Quantitative regularity estimates for compressible transport equations

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Preface

Those notes aim at presenting some recent quantitative estimates for transport equations with rough, *i.e.* non-smooth, velocity fields. Our final goal is to use those estimates to obtain new results on complex systems where the transport equation is coupled to other PDE's: A driving example is the compressible Navier-Stokes system.

We will therefore investigate in these notes the regularity of weak solutions ρ to the advection equation in conservative form

$$\partial_t \rho + \operatorname{div}(\rho\, u) = 0 \text{ in } (0,T) \times \Omega,$$

for a velocity field u that is not smooth with $u \in L^2([0,\ T],\ H^1(\Omega))$ as the typical example from Fluid Mechanics and where Ω is some smooth domain. There already exists a large body of literature around the well-posedness of such an equation with fields $u \in L^p([0,\ T],\ W^{1,p}(\Omega))$ where $p > 1$; see for example the surveys [4, 22]. What makes our investigation in these notes specific is that

- We require quantitative estimates of regularity. While linear advection equations are interesting in themselves, we ultimately want to consider general coupled systems for which quantitative bounds are easier to use.
- We want estimates that are compatible with strong compression effects, leading to large values of ρ, or rarefactions, leading to small values of ρ (or even vacuum in extreme cases). This means that we cannot impose a bound on $\operatorname{div} u$, upper or lower bounds on ρ. Instead we will only assume that $\rho \in L^\infty([0,\ T],\ L^p(\Omega))$ for some $p > 1$.

As we will see, there now exist several type of quantitative estimates that satisfy our first constraint. But many of those are not obviously compatible with our second constraint, so that we will only introduce our main such estimate in Chapter 2.

Because compression or rarefaction plays a strong role here, the conservative advection equation may have a different behavior from the advective form

$$\partial_t \phi + u \cdot \nabla \phi = 0.$$

The duality between those two equations and their appropriate combination will play an important role in our calculations. Finally we should mention here that most of the material and ideas presented here are valid for many type of spatial domains Ω: Either $\Omega = \mathbb{R}^d$ with appropriate decay at infinity or Ω a smooth bounded domain with appropriate boundary conditions. Because we want to focus on the main ideas behind the method, we will however work in the torus Π^d for simplicity.

Chapter 1

Lagrangian approaches

This chapter is devoted to the study of the trajectories of ODE's flows. We of course hope to derive the regularity of the solution to advection equations from the regularity of the trajectories, through the method of characteristics. For this reason, we consider all possible trajectories and consider the flow

$$\frac{d}{dt}X(t,x,s) = u(t, X(t,x,s)), \quad X(t=s,x,s) = x \in \Pi^d. \tag{1.1}$$

Our perspective here favors Eulerian approaches as they are easier to use when transport equations are coupled to other PDE's. But in other settings there can be many advantages to direct Lagrangian methods, not least of all the simplicity of the formulation. For reader's convenience, we recall some researchers (and corresponding dates) who have obtain important results on the subject: Lipschitz (1868), Peano (1886), Lindenhöf (1894), Osgood (1900), Nagumo (1926), Filippov (1960), Di Perna – Lions (1989), Ambrosio (2004), Crippa - De Lellis (2008) and others.

1.1 The Cauchy-Lipschitz theory

We start with the best known approach to well-posedness of ODE's and advection equations whose main result can be summarized by

Theorem 1.1.1. *Assume that* $u \in L^\infty([0,\ T],\ W^{1,\infty}(\Pi^d))$*, then there exists a unique solution* $X \in W^{1,\infty}([0,\ T] \times \Pi^d \times [0,\ T])$ *to* (1.1) *which satisfies*

$$|X(t,x,s) - X(t,y,s)| \le |x-y| \exp \int_{[s,\ t]} \|\nabla_x u(r,.)\|_{L^\infty(\Pi^d)}\, dr. \tag{1.2}$$

Moreover the map $x \longrightarrow X(t,x,s)$ *is an homeomorphism of* Π^d *for any fixed* t *and* s*, with*

$$X(t, X(s,x,r), s) = X(t,x,r), \quad \textit{in particular} \quad X(t, X(s,x,t), s) = x. \tag{1.3}$$

We do not give the proof of this theorem which is already well-known (even for $u \in L^1(0,T;W^{1,\infty}(\Pi^d))$, but we emphasize that the main point is to derive estimate (1.2) through the use of Gronwall lemma and the well-known inequality

$$|u(t,x)-u(t,y)| \leq \|\nabla u\|_{L^\infty([0,\ T]\times\Pi^d)}\,|x-y|. \tag{1.4}$$

We also observe that since we work on the torus, we are able to bypass all the discussion about trajectories going to infinity and the need for maximal solutions.

The method of characteristics allows to translate such regularity on the solution to advection equations as per

Theorem 1.1.2. *Assume that $u \in L^\infty([0,\ T],\ W^{1,\infty}(\Pi^d))$ and that $\phi^0 \in L^1(\Pi^d)$, then there exists a unique solution in the sense of distribution to*

$$\partial_t \phi + u\cdot\nabla\phi = 0, \quad \phi|_{t=0} = \phi^0, \tag{1.5}$$

which is given by

$$\phi(t,x) = \phi^0(X(0,x,t)).$$

In addition if $\phi^0 \in W^{s,p}(\Pi^d)$ then

$$\|\phi(t,.)\|_{W^{s,p}(\Pi^d)} \leq \|\phi(t,.)\|_{W^{s,p}(\Pi^d)}\, \exp\int_{[0,\ t]} \|\nabla_x u(r,.)\|_{L^\infty(\Pi^d)}\, dr.$$

We again skip the proof which is straightforward using Th. 1.1.1, and in particular Eq. (1.3). If one instead wishes to solve the conservative form of the transport equation, it is necessary to look at the Jacobian of the map X. Define hence

$$J(t,x,s) = \det \nabla_x X(t,x,s).$$

Then this Jacobian solves the ODE

$$\frac{d}{dt} JX(t,x,s) = J(t,x,s)\,\mathrm{div}\, u(t,X(t,x,s)), \tag{1.6}$$

which is well posed in this theory as $\mathrm{div}\, u$ is bounded. One then has

Corollary 1.1.3. *Assume that $u \in L^\infty([0,\ T],\ W^{1,\infty}(\Pi^d))$ and that $\phi^0 \in L^1(\Pi^d)$, then there exists a unique solution in the sense of distribution to*

$$\partial_t \rho + div\,(\rho\, u) = 0, \quad \rho|_{t=0} = \rho^0, \tag{1.7}$$

which is given by

$$\rho(t,x) = \frac{\rho^0(X(0,x,t))}{J(t,X(0,x,t))}.$$

While we have well-posedness under the same condition, one immediately sees that the regularity of the solution ρ also requires the corresponding regularity of $\mathrm{div}\, u$. This will be a recurring theme as the regularity for the convective or conservative form (1.7) will be consistently more difficult to obtain.

1.2 Lagrangian estimates for $u \in L^1(0,T;W^{1,p}(\Pi^d))$

It is possible to slightly extend the Gronwall's like estimates in the previous section, for example to log-Lipschitz velocity field which is critical for the uniqueness theory of 2d-Euler. But the first results on well-posedness for velocity fields in Sobolev spaces were obtained in [24] in an Eulerian framework that we will present in the next chapter.

Instead a corresponding Lagrangian approach was only introduced much later in [21]. In addition, [21] also provided the first explicit regularity estimates when $u \in W^{1,p}$, which makes it especially relevant for our purpose. The approach introduced in [21] has proved to be very fruitful with now many extensions. We only quote a few examples: [8] and [10] are concerned with velocity field u that are obtained from a singular integral or the Riesz transform of a measure; [18] and [33] apply to special Hamiltonian dynamics similar to Newton's second law and use this specific structure to require less than one derivative on u.

Let us from now on assume that $u \in L^1([0,\ T],\ W^{1,p}(\Pi^d))$ with $p > 1$. The first question is in what sense we can solve Eq. (1.1) as $u(t,x)$ may not be defined at every point x. We hence rely on some a priori estimates on the flow, namely we assume that for some exponent q with $1/p + 1/q \leq 1$, the Jacobian of the transform $x \longrightarrow X(t,x,s)$ is bounded in L^p in the sense that for any $\psi \in C^\infty(\Pi^d)$

$$\begin{aligned}&\int_{\Pi^d} \psi(X(t,x,s))\,dx = \int_{\Pi^d} \psi(x)\,w(t,x,s)\,dx,\\ &\text{with} \sup_{s\in[0,\ T]}\ \sup_{t\in[0,\ s]} \|w(t,.,s)\|_{L^q(\Pi^d)} = L < \infty.\end{aligned} \tag{1.8}$$

The function w can be interpreted as the law of the random variable $X(t,x,s)$ and would correspond to the previous $1/J(t,X(s,x,t),s)$. But it may not always be calculated directly like that as our flow may not be differentiable. In particular and contrary to the Lipschitz case, w may in fact vanish over large sets.

The reason for (1.8) will become apparent in the next section as it corresponds to natural L^q estimates on a solution ρ to (1.7). The original result in [21] instead assumed that w is bounded from below and from above. This allows to obtain additional properties from the regularity we present here, such as the full reversibility of the flow.

With (1.8), we can now define our notion of solution: $X(t,x,s)$ solves (1.1) iff for all test function $\psi \in C^\infty([0,s]\times\Pi^d)$,

$$\begin{aligned}&\int_{\Pi^d} \psi(t,X(t,x,s))\,dx = \int_{\Pi^d} \psi(s,x)\,dx\\ &-\int_t^s \int_{\Pi^d} \left(\partial_t\psi(r,X(t,x,s)) + u(r,X(r,x,s))\cdot\nabla_x\psi(r,X(r,x,s))\right)\,dx\,dr.\end{aligned} \tag{1.9}$$

The regularity estimate obtained in [21] reads

Theorem 1.2.1 ([21]). *Assume that $u \in L^1([0,\ T],\ W^{1,p}(\Pi^d))$ for some $1 < p < \infty$. Consider any solution $X(t,x,s)$ to* (1.1) *in the sense of* (1.9) *which also satisfies* (1.8) *with $1/p + 1/q \leq 1$. Then there exists a constant C depending only on the dimension s.t. for any $\omega \in \mathbb{S}^{d-1}$ and any h*

$$\int_{\Pi^d} \log\left(1 + \frac{|X(t,x,s) - X(t,x+h\,\omega,s)|}{h}\right) dx \leq C + C\,L \int_{[t,s]} \|\nabla u(r,.)\|_{L^p(\Pi^d)}\, dr.$$

Proof. Denote for simplicity

$$Q_h(t) = \int_{\Pi^d} \log\left(1 + \frac{|X(t,x,s) - X(t,x+h\,\omega,s)|}{h}\right) dx.$$

From the definition of a solution, one directly obtains that

$$Q_h(t) \leq Q_h(s) + \int_{[t,\ s]} \int_{\Pi^d} \frac{|u(r,X(r,x,s)) - u(r,X(r,x+h\,\omega,s))|}{h + |X(t,x,s) - X(t,x+h\,\omega,s)|}\, dx\, dr.$$

The key is here instead of using Lipschitz estimate to use the more precise inequality: There exists a constant C depending only on the dimension s.t. for any $u \in BV(\Pi^d)$

$$|u(x) - u(y)| \leq C\,|x-y|\,(M\,|\nabla u|(x) + M\,|\nabla u|(y)), \qquad (1.10)$$

where $M\,f$ is the maximal function of f

$$M\,f(x) = \sup_r \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\, dy.$$

We will later give a proof of (1.10) as we will require more precise estimates. Assuming it for the time being (one can also see [45]), this leads to

$$\begin{aligned} Q_h(t) &\leq Q_h(s) + C \int_t^s \int_{\Pi^d} (M\,|\nabla u(r,X(r,x,s))| + M\,|\nabla u(r,X(r,x+h\,\omega,s))|)\, dx\, dr \\ &= Q_h(s) + 2\,C \int_t^s \int_{\Pi^d} M\,|\nabla u(r,X(r,x,s))|\, dx\, dr. \end{aligned}$$

Using now (1.8), we obtain the precise intermediary estimate

$$Q_h(t) \leq Q_h(s) + 2\,C \int_t^s \int_{\Pi^d} M\,|\nabla u(r,x)|\, w(r,x,s)\, dx\, dr. \qquad (1.11)$$

By Hölder estimate, this implies

$$Q_h(t) \leq Q_h(s) + 2\,C \sup_{s\in[0,\ T]} \sup_{t\leq s} \|w(r,.,s)\|_{L^q} \int_t^s \int_{\Pi^d} \|M\,|\nabla u(r,.)\|_{L^p}\, dr,$$

which allows us to conclude the proof by recalling that the maximal function is bounded on L^p for $p > 1$, that is for some constant C depending only on d

$$\|M\,f\|_{L^p(\Pi^d)} \leq C\,\|f\|_{L^p(\Pi^d)}.$$

□

Compactness. While its proof is relatively straightforward, the regularity that is provided by Th. 1.2.1 may not be very clear at first. For example does it even imply compactness in L^1? From Rellich criterion an easy way to check for compactness and to measure regularity is to study for a given smooth convolution kernel K

$$\int_{\Pi^d} |X(t,x,s) - K_h \star X(t,.,s)|\,dx,$$

where as usual $K_h(x) = h^{-d}\,K(x/h)$. With a simple bound and the use of spherical coordinates, one may bound

$$\begin{aligned}&\int_{\Pi^d} |X(t,x,s) - K_h \star X(t,.,s)|\,dx\\ &\qquad\leq \int\int_{\mathbb{S}^{d-1}}\int_{\Pi^d} |X(t,x,s) - X(t,x+h\,r\,\omega,s)|\,dx\,K(r\,\omega)\,d\omega\,r^{d-1}\,dr.\end{aligned} \tag{1.12}$$

For any $t,\ s,\ h,\ r,\ \omega$, denote $I = \{x \in \Pi^d\,,\ |X(t,x,s) - X(t,x+h\,r\,\omega,s)\ \geq Rhr\}$, then

$$\begin{aligned}\int_{\Pi^d} \log\left(1+\frac{|X(t,x,s)-X(t,x+h\,r\,\omega,s)|}{h\,r}\right)dx &\geq \int_I \log\left(1+\frac{|X(t,x,s)-X(t,x+h\,r\,\omega,s)|}{h\,r}\right)dx\\ &\geq |I|\log(1+R)\end{aligned}$$

and therefore using Th. 1.2.1

$$|I| = |\{x\,,\ |X(t,x,s) - X(t,x+h\,r\,\omega,s)| \geq R\,h\,r\}| \leq \frac{C\,L\,\|\nabla u\|_{L^1([0,\ T],\ L^p(\Pi^d))} + C}{\log(1+R)}$$

Therefore wrtting $\Pi^d = I \cup (\Pi^d \setminus \bar{I})$ and using (1.12):

$$\int_{\Pi^d} |X(t,x,s) - K_h \star X(t,.,s)|\,dx \leq \frac{CL\|\nabla u\|_{L^1([0,\ T],\ L^p(\Pi^d))} + C}{\log(1+R)} + C_K\,R\,h,$$

where $C_K = \int |z|\,K(z)\,dz$. By optimizing in R, we finally obtain

$$\begin{aligned}&\int_{\Pi^d} |X(t,x,s) - K_h \star X(t,.,s)|\,dx\\ &\qquad\leq \frac{(C_K + C)\,L\,\|\nabla u\|_{L^1([0,\ T],\ L^p(\Pi^d))} + (C_K + C)}{\log 1/h},\end{aligned} \tag{1.13}$$

which proves that we control compactness through what is essentially a log of a derivative on X. This regularity may now be translated as a regularity on the transport equation in advective form

Corollary 1.2.2. *Assume that $u \in L^1([0,\ T],\ W^{1,p}(\Pi^d))$ for some $1 < p < \infty$. Assume that there exists a solution $X(t,x,s)$ to (1.1) in the sense of (1.9) which also satisfies (1.8) with $1/p + 1/q \leq 1$. Then for any $\phi^0 \in L^1(\Pi^d)$, there exists a solution ϕ to Eq. (1.5) given by*

$$\phi(t,x) = \phi^0(X(0,x,t)).$$

Moreover if $\phi^0 \in W^{s,q}$ for some $s > 0$ and $q \geq 1$ then

$$\|\phi(t,.) - K_h \star \phi(t,.)\|_{L^1} \leq \frac{C\, L|\nabla u\|_{L^1([0,\ T],\ L^p(\Pi^d))} + C}{\log 1/h},$$

for some C depending only on moments of K, s and q.

As before we only obtain directly the advective equation. Obtaining the conservative form would require to also solve the differential equation on the Jacobian and derive regularity from it which is not obvious in this framework.

1.3 A Eulerian formulation

A very natural question following from the previous analysis is whether one can find a Eulerian formulation of those Lagrangian estimates. One could think for example of trying Wasserstein distances; we will not pursue this idea here but refer for example to [44]. Instead here we will interpret the proof of Th. 1.2.1 as identifying the "good" trajectories where the flow has some regularity and then proving through sort of equivalent of (1.8) that those good trajectories have large probability.

The tracking of good trajectories may be done through an auxiliary equation on an appropriate weight $w(t,x)$ through

$$\partial_t w + u \cdot \nabla w = -\lambda\, M\, |\nabla u|\, w, \quad w|_{t=0} = 1. \tag{1.14}$$

Then one may prove as a first step

Proposition 1.3.1. *Assume that ϕ is a renormalized solution to the transport equation in advective form, Eq. (1.5). Then if w solves Eq. (1.14) with λ large enough, one has that for any $k > 0$, any h*

$$\int_{\Pi^{2\,d}} \frac{|\phi(t,x) - \phi(t,y)|}{(h + |x-y|)^k}\, w(t,x)\, w(t,y)\, dx\, dy \leq \int_{\Pi^{2\,d}} \frac{|\phi^0(x) - \phi^0(y)|}{(h + |x-y|)^k}\, dx\, dy.$$

Remark 1.3.2. We will define precisely what is meant by renormalized solutions in the next chapter. At this time, it should be interpreted as allowing to perform similar calculations as if u was smooth. Even in such a case, the proposition could for example be used to consider a sequence of solutions ϕ_n for a sequence of regularized velocity fields u_n. Quantitative regularity estimates would then be used to derive compactness.

Remark 1.3.3. Without weights, a quantity like

$$\int_{\Pi^{2\,d}} \frac{|\phi(t,x)-\phi(t,y)|}{(h+|x-y|)^k}\,dx\,dy$$

would actually control a Besov regularity of ϕ at order $k-d$ (and hence Sobolev regularity at any lower order). Unfortunately such regularity cannot hold for solutions to (1.5) with rough velocity fields, as the examples in [2] and [32].

Proof. Formal calculations show that

$$\partial_t|\phi(t,x)-\phi(t,y)| + u(t,x)\cdot\nabla_x|\phi(t,x)-\phi(t,y)| + u(t,y)\cdot\nabla_y|\phi(t,x)-\phi(t,y)| = 0.$$

Such an equality will again be justified in the next chapter. Hence still formally

$$\begin{aligned}\partial_t(|\phi(t,x)-\phi(t,y)|\,w(t,x)\,w(t,y)) + u(t,x)\cdot\nabla_x(|\phi(t,x)-\phi(t,y)|\,w(t,x)\,w(t,y))\\ + u(t,y)\cdot\nabla_y(|\phi(t,x)-\phi(t,y)|\,w(t,x)\,w(t,y))\\ = -\lambda\,(M\,|\nabla u|(t,x) + M\,|\nabla u|(t,y))\,|\phi(t,x)-\phi(t,y)|\,w(t,x)\,w(t,y).\end{aligned}$$

Multiplying by $(h+|x-y|)^k$ and integrating by parts yields

$$\begin{aligned}&\frac{d}{dt}\int_{\Pi^{2\,d}} \frac{|\phi(t,x)-\phi(t,y)|}{(h+|x-y|)^k}\,w(t,x)\,w(t,y)\,dx\,dy\\ &= k\int_{\Pi^{2\,d}} \frac{|\phi(t,x)-\phi(t,y)|}{(h+|x-y|)^{k+1}}\,w(t,x)\,w(t,y)\,(u(t,x)-u(t,y))\cdot\frac{x-y}{|x-y|}\,dx\,dy\\ &\quad+\int_{\Pi^{2\,d}} \frac{|\phi(t,x)-\phi(t,y)|}{(h+|x-y|)^k}\,w(t,x)\,w(t,y)\,(\operatorname{div} u(t,x)+\operatorname{div} u(t,y))\,dx\,dy\\ &\quad-\lambda\int_{\Pi^{2\,d}} \frac{|\phi(t,x)-\phi(t,y)|}{(h+|x-y|)^k}\,w(t,x)\,w(t,y)\,(M\,|\nabla u|(t,x)+M\,|\nabla u|(t,y))\,dx\,dy.\end{aligned}$$

Using inequality (1.4) and the symmetry in x and y, one finally deduces

$$\begin{aligned}&\frac{d}{dt}\int_{\Pi^{2\,d}} \frac{|\phi(t,x)-\phi(t,y)|}{(h+|x-y|)^k}\,w(t,x)\,w(t,y)\,dx\,dy\\ &\le \int_{\Pi^{2\,d}} \frac{|\phi(t,x)-\phi(t,y)|}{(h+|x-y|)^k}\,w(t,x)\,w(t,y)\,(\operatorname{div} u(t,x)-(\lambda-k)\,M\,|\nabla u|(t,x))\,dx\,dy.\end{aligned}$$

Since $M\,|\nabla u|(t,x) \ge |\nabla u(t,x)| \ge \operatorname{div} u(t,x)$, taking $\lambda \ge k+1$ gives the result. □

Prop. 1.3.1 is in itself insufficient as obviously if w vanishes everywhere for instance then it contains no information. It is hence necessary to control the set where w is small, as per

Lemma 1.3.4. *Assume that $u \in L^1([0,\ T],\ W^{1,p}(\Pi^{2d}))$ with $p > 1$ and that there exists a renormalized solution $\rho \in L^\infty([0,\ T],\ L^q(\Pi^{2d}))$ to Eq. (1.7) with $1/p + 1/q \le 1$, then*

$$\int_{\Pi^d} |\log w(t,x)|\,\rho(t,x)\,dx \le C_d\,\lambda\,\|u\|_{L^1([0,\ T],\ W^{1,p}(\Pi^{2d}))}\,\|\rho\|_{L^\infty([0,\ T],\ L^q(\Pi^{2d}))}.$$

Proof. Note that since w solves (1.14) then $w \leq 1$ *a.e.*. Hence $|\log w| = -\log w$ and

$$\partial_t |\log w(t,x)| + u\cdot\nabla_x |\log w(t,x)| = \lambda\, M\,|\nabla u|(t,x).$$

Multiplying by u and integrating yields

$$\int_{\Pi^d} |\log w(t,x)|\,\rho(t,x)\,dx \leq \lambda \int_0^t \int_{\Pi^d} M\,|\nabla u|(s,x)\,\rho(s,x)\,dx\,ds,$$

or by Hölder estimate

$$\int_{\Pi^d} |\log w(t,x)|\,\rho(t,x)\,dx \leq \lambda\,\|M\,|\nabla u|\|_{L^1([0,\ T],\ L^p(\Pi^{2d}))}\,\|\rho\|_{L^\infty([0,\ T],\ L^q(\Pi^{2d}))}.$$

Using the fact that the maximal operator is continuous on L^p for $p>1$ concludes the proof. □

It is now relatively simple to combine Lemma 1.3.4 to Prop. 1.3.1 to obtain

Theorem 1.3.5. *Assume that $u \in L^1([0,\ T],\ W^{1,p}(\Pi^{2d}))$ with $p>1$ and that there exists a renormalized solution $\rho \in L^\infty([0,\ T],\ L^q(\Pi^{2d}))$ to Eq. (1.7) with $1/p+1/q \leq 1$. Consider any renormalized solution $\phi \in L^\infty([0,\ T],\ L^r(\Pi^d))$ then for any $\alpha > 0$*

$$\begin{aligned}\int_{\Pi^{2\,d}} &\frac{|\phi(t,x)-\phi(t,y)|}{(h+|x-y|)^k}\,1\wedge\rho(t,x)\,1\wedge\rho(t,y)\,dx\,dy\\ &\leq h^{-\alpha}\int_{\Pi^{2\,d}} \frac{|\phi^0(x)-\phi^0(y)|}{(h+|x-y|)^k}\,dx\,dy\\ &+ C\,\frac{\lambda\,h^{d-k}}{|\log h|^{1-1/r}}\,\|\phi\|_{L^\infty([0,\ T],\ L^r(\Pi^d))}\,\|u\|^{1-1/r}_{L^1([0,\ T],\ W^{1,p}(\Pi^{2d}))}\,\|\rho\|^{1-1/r}_{L^\infty([0,\ T],\ L^q(\Pi^{2d}))},\end{aligned}$$

for some constant C depending only on the dimension d and α.

Remark 1.3.6. The theorem provides compactness on ϕ where ρ does not vanish. For instance if $\rho(t,x) \geq \bar\rho > 0$ and $\phi^0 \in W^{\alpha,1}$, then we may deduce that

$$\int_{\Pi^d} |\phi(t,x) - K_h \star \phi(t,x)|\,dx \leq L\,|\log h|^{-1},$$

where L depends on the various norm and $K_h = C^{-1}\,(h+|x|)^{d-k}$ can be interpreted as a convolution kernel. We hence have in that case an equivalent of Corollary 1.2.2.

Remark 1.3.7. In general however, and contrary to Corollary 1.2.2, we only control the regularity of ϕ where $\rho > 0$. This is because the assumption that there exists a ρ in L^q is much weaker than the assumption on the Jacobian (1.8). In fact, translated in Eulerian framework, (1.8) is equivalent to asking that for any $t_0 \in [0,\ T]$, there exists $\rho_{t_o} \in L^\infty_t L^p_x$ solving

$$\partial_t \rho_{t_0} + \mathrm{div}\,(\rho_{t_0}\,u) = 0, \quad \rho_{t_0}|_{t=t_0} = 1.$$

Unfortunately in our applications, we will not have such a family of solutions but only one...

Proof. The first part of the proof would be to prove that there exists an appropriate weight $w(t,x)$ solving (1.14) and such that we may apply Prop. 1.3.1 and Lemma 1.3.4. We will however skip this argument at the time being before we go back to the existence question for transport equations in the next chapter.

The rest of the proof is a simple interpolation, by decomposing Π^{2d} into the set $\{x,\, y \mid w(t,x)>\eta,\ w(t,y)>\eta\}$ and the complementary set

$$\begin{aligned}\int_{\Pi^{2\,d}} &\frac{|\phi(t,x)-\phi(t,y)|}{(h+|x-y|)^k}\, 1\wedge\rho(t,x)\, 1\wedge\rho(t,y)\,dx\,dy\\ &\leq \frac{1}{\eta^2}\int_{\Pi^{2\,d}} \frac{|\phi(t,x)-\phi(t,y)|}{(h+|x-y|)^k}\, w(t,x)\,w(t,y)\,dx\,dy\\ &\quad+\int_{x,\,y\ \mid\ w(t,x)>\eta,\ \text{or}\ w(t,y)>\eta} \frac{|\phi(t,x)|+|\phi(t,y)|}{(h+|x-y|)^k}\, 1\wedge\rho(t,x)\, 1\wedge\rho(t,y)\,dx\,dy.\end{aligned}$$

By symmetry the last term is bounded by

$$\begin{aligned}\int_{x,\,y\ \mid\ w(t,x)\leq\eta,\ \text{or}\ w(t,y)\leq\eta} &\frac{|\phi(t,x)|+|\phi(t,y)|}{(h+|x-y|)^k}\, 1\wedge\rho(t,x)\, 1\wedge\rho(t,y)\,dx\,dy\\ &\leq C\,h^{d-k}\int_{x,\ w(t,x)\leq\eta} (|\phi(t,x)|+K_h\star|\phi|)\, 1\wedge\rho(t,x)\,dx,\end{aligned}$$

where $K_h(x)=C^{-1}\,h^{k-d}\,(h+|x|)^k$ with C s.t. $\|K_h\|_{L^1}=1$. By Hölder estimate

$$\begin{aligned}\int_{x,\ w(t,x)\leq\eta} &(|\phi(t,x)|+K_h\star|\phi|)\, 1\wedge\rho(t,x)\,dx\\ &\leq \|\phi\|_{L^\infty([0,\ T],\ L^r(\Pi^d))}\left(\int_{x,\ w(t,x)\leq\eta} 1\wedge\rho(t,x)\,dx\right)^{1-1/r}.\end{aligned}$$

Now

$$\begin{aligned}\int_{x,\ w(t,x)\leq\eta} 1\wedge\rho(t,x)\,dx &\leq \frac{1}{|\log\eta}\int_{x,\ w(t,x)\leq\eta} |\log w(t,x)|\,\rho(t,x)\,dx\\ &\leq C_d\,\frac{\lambda}{|\log\eta|}\,\|u\|_{L^1([0,\ T],\ W^{1,p}(\Pi^{2d}))}\,\|\rho\|_{L^\infty([0,\ T],\ L^q(\Pi^{2d}))},\end{aligned}$$

by using Lemma 1.3.4. Using now Prop. 1.3.1 and combining our estimates, we find

$$\begin{aligned}\int_{\Pi^{2\,d}} &\frac{|\phi(t,x)-\phi(t,y)|}{(h+|x-y|)^k}\, 1\wedge\rho(t,x)\, 1\wedge\rho(t,y)\,dx\,dy\\ &\leq \frac{1}{\eta^2}\int_{\Pi^{2\,d}} \frac{|\phi(t,x)-\phi(t,y)|}{(h+|x-y|)^k}\, w(t,x)\,w(t,y)\,dx\,dy\\ &+C\,\frac{\lambda^{1-1/r}\,h^{d-k}}{|\log\eta|^{1-1/r}}\,\|\phi\|_{L^\infty([0,\ T],\ L^r(\Pi^d))}\,\|u\|^{1-1/r}_{L^1([0,\ T],\ W^{1,p}(\Pi^{2d}))}\,\|\rho\|^{1-1/r}_{L^\infty([0,\ T],\ L^q(\Pi^{2d}))},\end{aligned}$$

which gives the desired result after taking $\eta = h^{\alpha/2}$. $\square$

We can develop almost the same estimates and theory for the conservative form, starting with

Proposition 1.3.8. *Assume that ρ is a renormalized solution to the transport equation in conservative form, Eq. (1.7). Then if w solves Eq. (1.14) with λ large enough, one has that for any $k > 0$, any h*

$$\int_{\Pi^{2\,d}} \frac{|\rho(t,x)-\rho(t,y)|}{(h+|x-y|)^k}\, w(t,x)\, w(t,y)\, dx\, dy \le \int_{\Pi^{2\,d}} \frac{|\rho^0(x)-\rho^0(y)|}{(h+|x-y|)^k}\, dx\, dy$$
$$+ \frac{1}{2}\int_0^t \int_{\Pi^{2\,d}} \frac{|div\, u(s,x) - div\, u(s,y)|}{(h+|x-y|)^k}\, (\rho(s,x)+\rho(s,y))\, dx\, dy\, ds.$$

Proof. The argument follows exactly the same steps as before, starting with the modified equation

$$\partial_t|\rho(t,x)-\rho(t,y)| + u(x)\cdot\nabla_x|\rho(t,x)-\rho(t,y)| + u(y)\cdot\nabla_y|\rho(t,x)-\rho(t,y)|$$
$$\le \frac{|\mathrm{div}\, u(t,x) - \mathrm{div}\, u(t,y)|}{2}\,(\rho(t,x)+\rho(t,y))$$
$$+ \frac{|\rho(t,x)-\rho(t,y)|}{2}\,(|\mathrm{div}\, u(t,x)| + |\mathrm{div}\, u(t,y)|.$$

Hence, we obtain

$$\frac{d}{dt}\int_{\Pi^{2\,d}} \frac{|\rho(t,x)-\rho(t,y)|}{(h+|x-y|)^k}\, w(t,x)\, w(t,y)\, dx\, dy$$
$$= k\int_{\Pi^{2\,d}} \frac{|\rho(t,x)-\rho(t,y)|}{(h+|x-y|)^{k+1}}\, w(t,x)\, w(t,y)\, (u(t,x)-u(t,y))\cdot\frac{x-y}{|x-y|}\, dx\, dy$$
$$+ 2\int_{\Pi^{2\,d}} \frac{|\rho(t,x)-\rho(t,y)|}{(h+|x-y|)^k}\, w(t,x)\, w(t,y)\, (|\mathrm{div}\, u(t,x)| + |\mathrm{div}\, u(t,y)|)\, dx\, dy$$
$$- \lambda\int_{\Pi^{2\,d}} \frac{|\rho(t,x)-\rho(t,y)|}{(h+|x-y|)^k}\, w(t,x)\, w(t,y)\, (M\,|\nabla u|(t,x) + M\,|\nabla u|(t,y))\, dx\, dy$$
$$+ \frac{1}{2}\int_{\Pi^{2\,d}} \frac{|\mathrm{div}\, u(t,x) - \mathrm{div}\, u(t,y)|}{(h+|x-y|)^k}\, (\rho(t,x)+\rho(t,y))\, dx\, dy.$$

The rest follows as before with only the last term remaining. $\square$

Lemma 1.3.4 does not need to be modified and thus from Prop. 1.3.8, we may deduce

Theorem 1.3.9. *Assume that $u \in L^1([0,\ T],\ W^{1,p}(\Pi^{2d}))$ with $p > 1$ and that there exists a renormalized solution $\rho \in L^\infty([0,\ T],\ L^q(\Pi^{2d}))$ to Eq. (1.7) with*

$1/p + 1/q \leq 1$. *Then for any* $\alpha > 0$

$$\begin{aligned}
&\int_{\Pi^{2\,d}} \frac{|\rho(t,x)-\rho(t,y)|}{(h+|x-y|)^k}\, 1\wedge\rho(t,x)\, 1\wedge\rho(t,y)\,dx\,dy\\
&\quad\leq h^{-\alpha}\int_{\Pi^{2\,d}} \frac{|\rho^0(x)-\rho^0(y)|}{(h+|x-y|)^k}\,dx\,dy\\
&\qquad+\frac{1}{2\,h^\alpha}\int_0^t\int_{\Pi^{2\,d}} \frac{|div\, u(s,x)-div\, u(s,y)|}{(h+|x-y|)^k}\,(\rho(t,x)+\rho(t,y))\,dx\,dy\,ds\\
&\qquad+C\,\frac{\lambda\, h^{d-k}}{|\log h|^{1-1/r}}\,\|u\|^{1-1/q}_{L^1([0,\ T],\ W^{1,p}(\Pi^{2d}))}\,\|\rho\|^{2-1/q}_{L^\infty([0,\ T],\ L^q(\Pi^{2d}))},
\end{aligned}$$

for some constant C *depending only on the dimension* d *and* α.

Compared to the result for the advective equation (1.5), this new estimate includes a term with $\operatorname{div} u(t,x) - \operatorname{div} u(t,y)$. As we have seen early on, it is natural that the regularity of ρ involve the corresponding regularity of $\operatorname{div} u$.

As before the regularity is obtained only where ρ does not vanish. However there is an added twist that shows grounds for some optimism here, as now ρ is the same function for weight and for the regularity.

So for instance if $\rho(t,x) = \rho(t,y) = 0$ then obviously one has as well that $\rho(t,x) - \rho(t,y) = 0$ and there is nothing to control.

Unfortunately this does not quite work: The problem occurs when only one of $\rho(t,x)$ or $\rho(t,y)$ vanishes (or is small). If $\rho(t,y) = 0$ then Theorem 1.3.9 does not provide any bound and therefore $\rho(t,x) - \rho(t,y)$ could well be large.

The problem is that we are using the products, $1\wedge\rho(t,x)\,1\wedge\rho(t,y)$ and earlier $w(t,x)\,w(t,y)$, as weights. Instead one would like to work with weights which only vanish if both $\rho(t,x)$ and $\rho(t,y)$ vanish; a good example is the sum

$$1\wedge\rho(t,x) + 1\wedge\rho(t,y), \quad w(t,x)+w(t,y).$$

Contrary to what it may first seem, this will impose major changes in our approach. Theorem 1.3.9 compares between them "good" trajectories and we would now have compare a "good" to a "bad" trajectory. This will require proving that there are not too many bad trajectories around a good one and forces to move away from Lagrangian approaches.

Chapter 2

Examples of Eulerian approaches: Renormalized solutions

We now start by reviewing the classical notion of renormalized solutions. Those provide the basic tools to obtain well posedness for the various equations (or auxiliary equations) and are hence useful to justify our formal calculations. By emphasizing the notion of commutator estimates central to Eulerian approaches, they also lead to the method presented at the end of this chapter which is finally able to answer our main question.

2.1 Basic notions of renormalized solutions

Renormalized solutions were introduced in the seminal [24]. This was the first result to obtain well posedness for transport equations with velocity fields in $W^{1,p}$. And whereas almost all previous contributions were based on the study of the characteristics, [24] introduced a purely Eulerian method from which one could deduce the properties of the ODE and the flow if so desired.

We recall here our conservative or continuity equation

$$\partial_t \rho + \operatorname{div}(\rho\, u) = 0, \quad \rho|_{t=0} = \rho^0, \tag{2.1}$$

and its dual advective form

$$\partial_t \phi + u \cdot \nabla \phi = 0, \quad \phi|_{t=0} = \phi^0. \tag{2.2}$$

Following the presentation of the theory given in [22], one first defines

Definition 2.1.1. A function $\rho \in L^\infty([0,\ T],\ L^q(\Pi^d))$ is a renormalized solution to Eq. (2.1), where $u \in L^1([0,\ T],\ L^p(\Pi^d))$ and $\operatorname{div} u \in L^1([0,\ T],\ L^p(\Pi^d))$ with $1/p + 1/q \leq 1$, if ρ is a weak solution and for any $\chi \in C^1 \cap W^{1,\infty}(\mathbb{R})$, one has in the sense of distributions

$$\partial_t \chi(\rho) + \operatorname{div}(\chi(\rho)\, u) = \operatorname{div}\ \mathrm{u}\,(\chi(\rho) - \rho\, \chi'(\rho)). \tag{2.3}$$

Remark that the various products ρu, $\chi(\rho)\, u$ from the assumed bounds on ρ, u and $\operatorname{div} u$ since $|\chi(\rho)| \leq C + C\,|\rho|$. Of course a similar definition could be introduced for the advective form (2.2).

Ideally for a given velocity field u, all weak solutions would automatically be renormalized, leading to the definition

Definition 2.1.2. Assume that $u \in L^1([0,\ T],\ L^p(\Pi^d))$, $\operatorname{div} u \in L^1([0,\ T],\ L^p(\Pi^d))$. The equation 2.1 is said to have the renormalization property for this particular u if any weak solution $\rho \in L^\infty([0,\ T],\ L^q(\Pi^d))$ with $1/p + 1/q \leq 1$ is renormalized.

Readers will immediately perceive the convenience of having renormalized solution as it easily allows to manipulate various non-linear quantities. However the key point is that a renormalized equation is essentially well-posed. Starting with uniqueness

Theorem 2.1.3 ([24]). *Assume $u \in L^1([0,\ T],\ L^p(\Pi^d))$, $\operatorname{div} u \in L^1([0,\ T],\ L^p(\Pi^d))$. Assume moreover that the equation 2.1 has the renormalization property for u. Then there exists at most one weak solution $\rho \in L^\infty([0,\ T],\ L^q(\Pi^d))$ with $1/p + 1/q \leq 1$ for a given ρ^0.*

Proof. Given two solutions ρ_1 and ρ_2 in $L^\infty([0,\ T],\ L^q(\Pi^d))$, we define $\rho = \rho_1 - \rho_2$. ρ is also a weak solution and hence a renormalized one.

Choose a sequence $\chi_n \in C^1 \cap W^{1,\infty}$ s.t. $\chi_n(\xi) \to |\xi|$ in L^∞. By applying the definition of renormalized solution to $\chi_n(\rho)$ and passing to the limit in n, one finds that in the sense of distributions

$$\partial_t |\rho| + \operatorname{div}(|\rho|\, u) = 0.$$

Let us now use the function constant and equal to 1 as test function; one has that

$$\frac{d}{dt} \int_{\Pi^d} |\rho(t,x)|\, dx = 0.$$

Since $\rho^0 = 0$, we conclude that $\rho(t,x) = 0$ for *a.e.* $t,\ x$. □

Existence can be obtained trivially but is a priori more demanding as it requires a L^∞ bound on the divergence

Theorem 2.1.4. *Assume $u \in L^1([0,\ T],\ L^p(\Pi^d))$ for $p < \infty$, and assume now that $\operatorname{div} u \in L^1([0,\ T],\ L^\infty(\Pi^d))$. Then for a given $\rho^0 \in L^q(\Pi^d)$ with $q > 1$ and $1/q + 1/p \leq 1$, there exists at least one weak solution $\rho \in L^\infty([0,\ T],\ L^q(\Pi^d))$ to Eq. (2.1) .*

Proof. We consider a sequence of smooth (for example Lipschitz) u_n which converges to u in $L^1([0,\ T],\ L^p(\Pi^d))$. Since u_n is smooth, the Cauchy-Lipschitz theory provides a sequence ρ_n of solutions to

$$\partial_t \rho_n + \operatorname{div}(\rho_n\, u_n) = 0, \quad \rho_n|_{t=0} = \rho^0.$$

Since $\operatorname{div} u \in L^1([0,\ T],\ L^\infty(\Pi^d))$, it is possible to choose the sequence u_n s.t.

$$\sup_n \|\operatorname{div} u\|_{L^1([0,\ T],\ L^\infty(\Pi^d))} < \infty.$$

On the other hand a direct calculation shows that

$$\|\rho_n(t,.)\|^q_{L^q(\Pi^d)} \le (q-1)\, \|\rho_n(t,.)\|^q_{L^q(\Pi^d)} \exp \|\operatorname{div} u\|_{L^1([0,\ T],\ L^\infty(\Pi^d))}.$$

Therefore ρ_n is uniformly bounded in $L^\infty([0,\ T],\ L^q(\Pi^d))$. We may hence extract a weak-* subsequence converging to $\rho \in L^\infty([0,\ T],\ L^q(\Pi^d))$.

Passing to the limit in every term, one obtains a weak solution to (2.1). □

This existence result does not use the renormalization property and it is natural to ask if it can be improved so that we may obtain strong convergence of the sequence of approximation. We give such an argument below based on using $\chi(\xi) = \xi \log \xi$ which forms the basis of the compactness method introduced in [37] for the compressible Navier-Stokes.

Theorem 2.1.5 ([24, 37]). *Consider a sequence u_n converging strongly to $u \in L^1([0,\ T],\ L^p(\Pi^d))$ for $p < \infty$, and assume moreover that $div\, u_n$ converges to $div\, u \in L^1([0,\ T],\ L^\infty(\Pi^d))$. Consider further any sequence ρ_n of renormalized solutions to*

$$\partial_t \rho_n + div\,(\rho_n\, u_n) = 0.$$

Assume that ρ_n is uniformly bounded in $\rho \in L^\infty([0,\ T],\ L^q(\Pi^d))$ with $q > 1$ and $1/q + 1/p \le 1$, that ρ_n converges weak- to ρ, that ρ_n^0 converges strongly to $\rho^0 \in L^q(\Pi^d)$ and that ρ is a renormalized solution to (2.1). Then ρ_n converges strongly to ρ in $L^1([0,\ T] \times \Pi^d)$.*

Proof. First of all remark that we may use $\chi(\xi) = \xi \log \xi$ in the definition of a renormalized solution even though $\chi \notin W^{1,\infty}$. Consider any velocity field $u \in L^1([0,\ T],\ L^p(\Pi^d))$ with $\operatorname{div} u \in L^1([0,\ T],\ L^p(\Pi^d))$, and a renormalized solution to (2.1), $\rho \in L^\infty([0,\ T],\ L^q(\Pi^d))$ with $1/p + 1/q < 1$.

Choose any sequence $\chi_k \in C^1 \cap W^{1,\infty}$ converges pointwise to χ and bounded by $\chi(\xi)$ for ξ large. Since $1/p + 1/q < 1$, it is straightforward to check that $\chi_k(\rho)\, u$ converges to $\chi(\rho)\, u$ and that Eq. (2.3) holds for $\chi(\xi) = \xi \log \xi$.

We first apply this to ρ_n and u_n to find that

$$\partial_t \rho_n \, \log \rho_n + \operatorname{div}(\rho_n \, \log \rho_n \, u_n) = -\operatorname{div} u_n \, \rho_n.$$

We may pass to the limit in this equation. But of course since we have not proved compactness of ρ_n yet, we cannot identify the weak-* limit of ρ_n. Let us hence denote

$$\overline{\rho\log\rho} = \text{weak-* }\lim \rho_n\, \log \rho_n.$$

We obtain

$$\partial_t \overline{\rho\,\log\rho} + \operatorname{div}\,(\overline{\rho\,\log\rho}\,u) = -\operatorname{div} u\,\rho.$$

From the proof of Theorem 2.1.4, we know that ρ and u solve (2.1) which by assumption has the renormalization property. Therefore we also have

$$\partial_t \rho\,\log\rho + \operatorname{div}\,(\rho\,\log\rho\,u) = -\operatorname{div} u\,\rho.$$

By taking the difference and integrating over Π^d, this leads to

$$\frac{d}{dt}\int_{\Pi^d} (\overline{\rho\,\log\rho} - \rho\,\log\rho)\,dx = 0.$$

By the compactness of ρ_n^0, we finally deduce that

$$\int_{\Pi^d} (\overline{\rho\,\log\rho} - \rho\,\log\rho)\,dx = 0.$$

But by the convexity of $\chi = \xi\,\log\xi$, one has that

$$\overline{\rho\,\log\rho} - \rho\,\log\rho \geq 0,$$

concluding that

$$\overline{\rho\,\log\rho} = \rho\,\log\rho,$$

and proving the compactness of ρ_n. □

Th. 2.1.5 is our first result proving compactness of a sequence ρ_n without any assumption of boundedness on the divergence or any comparable assumption on the vacuum. Of course it does not provide a quantitative regularity estimate and it relies explicitly on the structure of the limit equation, which can be a clear drawback to study non-linear coupled models. It also remains an if-theorem at this stage as we have not yet found any sufficient condition on u to guarantee that Eq. (2.1) has the renormalization property. This will be the object of the next chapter around the so-called commutator estimates.

One may make a last remark on our approach so far, which is the requirement that $1/p + 1/q \leq 1$. We of course need to make sense of the product $\rho\,u$ but also of the product $\rho \operatorname{div} u$. However this last requirement does not seem optimal as a more clever use of the renormalization χ should make it unnecessary. This is in fact the basis for the improvement on the Lions theory developed in particular in [28, 29].

2.2 Proving the renormalization property: Commutator estimates

The main breakthrough of [24] was to present a very straightforward proof of

Theorem 2.2.1 ([24]). *Assume that $u \in L^1([0,\ T],\ W^{1,p}(\Pi^d))$, then Eq.* (2.1) *has the renormalization property.*

Obviously if $u \in L^1([0,\ T],\ W^{1,p}(\Pi^d))$ then $\operatorname{div} u \in L^1([0,\ T],\ L^p(\Pi^d))$ and all the results of the previous section automatically apply. There are even more consequences to having the renormalization property and we refer again to [4, 22] for a more thorough treatment.

The ideas introduced in [24] started a now very active field of research about the minimal conditions on u guaranteeing that (2.1) has the renormalization property. A crucial initial effort culminated in [3] (after corresponding results in the kinetic case in [9]) to lower the requirement to $u \in L^1_t BV_x$ with $\operatorname{div} u \in L^1_{t,x}$, which is critical to many applications to hyperbolic systems. In view of the counterexample developed in [23], the BV regularity seems to be optimal in such a general setting.

The commutator estimates can also be partially translated on the characteristics as in [31] and renormalized solutions applied to various settings such as degenerate diffusion in [34].

It is possible to study further the regularity of renormalized solutions to (2.1); almost everywhere differentiability in [5] for instance. But as we mentioned before the first quantitative regularity estimate had been obtained in [21].

Proof. Consider any $\rho \in L^\infty([0,\ T],\ L^q(\Pi^d))$, weak solution to (2.1). For any $\chi \in C^1 \cap W^{1,\infty}$, we have to prove that (2.3) holds. If ρ was smooth then showing (2.3) would be a straightforward consequence of the chain rule. The main idea in the proof of Th. 2.2.1 is hence simply to regularize ρ by convolution.

Hence choose any smooth $K \in C^1(\Pi^d)$ with $\operatorname{supp} K$ concentrated near 0 so that $K_\varepsilon(x) = \varepsilon^{-d}\, K(x/\varepsilon)$ is an approximation of the identity as $\varepsilon \to 0$.

Denote $\rho_\varepsilon = K_\varepsilon \star \rho$. ρ_ε cannot solve (2.1) exactly (unless u is constant) but one may write

$$\partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon\, u) = R_\varepsilon, \tag{2.4}$$

where the commutator reads

$$R_\varepsilon(x) = \int_{\Pi^d} \nabla K_\varepsilon(x-y)\cdot(u(t,x)-u(t,y))\,\rho(t,y)\,dy + \rho_\varepsilon(t,x)\operatorname{div} u(t,x). \tag{2.5}$$

The heart of the method is hence to prove through a commutator estimate that R_ε converges strongly to 0. For a fixed ρ and u, this is straightforward through

Proposition 2.2.2 (Commutator estimate from [24]). *$u \in L^1([0,\ T],\ W^{1,p}(\Pi^d))$ and that $\rho \in L^\infty([0,\ T],\ L^q(\Pi^d))$ then $R_\varepsilon \longrightarrow 0$ in $L^1([0,\ T]\times\Pi^d)$ as $\varepsilon \to 0$ where R_ε is defined by* (2.5).

Assuming for the time being that Prop. 2.2.2 holds, we can easily conclude. Now ρ_ε is smooth and we may apply the chain rule on Eq. (2.4) to find

$$\partial_t \chi(\rho_\varepsilon) + \mathrm{div}\,(\chi(\rho_\varepsilon)\,u) = \mathrm{div}\,u\,(\chi(\rho_\varepsilon) - \rho_\varepsilon\,\chi'(\rho_\varepsilon)) + \chi'(\rho_\varepsilon)\,R_\varepsilon.$$

Since χ' is bounded we know from the proposition that $\chi'(\rho_\varepsilon)\,R_\varepsilon \to 0$.

Since K_ε is an approximation of the identity as $\varepsilon \to 0$, then ρ_ε converges strongly to ρ in $L^r([0,\ T],\ L^q(\Pi^d))$ for any $r < \infty$. Therefore $\chi(\rho_\varepsilon)$ converges strongly to $\chi(\rho_\varepsilon)$ in the same space. Therefore $\chi(\rho_\varepsilon)$ converges weak-* to $\chi(\rho)$ in $L^\infty([0,\ T],\ L^q(\Pi^d))$. Passing to the limit in each term in Eq. (2.4), we deduce Eq. (2.3).

Proof of proposition 2.2.2. There only remains to prove Prop. 2.2.2. Note that for *a.e.* $x,\ y$

$$u(t,x) - u(t,y) = \int_0^1 (x-y)\cdot\nabla u(t,\theta\,x + (1-\theta)\,y)\,d\theta,$$

which lets us write

$$\int_{\Pi^d} \nabla K_\varepsilon(x-y)\cdot(u(t,x)-u(t,y))\,\rho(t,y)\,dy$$
$$= \int_0^1 \int_{\Pi^d} (x-y)\otimes\nabla K_\varepsilon(x-y)\ :\ \nabla u(t,\theta\,x+(1-\theta)\,y)\,\rho(t,y)\,dy\,d\theta.$$

Remark that

$$(x-y)\otimes\nabla K_\varepsilon(x-y) = \varepsilon^{-d}\,\frac{x-y}{\varepsilon}\otimes\nabla K((x-y)/\varepsilon) = \varepsilon^{-d}\,L((x-y)/\varepsilon) = L_\varepsilon(x-y),$$

with $L(x) = x\otimes\nabla K(x)$. Observe that by integration by parts

$$\int_{\Pi^d} L_{ij}(x)\,dx = \int_{\Pi^d} x_i\,\partial_j K(x)\,dx = -\delta_{ij}\int_{\Pi^d} K(x)\,dx = -\delta_{ij}.$$

Hence as a convolution operator, L_ε is an approximation of $\delta\,I$ with I the identity matrix. Therefore strongly in L^1

$$\int_0^1\int_{\Pi^d} (x-y)\otimes\nabla K_\varepsilon(x-y)\ :\ \nabla u(t,\theta\,x+(1-\theta)\,y)\,\rho(t,y)\,dy\,d\theta \longrightarrow -\mathrm{div}\,u(t,x)\,\rho(t,x),$$

proving that $R_\varepsilon \longrightarrow 0$ in L^1. □

As simple as the previous proof is, since we are looking for quantitative estimates, a natural question is whether it would be possible to quantify the previous argument and in particular Prop. 2.2.2. This does not seem easy as it would imply giving an explicit rate of convergence on the commutator R_ε without using any additional regularity on ρ or ∇u.

Such an approach was nevertheless initiated in [6] and simplified in [7] from which we quote

Proposition 2.2.3. *Let* $1 < p < \infty$, $\exists C < \infty$ *depending only on* p *and the dimension s.t.* $\forall u \in W^{1,p}(\Pi^d)$ *with* $1 \le p \le 2$ *and* $\forall g \in L^{2p^*}$ *with* $1/p^* = 1 - 1/p$,

$$\begin{aligned}\int_{\Pi^{2d}} &\nabla K_h(x-y)\,(u(x)-u(y))\,|g(x)-g(y)|^2\,dx\,dy\\ &\le C\,\|\nabla u\|_{B^0_{p,q}}\,|\log h|^{1-1/q}\,\|g\|^2_{L^{2\,p^*}}\\ &\quad + C\,\|div\,u\|_{L^\infty}\int_{\mathbb{R}^{2d}} K_h(x-y)\,|g(x)-g(y)|^2\,dx\,dy,\end{aligned}$$

where $K_h(x) = (h+|x|)^d$ *for* x *small enough.*

In particular using $q = 2$,

$$\begin{aligned}\int_{\Pi^{2d}} &\nabla K_h(x-y)\,(u(x)-u(y))\,|g(x)-g(y)|^2\,dx\,dy\\ &\le C\,\|\nabla u\|_{L^p}\,|\log h|^{1/2}\,\|g\|_{L^{2\,p^*}}\\ &\quad + C\,\|div\,u\|_{L^\infty}\int_{\mathbb{R}^{2d}} K_h(x-y)\,|g(x)-g(y)|^2\,dx\,dy.\end{aligned}$$

The proof of this proposition will not be given here, as it is rather complex and requires a careful analysis of the cancellations in the expression. We emphasize that this commutator estimate only works for kernels with the critical singularity in $|x|^{-d}$ at $x = 0$; we will understand better the reason for that in the next section.

The straightforward estimate would give

$$\int_{\Pi^{2d}} \nabla K_h(x-y)\,(a(x)-a(y))\,|g(x)-g(y)|^2\,dx\,dy \sim |\log h|,$$

and therefore Prop. 2.2.3 gains a factor $|\log h|^{1/2}$ as a rate of convergence and would later yield a corresponding gain of derivative. We are hence again, as in the first chapter, at a log scale for the gain of regularity.

The underlying result behind Prop. 2.2.3 has recently been improved in [43] to a gain of a full $|\log h|$ (at the cost of a much more complicated analysis), see also [35]. This kind of critical semi-norm has also been used in other contexts, see for example [11].

However from our point of view in these notes, the major drawback of Prop. 2.2.3 is that it requires $\operatorname{div} u$ to be bounded. There are major benefits to having a self contained and quantitative commutator estimate, which would be more apparent if we were to consider vanishing viscosity or other approximations of (2.1). But our goal of obtaining estimates that do not require a bounded divergence will instead lead us to combine some of the ideas in the proof of Prop. 2.2.3 with the Lagrangian approach (or the Eulerian formulation of the Lagrangian approach) explained in the previous chapter.

2.3 The $\log \log$ scale for compressible transport equations

We here present the main estimate of these notes for the linear convective equation (2.1). This estimate will also form the basis for the analysis of some simple non-linear models in the next chapter. We follow closely here [16, 17] where the method has been introduced.

2.3.1 Technical preliminaries

As we had seen in the previous chapter, there is a technical difficulty if we try to use weights like $w(t,x)+w(t,y)$. To be more precise here, we would have to try (and fail) to control $M|\nabla u_k|(y)$ by $M|\nabla u_k|(x)$. Instead we have to be more precise than (1.10) in order to avoid this and use more sophisticated tools. First one replaces (1.10) by

Lemma 2.3.1. *There exists $C>0$ s.t. for any $u \in W^{1,1}(\Pi^d)$, one has*

$$|u(x)-u(y)| \leq C\,|x-y|\,(D_{|x-y|}u(x)+D_{|x-y|}u(y)),$$

where we denote

$$D_h u(x) = \frac{1}{h}\int_{|z|\leq h} \frac{|\nabla u(x+z)|}{|z|^{d-1}}\,dz.$$

Proof. A full proof of such well known result can for instance be found in [18] in a more general setting namely $u \in BV$. One possibility is simply to consider trajectories $\gamma(t)$ from x to y which stays within the ball of diameter $|x-y|$ to control

$$|u(x)-u(y)| \leq \int_0^1 \gamma'(t)\cdot\nabla u(\gamma(t))\,dt.$$

And then to average over all such trajectories with length of order $|x-y|$. □

Note that this result actually implies the estimate (1.10) as one can check, through a simple dyadic decomposition, that there exists $C>0$, for any $u \in W^{1,p}(\Pi^d)$ with $p \geq 1$

$$D_h\,u(x) \leq C\,M|\nabla u|(x). \tag{2.6}$$

We leave such proof to the reader and instead emphasize that the key improvement in using D_h is that small translations of the operator D_h are actually easy to control.

Let us first specify precisely the kernel K_h that we will use from now on. Choosing $[-1,\ 1]^d$ as a representative of the torus Π^d, we choose $K_h \in W^{1,\infty}$ with

$$K_h(x) = \frac{1}{(h+|x|)^d}, \quad \text{for } |x| \leq \frac{1}{2}, \quad 0 \leq K_h(x) \leq 1, \quad \operatorname{supp} K_h \subset [-3/4,\ 3/4]^d. \tag{2.7}$$

We insist here about the precise exponent d in K_h which is critical for integrability and that we have seen in Prop. 2.2.3. In particular

$$\frac{|\log h|}{C} \leq \int_{\Pi^d} K_h(x)\,dx \leq C\,|\log h|.$$

We hence also define the normalized kernel

$$\overline{K}_h(x) = \frac{K_h(x)}{\int_{\Pi^d} K_h(y)\,dy}, \tag{2.8}$$

which is now a standard convolution kernel or approximation of identity.

The main point here is the estimate

Lemma 2.3.2. *For any $1 < p < \infty$, there exists $C > 0$ s.t. for any $u \in H^1(\Pi^d)$*

$$\int_{\Pi^d} K_h(z)\,\|D_{|z|}\,u(.) - D_{|z|}\,u(.+z)\|_{L^p}\,dz \leq C\;\|u\|_{B^1_{p,1}}, \tag{2.9}$$

where $B^1_{p,1}$ is the classical Besov space. As a consequence for any $1 < p < 2$

$$\int_{\Pi^d} K_h(z)\,\|D_{|z|}\,u(.) - D_{|z|}\,u(.+z)\|_{L^p}\,dz \leq C\,|\log h|^{1/2}\,\|u\|_{W^{1,p}}. \tag{2.10}$$

This lemma is in fact a corollary of a classical result

Lemma 2.3.3. *For any $1 < p < \infty$, any family L_r of kernels satisfying for some $s > 0$*

$$\int L_r = 0, \quad \sup_r\,(\|L_r\|_{L^1} + r^s\,\|L_r\|_{W^{s,1}}) \leq C_L, \quad \sup_r r^{-s}\int |z|^s\,|L_r(z)|\,dz \leq C_L. \tag{2.11}$$

Then there exists $C > 0$ depending only on C_L above s.t. for any $u \in L^p(\Pi^d)$

$$\int_0^1 \|L_r \star u\|_{L^p}\,\frac{dr}{h+r} \leq C\;\|u\|_{B^0_{p,1}}. \tag{2.12}$$

As a consequence for $p \leq 2$

$$\int_0^1 \|L_r \star\,u\|_{L^p}\,\frac{dr}{h+r} \leq C\,|\log h|^{1/2}\,\|u\|_{L^p}. \tag{2.13}$$

Remark. The bounds (2.10) and (2.13) could also be obtained by straightforward application of the so-called square function, see the book written by E.M. Stein [45].

Proof of Lemma 2.3.2 assuming Lemma 2.3.3. Using spherical coordinates

$$\int_{\Pi^d} K_h(z)\, \|D_{|z|}\, u(.) - D_{|z|}\, u(.+z)\|_{L^p}\, dz \\ \leq C \int_{S^{d-1}} \int_0^{1/2} \|D_r\, u(.) - D_r\, u(.+r\,\omega)\|_{L^p}\, \frac{dr}{r+h}\, d\omega.$$

Denote

$$L_\omega(x) = \frac{\mathbb{I}_{|x|\leq 1/2}}{|x|^{d-1}} - \frac{\mathbb{I}_{|x-\omega|\leq 1/2}}{|x-\omega|^{d-1}}, \quad L_{\omega,r}(x) = r^{-d}\, L_\omega(x/r),$$

and remark that $L_\omega \in W^{s,1}$ with a norm uniform in ω and with support in the unit ball. Moreover

$$D_r u(x) - D_r u(x+r\omega) = \int |\nabla u|(x - r\, z)\, L_\omega(z)\, dz = L_{\omega,r} \star |\nabla u|.$$

We hence apply Lemma 2.3.3 since the family $L_{\omega,r}$ satisfies the required hypothesis and we get

$$\int_{h_0}^1 \|L_{\omega,r} \star \nabla\, u\|_{L^p}\, \frac{dr}{r} \leq C\, \|u\|_{B^1_{1,p}},$$

with a constant C independent of ω and so

$$\int_{h_0}^1 \int_{\Pi^d} K_h(z)\, \|D_{|z|}\, u(.) - D_{|z|}\, u(.+z)\|_{L^p}\, dz\, dh \\ \leq C \int_{S^{d-1}} \int_{h_0}^1 \|L_{\omega,r} \star \nabla u\|_{L^p}\, \frac{dr}{r}\, d\omega \leq C \int_{S^{d-1}} \|u\|_{B^1_{1,p}}\, d\omega,$$

yielding (2.9). The bound (2.10) is deduced in the same manner. □

2.3.2 Propagating regularity with weights

We now come back to the basic strategy outlined at the end of the previous chapter and consider again the auxiliary equation on the weights

$$\partial_t w + u \cdot \nabla_x w = -\lambda\, M\, |\nabla u|\, w - D\, w, \tag{2.14}$$

where we allow for an abstract additional penalization $D(t,x)$ which we will need in the next chapter.

By using the tools for renormalized solutions that we briefly explained at the beginning of the chapter, one can ensure

Lemma 2.3.4. *Assume $D \geq 0$, $u \in L^1([0,\ T],\ W^{1,p}(\Pi^d))$, then there exists a renormalized solution to Eq.* (2.14).

We skip the proof of Lemma 2.3.4 which essentially follows the existence strategy of Th. 2.1.4 while using Th. 2.2.1 for the renormalization property.

We are now ready to prove an equivalent of Prop. 1.3.1 or Prop. 1.3.8 but for the weight $w(t,x)+w(t,y)$.

Proposition 2.3.5. *Assume $u \in L^1([0,\ T],\ W^{1,p}(\Pi^d))$, $\rho \in L^\infty([0,\ T],\ L^q(\Pi^d))$ is a renormalized solution to Eq.* (2.1) *with $1/p+1/q \leq 1$. Then for w a renormalized solution to Eq.* (1.14) *with λ large enough, one has that for any h*

$$\begin{aligned}
&\int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\, K_h(x-y)\,(w(t,x)+w(t,y))\,dx\,dy\\
&\leq \int_{\Pi^{2\,d}} |\rho^0(x)-\rho^0(y)|\, K_h(x-y)\,dx\,dy\\
&+ C\,|\log h|^{1/2}\,\|u\|_{L^1([0,\ T],\ W^{1,p}(\Pi^d))}\,\|\rho\|_{L^\infty([0,\ T],\ L^q(\Pi^d))}\\
&-2\int_0^t\int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\, K_h(x-y)\,w(t,x)\,dx\,dy\,dt\\
&-2\int_0^t\int_{\Pi^{2\,d}} (div\,u(t,x)-div\,u(t,y))\,K_h\,w(t,x)\,(\rho(t,x)+\rho(t,y))\,s(x,y)\,dx\,dy\,dt,
\end{aligned}$$

where $s(x,y)=sign(\rho(t,x)-\rho(t,y))$.

Proof. The argument initially follows the same steps as Prop. 1.3.1 or Prop. 1.3.8. We first specify more (for further use in the next chapter) the equation

$$\begin{aligned}
\partial_t|\rho(t,x)-\rho(t,y)| &+ u(x)\cdot\nabla_x|\rho(t,x)-\rho(t,y)| + u(y)\cdot\nabla_y|\rho(t,x)-\rho(t,y)|\\
&= \frac{\mathrm{div}\,u(t,y)-\mathrm{div}\,u(t,x)}{2}\,(\rho(t,x)+\rho(t,y))\,s(x,y)\\
&\quad-\frac{|\rho(t,x)-\rho(t,y)|}{2}\,(\mathrm{div}\,u(t,x)+\mathrm{div}\,u(t,y)),
\end{aligned}$$

where again $s(x,y)=\mathrm{sign}(\rho(t,x)-\rho(t,y))$ and where we can now fully justify the calculations as ρ is a renormalized solution. Multiplying by $w(t,x)+w(t,y)$ and using Eq. (2.14) and the symmetry between x and y, we find the modified

$$\begin{aligned}
&\frac{d}{dt}\int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\, K_h(x-y)\,(w(t,x)+w(t,y))\,dx\,dy\\
&\leq 2\int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\, w(t,x)\,\nabla K_h(x-y)\cdot(u(t,x)-u(t,y))\,dx\,dy\\
&\quad+\int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\, K_h\,w(t,x)\,(\mathrm{div}\,u(t,x)+\mathrm{div}\,u(t,y))\,dx\,dy\\
&\quad-2\int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\, K_h\,w(t,x)\,(D+\lambda\,M\,|\nabla u|(t,x))\,dx\,dy\\
&\qquad-\int_{\Pi^{2\,d}} (\mathrm{div}\,u(t,x)-\mathrm{div}\,u(t,y))\,K_h\,w(t,x)\,(\rho(t,x)+\rho(t,y))\,s(x,y)\,dx\,dy.
\end{aligned}$$

Remark that

$$\begin{aligned}
&\int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\, K_h\, w(t,x)\, \mathrm{div}_y\, u(t,y))\, dx\, dy \\
&\quad = \int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\, K_h\, w(t,x)\, \mathrm{div}\, u(t,x))\, dx\, dy \\
&\quad - \int_{\Pi^{2\,d}} (\mathrm{div}\, u(t,x)-\mathrm{div}\, u(t,y))\, K_h\, w(t,x)\, (\rho(t,x)+\rho(t,y))\, s(x,y)\, dx\, dy.
\end{aligned}$$

Recalling that div u$(t,x) \le M\,|\nabla u|(t,x)$, we may thus simplify for λ large enough

$$\begin{aligned}
&\frac{d}{dt}\int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\, K_h(x-y)\, (w(t,x)+w(t,y))\, dx\, dy \\
&\le 2\int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\, w(t,x)\, \nabla K_h(x-y)\cdot(u(t,x)-u(t,y))\, dx\, dy \\
&\quad -\lambda \int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\, K_h\, w(t,x)\, M\,|\nabla u|(t,x)\, dx\, dy \\
&\quad -2\int_0^t\int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\, K_h(x-y)\, w(t,x)\, dx\, dy\, dt \\
&\qquad -2\int_{\Pi^{2\,d}} (\mathrm{div}\, u(t,x)-\mathrm{div}\, u(t,y))\, K_h\, w(t,x)\, (\rho(t,x)+\rho(t,y))\, s(x,y)\, dx\, dy,
\end{aligned}$$

and we are back to our commutator estimate. However now we cannot use estimate (1.10) as we would then have to bound $w(t,x)\, M|\nabla u|(t,y)$ by $w(t,x)\, M|\nabla u|(t,x)$ which is simply not possible absent some more regularity on ∇u.

Instead we use Lemma 2.3.1 to bound

$$\begin{aligned}
&\int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\, w(t,x)\, \nabla K_h(x-y)\cdot(u(t,x)-u(t,y))\, dx\, dy \\
&\le C\int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\, w(t,x)\, (1+K_h)\, (D_{|x-y|}u(t,x)+D_{|x-y|}u(t,y))\, dx\, dy,
\end{aligned}$$

since we recall that for small x, $|\nabla K_h(x)| \le C\,|x|^{-1}\, K_h(x)$ and that K_h is smooth for x of order 1.

By (2.6), we may bound directly the term without K_h

$$\begin{aligned}
&\int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\, w(t,x)\, (D_{|x-y|}u(t,x)+D_{|x-y|}u(t,y))\, dx\, dy \\
&\qquad \le \|\rho(t,.)\|_{L^q(\Pi^d)}\, \|M\,|\nabla u(t,.)|\|_{L^p(\Pi^d)}.
\end{aligned}$$

As for the other term, we may now use Lemma 2.3.2 to move $D_{|x-y|}u(t,y)$ to

$D_{|x-y|}u(t,x)$. By a change of variable

$$\begin{aligned}
&\int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\, w(t,x)\, K_h\, D_{|x-y|}u(t,y)\, dx\, dy\\
&= \int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\, w(t,x)\, K_h\, D_{|x-y|}u(t,y)\, dx\, dy\\
&\quad + \int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,x+z)|\, w(t,x)\, K_h(z)\, (D_{|z|}u(t,x+z)-D_{|z|}u(t,x))\, dx\, dz.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,x+z)|\, w(t,x)\, K_h(z)\, (D_{|z|}u(t,x+z)-D_{|z|}u(t,x))\, dx\, dz\\
&\quad\leq \int_{\Pi^d} \|\rho(t,.)-\rho(t,.+z)\|_{L^q}\, K_h(z)\, \|D_{|z|}u(t,.+z)-D_{|z|}u(t,.)\|_{L^p}\, dz\\
&\quad\leq C\,|\log h|^{1/2}\, \|\rho(t,.)\|_{L^q(\Pi^d)}\, \|u\|_{W^{1,p}},
\end{aligned}$$

by bounding $\|\rho(t,.)-\rho(t,.+z)\|_{L^q} \leq 2\|\rho(t,.)\|_{L^q(\Pi^d)}$ and a direct application of Lemma 2.3.2. We want to emphasize here that this is the key part of the proof. Even though it remains relatively straightforward technically (also thanks to the preliminaries), this is what forces us to use this specific K_h.

Combining those estimates, we get that

$$\begin{aligned}
&\frac{d}{dt}\int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\, K_h(x-y)\, (w(t,x)+w(t,y))\, dx\, dy\\
&\leq 2\int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\, w(t,x)\, K_h(x-y)\, D_{|x-y|}u(t,x)\, dx\, dy\\
&\quad - 2\int_0^t\int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\, K_h(x-y)\, w(t,x)\, dx\, dy\, dt\\
&\quad - \lambda\int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\, K_h\, w(t,x)\, M\,|\nabla u|(t,x)\, dx\, dy\\
&\quad + C\,|\log h|^{1/2}\, \|\rho(t,.)\|_{L^q(\Pi^d)}\, \|u\|_{W^{1,p}}\\
&\quad - 2\int_{\Pi^{2\,d}} (\mathrm{div}, u(t,x)-\mathrm{div}\, u(t,y))\, K_h\, w(t,x)\, (\rho(t,x)+\rho(t,y))\, s(x,y)\, dx\, dy,
\end{aligned}$$

which lets us conclude the proof by applying (2.6) and integrating in time. □

2.3.3 The final estimate

We are now ready to state the concluding result of our linear analysis,

Theorem 2.3.6. *Assume $u \in L^1([0,\ T],\ W^{1,p}(\Pi^d))$, $\rho \in L^\infty([0,\ T],\ L^q(\Pi^d))$ is a renormalized solution to Eq.* (2.1) *with $1/p+1/q \leq 1$. Assume that*

$$\int_{\Pi^d} \|div\, u(t,.)-div\, u(t,.+z)\|_{L^1([0,\ T],\ L^p(\Pi^d))}\, K_h(z)\, dz \leq L,$$

and that

$$\int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\,K_h(x-y)\,dx\,dy \leq L.$$

Then there exists a constant C depending only the dimension and L such that one has that for any h

$$\int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\,K_h(x-y)\,dx\,dy \leq C\,N\,\frac{|\log h|}{\log|\log h|},$$

with

$$N=(1+\|\rho\|_{L^\infty([0,\,T],\,L^q(\Pi^d))})\,(1+\|u\|_{L^1([0,\,T],\,W^{1,p}(\Pi^d))}).$$

This is the result that we had been looking for:

- It does not require any bound on $\operatorname{div} u$ or on the Jacobian of the flow in general. It only requires *one* solution ρ bounded in some L^q.
- It provides an explicit regularity estimate on the solution ρ. And it only requires minimal regularity on $\operatorname{div} u$ (in fact any compactness on $\operatorname{div} u$ would give compactness on ρ by an easy modification of the proof).

Th. 2.3.6 essentially provides a $\log\log$ derivative on ρ. This appears to be a new scale in the problem (recall that we had a log scale previously), one that is due to possible concentration or vacuum.

Proof. Remark that by a change of variable

$$\begin{aligned}&\int_0^t\int_{\Pi^{2\,d}} (\operatorname{div} u(t,x)-\operatorname{div} u(t,y))\,K_h\,w(t,x)\,(\rho(t,x)+\rho(t,y))\,s(x,y)\,dx\,dy\,dt\\ &\leq 2\int_{\Pi^d} \|\operatorname{div} u(t,.)-\operatorname{div} u(t,.+z)\|_{L^1([0,\,T],\,L^p(\Pi^d))}\,K_h(z)\,\|\rho\|_{L^\infty([0,\,T],\,L^q(\Pi^d))}\,dz\\ &\quad\leq L\,\|\rho\|_{L^\infty([0,\,T],\,L^q(\Pi^d))}\end{aligned}$$

Then we choose $D=0$ and since there exists a weight by Lemma 2.3.4, we may directly apply Prop. 2.3.5 to find

$$\begin{aligned}&\int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\,K_h(x-y)\,(w(t,x)+w(t,y))\,dx\,dy\\ &\quad\leq L\,(1+\|\rho\|_{L^\infty([0,\,T],\,L^q(\Pi^d))})\\ &\qquad+C\,|\log h|^{1/2}\,\|u\|_{L^1([0,\,T],\,W^{1,p}(\Pi^d))}\,\|\rho\|_{L^\infty([0,\,T],\,L^q(\Pi^d))}\\ &\quad\leq N\,(L+C\,|\log h|^{1/2}).\end{aligned}$$

where we recall the definition of N

$$N=(1+\|\rho\|_{L^\infty([0,\,T],\,L^q(\Pi^d))})\,(1+\|u\|_{L^1([0,\,T],\,W^{1,p}(\Pi^d))}).$$

Now it only remains to remove the weight $w(t,x)+w(t,y)$. But those only vanish if both $w(t,x)$ and $w(t,y)$ vanish. Defining

$$\Omega = \{x,\ w(t,x) \leq \eta\},$$

we may simply write

$$\int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\, K_h(x-y)\, dx\, dy \leq \int_{\Omega^2} \ldots + \int_{x\notin\Omega \text{ or } y\notin\Omega} \ldots$$

If $x \notin \Omega$ or $y \notin \Omega$ then $w(t,x)+w(t,y) \geq \eta$, thus

$$\begin{aligned}
&\int_{x\notin\Omega \text{ or } y\notin\Omega} |\rho(t,x)-\rho(t,y)|\, K_h(x-y)\, dx\, dy \\
&\quad \leq \frac{1}{\eta} \int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\, K_h(x-y)\, (w(t,x)+w(t,y))\, dx\, dy \\
&\quad \leq \frac{N}{\eta}\, (L + C\, |\log h|^{1/2}).
\end{aligned}$$

On the other hand by symmetry

$$\begin{aligned}
\int_{\Omega^2} |\rho(t,x)-\rho(t,y)|\, K_h(x-y)\, dx\, dy &\leq C\, |\log h| \int_{\Omega} \rho(t,x)\, dx \\
&\leq C\, \frac{|\log h|}{|\log \eta|} \int_{\Pi^d} |\log w|\, \rho(t,x)\, dx \\
&\leq C\, \frac{|\log h|}{|\log \eta|}\, N,
\end{aligned}$$

by Lemma 1.3.4 which we may directly use as we chose $D = 0$.

Finally

$$\int_{\Pi^{2\,d}} |\rho(t,x)-\rho(t,y)|\, K_h(x-y)\, dx\, dy \leq C\, \frac{|\log h|}{|\log \eta|}\, N + \frac{N}{\eta}\, (L + C\, |\log h|^{1/2}),$$

which finishes the proof by choosing for example $\eta = |\log h|^{-1/4}$. □

Chapter 3

Example of application: A coupled Stokes system

3.1 The compressible Navier-Stokes' system

The theory introduced in the last section of the previous chapter had in fact been developed in [16] for the study of the compressible Navier-Stokes system in various unstable regimes such as non monotone pressure laws or anisotropic stress tensors.

In its simplest form the Navier-Stokes system reads

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) - \mu \Delta u - (\lambda + \mu)\nabla \mathrm{div} u + \nabla p(\rho) = \rho f, \end{cases} \tag{3.1}$$

with $2\,\mu/d + \lambda$, and p is the barotropic pressure law ($s \mapsto p(s)$ given) which is typically continuous on $[0, +\infty)$, and locally Lipschitz on $(0, +\infty)$. The initial condition reads

$$\rho|_{t=0} = \rho_0, \qquad (\rho u)|_{t=0} = m_0.$$

The main difficulty in obtaining global existence for system (3.1) is to prove compactness of the density ρ which exactly solves the continuity equation that was the object of our previous investigations.

The first global existence result has been obtained in [37], using the (non-quantitative) theory for renormalized solution introduced at the beginning of the second chapter. This was the start of many works, for instance [19, 47, 20], pushing the theory and in particular the required growth at infinity of the pressure. Those culminated in the estimates in [27, 30] and exposed at length in [28] (see also [39]). This also enabled to treat the more physically realistic Navier-Stokes-Fourier system for which we refer to [29]. We also mention the recent [42] which is able to handle the isothermal system.

While system (3.1) is written with a constant viscosity, realistic physical settings often involve density dependent viscosities. This requires a different type of approach with new regularity estimates explicited in [14], new integrability bounds in [38], and leading to the existence of weak solutions in this setting in [15, 48]. Those regularity estimates are based on a two-velocity interpretation of the Navier-Stokes system, which has several other applications as in [40].

The Navier-Stokes system is also a classical model for geophysical flows as illustrated in [26] and [46]. We finally refer to [12] for an example of recent important topics.

Because the classical theory of existence relies on non-quantitative regularity estimates for ρ, it requires pressure laws that are thermodynamically stable. We hence conclude this introduction by quoting, as an illustration, one of the results from [16]. The assumption on the pressure are only that there exists $C > 0$ with

$$C^{-1}\rho^\gamma - C \le p(\rho) \le C\rho^\gamma + C \tag{3.2}$$

and for all $s \ge 0$

$$|p'(s)| \le \bar{p} s^{\tilde{\gamma}-1}. \tag{3.3}$$

This allows oscillating pressure laws, alternating stable and unstable regions. Nevertheless this still leads to global existence as per

Theorem 3.1.1. *Assume that the initial data u_0 and ρ_0 satisfies the bound*

$$E_0 = \int_{\Pi^d} \big(\rho^0 \frac{|u_0|^2}{2} + \rho_0 e(\rho_0)\big)\, dx < +\infty.$$

Let the pressure law p satisfies (3.2) *and* (3.3) *with*

$$\gamma > \big(\max(2,\tilde{\gamma}) + 1\big)\,\frac{d}{d+2}. \tag{3.4}$$

Then there exists a global weak solution of the compressible Navier–Stokes system (3.1)*. Moreover the solution satisfies the explicit regularity estimate*

$$\int_{\Pi^{2d}} 1_{\rho_k(x)\ge\eta}\, 1_{\rho_k(y)\ge\eta}\, K_h(x-y)\,\chi(\delta\rho_k) \le \frac{C\,\|K_h\|_{L^1}}{\eta^{1/2}\,|\log h|^{\theta/2}},$$

for some $\theta > 0$.

3.2 The result on the Stokes' system

In the rest of this chapter, we mostly follow the presentation in [17] and focus on a example of application, namely the coupled Stokes' system

$$\left\{ \begin{array}{l} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ -\mu\Delta u + \alpha u + \nabla p(\rho) = S, \end{array} \right. \tag{3.5}$$

with $\mu, \alpha > 0$ endowed with the following initial condition

$$\rho|_{t=0} = \rho_0. \tag{3.6}$$

In addition of being a limit of the compressible Navier-Stokes system (3.1) in some regime, system (3.5) (with many variants) is commonly used to model various biological systems, tumor for example in [13, 25, 41].

We consider a pressure law p which is continuous on $[0, +\infty)$, p locally Lipschitz on $(0, +\infty)$ with $p(0) = 0$ such that there exists $C > 0$ with

$$C^{-1}\rho^\gamma - C \le p(\rho) \le C\rho^\gamma + C, \tag{3.7}$$

and for all $s \ge 0$

$$|p'(s)| \le \bar{p}s^{\gamma-1}. \tag{3.8}$$

One can then used the linear theory that we previously developed to prove

Theorem 3.2.1. *Assume that $S \in L^2(0,T;\ H^{-1}(\Pi^d))$ and the initial data ρ_0 satisfies the bound*

$$\rho^0 \ge 0, \qquad 0 < M_0 = \int_{\Pi^d} \rho^0 < +\infty, \qquad E_0 = \int_{\Pi^d} \rho^0\, e(\rho^0)\, dx < +\infty,$$

where $e(\rho) = \int_{\rho^\star}^{\rho} p(s)/s^2 ds$ with $\rho^\star$ a constant reference density. Let the pressure law p satisfies (3.7) *and* (3.8) *with $\gamma > 1$. Then there exists a global weak solution (ρ, u) of the compressible system* (3.5)–(3.6)*with*

$$\rho \in L^\infty(0,T; L^\gamma(\Pi^d)) \cap L^{2\gamma}((0,T) \times \Pi^d), \qquad u \in L^2(0,T; H^1(\Pi^d)).$$

Remark. As noted in [17], the regularity of S is not optimized and could be far less smooth.

3.3 Sketch of the proof of Theorem 3.2.1

The proof of global weak solutions of PDEs is usually divided in three steps:

- *A priori* energy estimates,
- Stability of weak sequences: Compactness,
- Construction of approximate solutions.

We mostly focus on the first two points here as they best illustrate the main ideas. We refer to [16, 17] for more technical precisions.

3.3.1 Construction of approximate solutions.

To keep our analysis simple, we in fact consider a sequence $\rho_k,\ u_k$ of solutions to the exact system (3.5) and will prove that the limit of the sequence is also a solution to (3.5). Even though it is not a complete proof, such a result of weak stability gives the main ideas behind Theorem 3.2.1.

We only briefly indicate in this subsection what would be the approximate system from which (3.5) is obtained, namely

$$\begin{cases} \partial_t \rho_k + \mathrm{div}(\rho_k u_k) = \alpha_k \Delta \rho_k, \\ -\mu \Delta u_k - (\lambda+\mu)\nabla \mathrm{div} u_k + \nabla p_\epsilon(\rho_k) + \alpha_k \nabla \rho_k \cdot \nabla u_k = S, \end{cases} \tag{3.9}$$

with the fixed source term S and the fixed initial data

$$\rho_k|_{t=0} = \rho^0. \tag{3.10}$$

The pressure p_ϵ is defined as follows:

$$p_\epsilon(\rho) = p(\rho) \text{ if } \rho \le c_{0,\epsilon}, \qquad p_\epsilon(\rho) = p(C_{0,\epsilon}) + C(\rho - c_{0,\epsilon})^\beta \text{ if } \rho \ge c_{0,\epsilon},$$

with a large enough β. We defer to [16, 17] and the references therein for the existence of such an approximate system.

3.3.2 Energy estimates

Let us start with the basic kinetic energy estimate. Multiply the Stokes equation by u and integrate by parts,

$$\mu \int_{\Pi^d} |\nabla u_k|^2 + \alpha \int_{\Pi^d} |u_k|^2 + \int_{\Pi^d} \nabla p(\rho_k) \cdot u = \int_{\Pi^d} S_k \cdot u_k.$$

Now we write the equation satisfied by $\rho_k e(\rho_k)$ where $e(\rho_k) = \int_{\rho_{\mathrm{ref}}}^{\rho_k} p(s)/s^2 ds$, with ρ_{ref} a constant reference density,

$$\partial_t(\rho e(\rho)) + \mathrm{div}(\rho e(\rho) u) + p(\rho)\mathrm{div} u = 0.$$

Integrating in space and adding to the first equation we get

$$\frac{d}{dt} \int_{\Pi^d} \rho_k e(\rho_k) + \mu \int_{\Pi^d} |\nabla u_k|^2 = \int_{\Pi^d} S_k \cdot u_k.$$

One only needs $S_k \in L^2([0,\ T],\ H^{-1}(\Pi^d))$ uniformly and using the behavior of p, then we get the uniform bound

$$\rho_k^\gamma \in L^\infty(0,T; L^1(\Pi^d)), \qquad u_k \in L^2(0,T; H^1(\Pi^d)).$$

When now considering the compressible system (3.5), the divergence $\mathrm{div} u_k$ is given by

$$\mathrm{div} u_k = \frac{1}{\mu} p(\rho_k) + \frac{1}{\mu} \Delta^{-1} \mathrm{div} R_k$$

with $R_k = S_k - \alpha u_k$. Therefore, since $\rho_k \in L^\infty(0,T; L^\gamma(\Pi^d))$, if we multiply by ρ_k^θ, we obtain

$$I = \int_0^T \int_{\Pi^d} p(\rho_k)\rho_k^\theta = \mu \int_0^T \int_{\Pi^d} \mathrm{div} u_k \rho_k^\theta - \int_0^T \int_{\Pi^d} \Delta^{-1} \mathrm{div} R_k\, \rho_k^\theta,$$

which is easily bounded as follows

$$I \le \big[\mu\|\mathrm{div} u_k\|_{L^2((0,T)\times\Pi^d)} + \|\Delta^{-1}\mathrm{div} R_k\|_{L^2((0,T)\times\Pi^d)}\big]\,\|\rho_k^\theta\|_{L^2((0,T)\times\Pi^d)}$$

Thus using the behavior of p and information on u_k and R_k, we get for large density

$$\int_0^T\int_{\Pi^d}(\rho^{\gamma+\theta}) \le C + \varepsilon\int_0^T\int_{\Pi^d}(\rho^{2\theta}).$$

Thus we get a control on $\rho_k^{\gamma+\theta}$ if $\theta\le\gamma$. Therefore, we get $\rho_k \in L^p((0,T)\times\Pi^d)$ with for any $p\le 2\,\gamma$ and in particular some $p>2$ if $\gamma>1$.

3.3.3 Stability of weak sequences: Compactness

From the energy estimates we can extract converging subsequences

$$\begin{aligned} &S_k \longrightarrow S \text{ in } L^2_t H^{-1}_x,\\ &\rho_k \longrightarrow \rho \text{ weak-* in } L^\infty_t L^\gamma_x,\ L^{2\gamma}_{t,x}\\ &u_k \longrightarrow \rho \text{ weak-* in } L^2_t H^1_x. \end{aligned}$$

This is enough to pass to the limit in every term of system (3.5) except for $p(\rho_k)$. This requires the compactness of ρ_k for which we prove the following result which is the main part of the proof

Proposition 3.3.1. *Assume* (ρ_k, u_k) *are weak solutions to system* (3.5) *with a pressure law satisfying* (3.7)–(3.8) *and with the following uniform bounds*

$$\sup_k \|\rho_k^\gamma\|_{L^\infty_t L^1_x} < \infty, \qquad \sup_k \|\rho_k\|_{L^p_{t,x}} < \infty \quad \textit{with } p\le 2\gamma,$$

and

$$\sup_k \|u_k\|_{L^2_t H^1_x} < \infty.$$

Assume moreover that the source term S_k *is compact in* $L^2([0,\ T],\ H^{-1}(\Pi^d))$ *and that the initial density sequence* $(\rho_k)_0$ *is compact and hence satisfies*

$$\limsup_k\Big[\frac{1}{|\log h|}\int_{\Pi^{2d}} K_h(x-y)|(\rho_k^x)_0 - (\rho_k^y)_0|\Big] = \epsilon(h)\to 0 \ \textit{as}\ h\to 0,$$

then ρ_k *is locally compact.*

Remark 3.3.2. Here and in the following, we use the convenient notation $\rho_k^x = \rho_k(t,x)$, $\rho_k^y = \rho_k(t,y)$ and $(\rho_k^x)_0 = \rho_k(t=0,x)$, $(\rho_k^y)_0 = \rho_k(t=0,y)$. Similarly $w_k^x = w_k(t,x)$, $u_k^x = u_k(t,x)$, $w_k^y = w_k(t,y)$, $u_k^y = u_k(t,y)$.

Proof. Introduce as before the auxiliary equation on the weight w_k

$$\partial_t w_k + u_k \cdot \nabla w_k = -\lambda\, M\, |\nabla u|\, w - (1+\rho_k^\gamma)\, w. \tag{3.11}$$

We start by using Prop. 2.3.5 from last chapter to find that

$$\begin{aligned}\int_{\Pi^{2d}} K_h(x-y)\, |\rho_k^x - \rho_k^y|\, (w_k^x + w_k^y)\, dx\, dy \le |\log h|\, \epsilon(h) + C\, |\log h|^{1/2}\, N + A \\ - 2\int_0^t \int_{\Pi^{2d}} K_h(x-y)\, (1+(\rho_k^x)\gamma)|\rho_k^x - \rho_k^y|\, w_k^x\, dx\, dy\, dt\end{aligned} \tag{3.12}$$

where

$$N = \sup_k \|\rho_k\|_{L^{2\gamma}_{t,x}}\, \|u_k\|_{L^2_t H^1_x},$$

$$A = -2\int_0^t \int_{\Pi^{2d}} (\operatorname{div} u_k(t,x) - \operatorname{div} u_k(t,y))\, K_h(x-y)\, w_k^x\, (\rho_k^x + \rho_k^y)\, s_k(x,y)\, dx\, dy,$$

with $s_k(x,y) = \operatorname{sign}(\rho_k^x - \rho_k^y)$.

Use the relation between $\operatorname{div} u_k^x$ (respectively $\operatorname{div} u_k^y$) with ρ_k^x (respectively ρ_k^y), to obtain

$$A = -2\int_0^t \int_{\Pi^{2d}} K_h(x-y)\, (p(\rho_k^x) - p(\rho_k^y))\, (\rho_k^x + \rho_k^y)\, s_k\, w^x\, dx\, dy\, dt - \frac{2}{\mu}\, Q_h$$

where

$$Q_h = \int_0^t\!\!\int_{\Pi^{2d}} K_h(x-y)\, (\Delta^{-1}\operatorname{div} R_k(t,x) - \Delta^{-1}\operatorname{div} R_k(t,y))\, (\rho_k^x + \rho_k^y)\, s_k\, w^x\, dx\, dy\, dt,$$

encodes the compactness in space of $\Delta^{-1}\operatorname{div} R_k$ and therefore has the right behavior. Indeed in particular

$$\frac{1}{|\log h|}\int_0^t \int_{\Pi^{2d}} K_h(x-y)\, (\Delta^{-1}\operatorname{div} R_k(t,x) - \Delta^{-1}\operatorname{div} R_k(t,y))\, dx\, dy\, dt \to 0,$$

as $h \to 0$ since R_k is compact in $L^2_t H^{-1}_x$ and hence $\Delta^{-1}\operatorname{div} R_k$ is compact in $L^2_{t,x}$ by the gain of one derivative.

However the "bad" term $p(\rho_k^y) w_k^x$ cannot a priori be bounded directly with weights, again because it mixes points x and y. We review the various configurations

First note that we have $\rho_k^x + \rho_k^y \ge |\rho_k^x - \rho_k^y|$.

– Case 1: $p(\rho_k^x) - p(\rho_k^y))(\rho_k^x - \rho_k^y) \ge 0$. We then directly have that $p(\rho_k^x) - p(\rho_k^y))\, s_k$ and this yields the right sign and a dissipation term in A.

– Case 2: $p(\rho_k^x) - p(\rho_k^y))(\rho_k^x - \rho_k^y) < 0$ and $\rho_k^y \le \rho_k^x/2$ or $\rho_k^y \ge 2\rho_k^x$.

Assume we are in the case $\rho_k^y \geq 2\rho_k^x$, then

$$(p(\rho_k^x) - p(\rho_k^y))(\rho_k^x + \rho_k^y)s_k \geq -C\,(\rho_k^x)^\gamma\,|\rho_k^x - \rho_k^y|,$$

since $p(\xi) \leq p(0) + C\xi^{\gamma-1}\xi \leq C\xi^\gamma$. If we now look at the case $p(\rho_k^x) \leq p(\rho_k^y)$ and $\rho_k^y \leq \rho_k^x/2$, then we again bound

$$(p(\rho_k^x) - p(\rho_k^y))(\rho_k^x + \rho_k^y)s_k \geq -C\,(\rho_k^x)^\gamma\,|\rho_k^x - \rho_k^y|.$$

— Case 3: The case where $p(\rho_k^x) - p(\rho_k^y)$ and $\rho_k^x - \rho_k^y$ have different signs but $\rho_k^x/2 \leq \rho_k^y \leq 2\rho_k^x$. Then it is easy to get again

$$(p(\rho_k^x) - p(\rho_k^y))(\rho_k^x + \rho_k^y)s_k \geq -C\,(1 + (\rho_k^x)^\gamma)\,|\rho_k^x - \rho_k^y|.$$

Therefore combining all cases, we obtain

$$A \leq C\int_0^t\int K_h(x-y)\,(1 + (\rho_k^x)^\gamma))\,|\rho_k^x - \rho_k^y|\,w_k^x\,dx\,dy\,dt - \frac{2}{\mu}\,Q_h,$$

with $Q_h/|\log h| \to 0$. Inserting this in (3.12), we deduce that

$$\int_{\Pi^{2d}} K_h(x-y)\,|\rho_k^x - \rho_k^y|\,(w_k^x + w_k^y)\,dx\,dy \leq |\log h|\,\tilde{\epsilon}(h), \tag{3.13}$$

with

$$\tilde{\epsilon}(h) = \epsilon(h) + C\,|\log h|^{-1/2}\,N - \frac{2}{\mu}\,Q_h \longrightarrow 0, \quad \text{as } h \to 0.$$

We now need to remove the weight just as in the proof of Theorem 2.3.6. First of all since Eq. (3.11) has an additional term with respect to (1.14), we remark that we have the easy extension of Lemma 1.3.4, namely

Lemma 3.3.3. *Assume that* $u_k \in L^2([0,\ T],\ H^1(\Pi^{2d}))$ *and that* $\rho_k \in L^{2\gamma}([0,\ T] \times \Pi^{2d}))$ *with* $\gamma > 1$ *then if* w_k *solves* (3.11)

$$\int_{\Pi^d} |\log w_k(t,x)|\,\rho_k(t,x)\,dx \leq C\,(\|u_k\|_{L_t^2 H_x^1} + \|\rho_k\|_{L_{t,x}^{2\gamma}})\,\|\rho_k\|_{L_{t,x}^{2\gamma}}.$$

The proof of Lemma 3.3.3 is essentially identical to the one of Lemma 1.3.4 and we skip it here.

Using the same decomposition as in the proof of Theorem 2.3.6, we obtain from (3.13) that

$$\int_{\Pi^{2d}} K_h(x-y)\,|\rho_k^x - \rho_k^y|\,(w_k^x + w_k^y)\,dx\,dy \leq C\,\frac{|\log h|}{|\log \eta|}\,N^2 + C\,|\log h|\,\tilde{\epsilon}(h),$$

which finishes the proof by optimizing in η. □

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