solution of the system (5.11). Circle: numerical solution of the system (5.10), both use gPC order $K = 4$.

solutions. One can observe a satisfactory agreement between the two solutions when ϵ is really small, which indicates that our proposed gPC scheme is stochastic AP.

In Fig. 9, the semilog plot of the errors of mean and standard deviation of physical quantities at $\epsilon = 10^{-5}$, $T = 0.03$, $\Delta t = 5 \times 10^{-5}$, with respect to increasing gPC orders are shown at different levels of grid resolutions, $\Delta x = 0.02$ (squares) and $\Delta x = 0.01$ (circles). We demonstrate a fast exponential convergence with respect to increasing K. The errors quickly saturate at modest gPC orders when $K = 4$, which implies that the errors arisen from the temporal and spatial discretization contribute more than that from the gPC expansion.

References

[1] E. F. A.K. Prinja and J. S. Warsa, *Stochastic methods for uncertainty quantification in radiation transport*, In International Conference on Mathematics, Computational Methods & Reactor Physics (M&C 2009), (2009).

[2] C. Cercignani, *The Boltzmann equation and its applications*, vol. 67 of Applied Mathematical Sciences, Springer-Verlag, New York, 1988.

[3] E. Gabetta, L. Pareschi, and G. Toscani, *Relaxation schemes for nonlinear kinetic equations*, SIAM J. Numer. Anal, 34 (1997).

[4] R. G. Ghanem and P. D. Spanos, *Stochastic Finite Elements: A Spectral Approach*, Springer-Verlag, New York, 1991.

[5] J. Hu and S. Jin, *A stochastic galerkin method for the boltzmann equation with uncertainty*, preprint, (2015).

[6] S. Jin, *Efficient asymptotic-preserving schemes for some multiscale kinetic equations*, SIAM J. Sci. Comput., 21(2) (1999), pp. 441–454 (electronic).

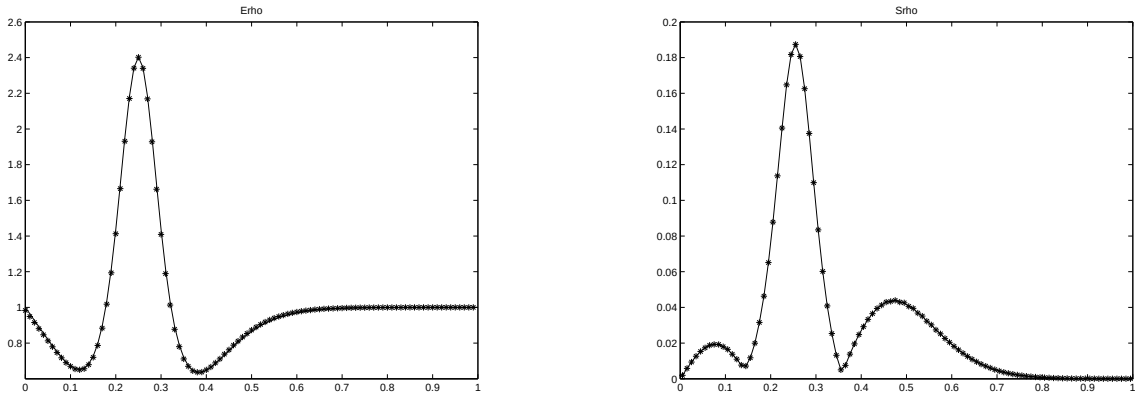
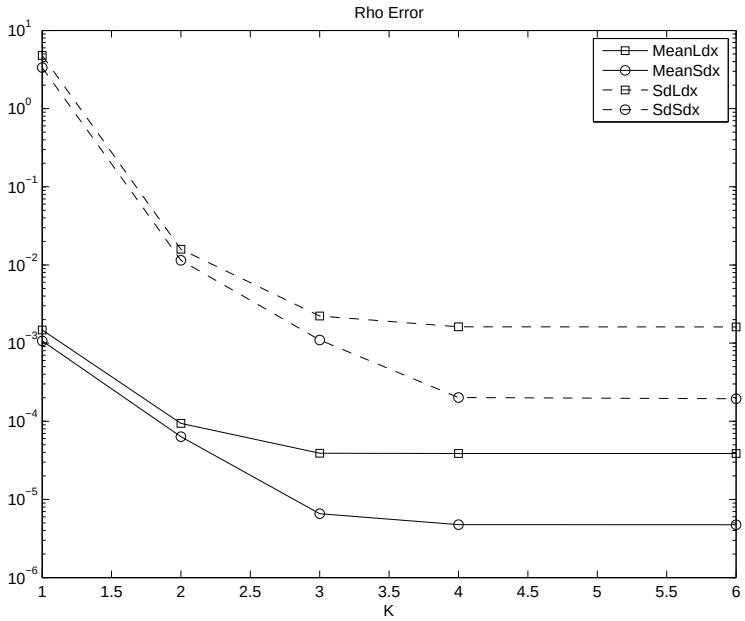


Figure 8: Test I with one-dimensional random variable. Mean and standard deviation of ρ, at $t = 0.025$, $\Delta x = 0.01$, $\Delta t = 5 \times 10^{-5}$, $\epsilon = 10^{-6}$. Solid line: numerical solution of the limit diffusion equation (5.10). Star: gPC-SG solution of the proposed scheme, both use gPC order $K = 4$.



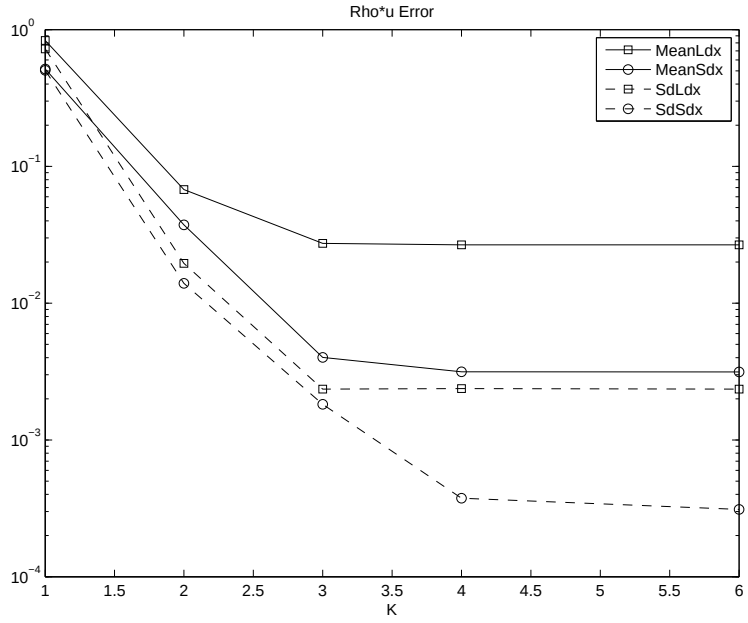


Figure 9: Test I with one-dimensional random variable. Errors of the mean (solid line) and standard deviation (dash line) of ρ and ρu with respect to gPC order K at $\epsilon = 10^{-5}$, $T = 0.03$ with $\Delta x = 0.02$ (squares), $\Delta x = 0.01$ (circles), $\Delta t = 5 \times 10^{-5}$.

[7] S. Jin, *Asymptotic preserving schemes for multiscale kinetic and hyperbolic equations: A review*, Rivista di Matematica della Universita di Parma, 3 (2012), pp. 177–216.

[8] S. Jin and L. Pareschi, *Discretization of the multiscale semiconductor Boltzmann equation by diffusive relaxation schemes*, J. Comput. Phys., 161 (2000), pp. 312–330.

[9] S. Jin, L. Pareschi, and G. Toscani, *Uniformly accurate diffusive relaxation schemes for multiscale transport equations*, SIAM J. Num., 38.3 (2000), pp. 913–936.

[10] S. Jin, D. Xiu, and X. Zhu, *Asymptotic-preserving methods for hyperbolic and transport equations with random inputs and diffusive scalings*, J. Comp. Phys, 289 (2014), pp. 35–52.

[11] A. Klar, *An asymptotic-induced scheme for nonstationary transport equations in the diffusive limit*, SIAM J. Numer. Anal., 35 (1998), pp. 1073–1094 (electronic).

[12] E. W. Larsen and J. E. Morel, *Asymptotic solutions of numerical transport problems in optically thick, diffusive regimes. ii*, J. Comput. Phys., 83(1) (1989), pp. 212–236.

[13] O. P. Le Maître and O. M. Knio, *Spectral Methods for Uncertainty Quantification*, Scientific Computation, Springer, New York, 2010. With applications to computational fluid dynamics.

[14] M. Lo'eve, *Probability Theory*, vol. fourth edition, Springer-Verlag, New York, 1977.

[15] P. Markowich, C. Ringhofer, and C. Schmeiser, *Semiconductor Equations*, Springer-Verlag, Wien-New York, 1989.

[16] E. Poupaud, *Diffusion approximation of the linear semiconductor boltzmann equation: Analysis of boundary layers*, Asymptot. Anal, 4,293 (1991), pp. 293–317.

[17] C. Ringhofer, C. Schmeiser, and A. Zwirchmayr, *Moment methods for the semiconductor boltzmann equation on bounded position domains*, SIAM J. Num. Anal., 39.3 (2001), pp. 1078–1095.

[18] D. Xiu, *Numerical Methods for Stochastic Computations*, Princeton University Press, Princeton, New Jersey, 2010.

[19] D. Xiu and J. Shen, *Efficient stochastic galerkin methods for random diffusion equations*, J. Comput. Phys, 228.2 (2009), pp. 226–281.

[20] A. Zwirchmayr and C. Schmeiser, *Convergence of moment methods for linear kinetic equations*, SIAM J. Num. Anal, 36 (1998), pp. 74–88.