

Global Existence of Weak Solutions for Compressible Navier–Stokes Equations: Thermodynamically unstable pressure and anisotropic viscous stress tensor

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Abstract

We prove global existence of appropriate weak solutions for the compressible Navier–Stokes equations for more general stress tensor than those covered by P.–L. LIONS and E. FEIREISL’s theory. More precisely we focus on more *general pressure laws* which are *not thermodynamically stable*; we are also able to handle some *anisotropy in the viscous stress tensor*. To give answers to these two longstanding problems, we revisit the classical compactness theory on the density by obtaining precise quantitative regularity estimates: This requires a more precise analysis of the structure of the equations combined to a novel approach to the compactness of the continuity equation. These two cases open the theory to important physical applications, for instance to describe solar events (virial pressure law), geophysical flows (eddy viscosity) or biological situations (anisotropy).

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1 Introduction

The question of global in time existence of solutions to fluid dynamics’ models goes back to the pioneering work by J. LERAY (1933) where he introduced the concept of weak (turbulent) solutions to the Navier–Stokes systems describing the motion of an INCOMPRESSIBLE fluid; this work has become the basis of the underlying mathematical theory up to present days. The theory for viscous COMPRESSIBLE fluids in a barotropic regime has, in comparison, been developed more recently in the monograph by P.–L. LIONS [49] (1993-1998), later extended by E. FEIREISL and collaborators [36] (2001) and has been since then a very active field of study.

When changes in temperature are not taken into account, the barotropic Navier–Stokes system reads

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div} \sigma = \rho f, \end{cases} \quad (1.1)$$

where ρ and u denote respectively the density and the velocity field. The stress tensor σ of a general fluid obeys Stokes’ law $\sigma = \mathcal{S} - P \operatorname{Id}$ where P is a scalar function termed pressure (depending on the density in the compressible barotropic setting or being an unknown in the incompressible setting) and \mathcal{S} denotes the viscous stress tensor which characterizes the measure of resistance of the fluid to flow.

Our approach also applies to the Navier–Stokes–Fourier system, as explained in section 10, which is considered more physically relevant. But our main purpose here is to explain how the new regularity method that we introduce can be applied to a wide range of Navier–Stokes like models and not to focus on a particular system. For

this reason, we discuss the main features of our new theory on the simpler (1.1). We indicate later in the article after the main ideas how to extend the result to some Navier-Stokes-Fourier systems.

In comparison with Leray's work on incompressible flows, which is nowadays relatively "simple" at least from the point of view of modern functional analysis (and in the linear viscous stress tensor case), the mathematical theory of weak solutions to compressible fluids is quite involved, bearing many common aspects with the theory of nonlinear conservation laws.

Our focus is on the global existence of weak solutions. For this reason we will not refer to the important question of existence of strong solutions or the corresponding uniqueness issues.

Several important problems about global existence of weak solutions for compressible flows remain open. We consider in this article the following questions

- General pressure laws, in particular without any monotonicity assumption;
- Anisotropy in the viscous stress tensor which is especially important in geophysics.

In the current LIONS-FEIREISL theory, the pressure law P is often assumed to be of the form $P(\rho) = a\rho^\gamma$ but this can be generalized, a typical example being

$$P \in \mathcal{C}^1([0, +\infty)), \quad P(0) = 0 \text{ with} \\ a\rho^{\gamma-1} - b \leq P'(\rho) \leq \frac{1}{a}\rho^{\gamma-1} + b \text{ with } \gamma > d/2 \quad (1.2)$$

for some constants $a > 0, b \geq 0$: See B. DUCOMET, E. FEIREISL, H. PETZELTOVA, I. STRASKRABA [29] or E. FEIREISL [33] for slightly more general assumptions. However it is always required that $P(\rho)$ be *increasing* after a certain critical value of ρ .

This monotonicity of P is connected to several well known difficulties

- The monotonicity of the pressure law is required for the stability of the thermodynamical equilibrium. Changes in monotonicity in the pressure are typically connected to intricate phase transition problems.
- At the level of compressible Euler, *i.e.* when $\mathcal{S} = 0$, non-monotone pressure laws may lead to a loss of hyperbolicity in the system, possibly leading to corrected systems (as by KORTEWEG in particular).

In spite of these issues, we are able to show that compressible Navier-Stokes systems like (1.1) are *globally well posed without monotonicity assumptions on the pressure law*; instead only rough growth estimates are required. This allows to *consider for the first time several famous physical laws such as modified virial expansions*.

As for the pressure law, the theory initiated in the multi-dimensional setting by P.-L. LIONS and E. FEIREISL requires that the stress tensor has the very specific form

$$\sigma = 2\mu D(u) + \lambda \operatorname{div} u \operatorname{Id} - P(\rho) \operatorname{Id}$$

with $D(u) = (\nabla u + \nabla u^T)/2$, μ and λ such that $\lambda + 2\mu/d \geq 0$. The coefficients λ and μ do not need to be constant but require some explicit regularity, see for instance [35] for temperature dependent coefficients.

Unfortunately several physical situations involve some anisotropy in the stress tensor; geophysical flows for instance use and require such constitutive laws, see for instance [61] and [16] with eddy viscosity in turbulent flows.

We present in this article the first results able to handle *more general viscous stress tensor* of the form

$$\sigma = A(t)\nabla u + \lambda \operatorname{div} u \operatorname{Id} - P(\rho) \operatorname{Id}$$

with a $d \times d$ symmetric matrix A with regular enough coefficients. The matrix A can incorporate anisotropic phenomena in the fluid. Note that our result also applies to the case

$$\sigma = A(t)D(u) + \lambda \operatorname{div} u \operatorname{Id} - P(\rho) \operatorname{Id}$$

where $D(u) = (\nabla u + \nabla u^T)/2$ still.

Our new results therefore significantly expand the reach of the current theory for compressible Navier-Stokes and make it more robust with respect to the large variety of laws of state and stress tensors that are used. This is achieved through a complete revisiting of the classical compactness theory by obtaining *quantitative regularity estimates*. The idea is inspired by estimates obtained for nonlinear continuity equations in [7], though with a different method than the one introduced here. Those estimates correspond to critical spaces, also developed and used for instance in works by J. BOURGAIN, H. BRÉZIS and P. MIRONESCU and by A.C. PONCE, see [12] and [58].

Because of the weak regularity of the velocity field, the corresponding norm of the critical space cannot be propagated. Instead the norm has to be modified by weights based on a auxiliary function which solves a kind of dual equation adapted to the compressible Navier–Stokes system under consideration. After proving appropriate properties of the weights, we can prove compactness on the density.

The plan of the article is as follows

- Section 2 presents the classical theory by P. – L. LIONS and E. FEIREISL, with the basic energy estimates. It explains why the classical proof of compactness does not seem able to handle the more general equations of state that concern us here. We also summarize the basic physical discussions on pressure laws and stress tensors choices which motivates our study. This section can be skipped by readers which are already familiar with the state of the art.
- In Section 3, we present the equations and the corresponding main results concerning global existence of weak solutions for non-monotone pressure law and then for anisotropic viscous stress tensor. Those are given in the barotropic setting.
- Section 4 is devoted to an introduction to our new method. We give our quantitative compactness criterion and we show the basic ideas in the simplex

context of linear uncoupled transport equations and a very rough sketch of proof in the compressible Navier–Stokes setting.

- Section 5 states the stability results which constitute the main contribution of the paper.
- Section 6 states technical lemmas which are needed in the main proof and are based on classical harmonic analysis tools: Maximal and square functions properties, translation of operators.
- Section 7 and 8 constitutes the heart of the proof. Section 7 is devoted to renormalized equation with definitions and properties of the weights. Section 8 is devoted to the proof of the stability results of section 5 both concerning more general pressure laws and concerning the anisotropic stress tensor.
- Section 9 concerns the construction of the approximate solutions. It uses the stability results of section 5 to conclude the proof of the existence theorems of section 3.
- The extension for non-monotone pressure laws (with respect to density) to the Navier–Stokes–Fourier system is discussed in Section 10. It contains a discussion of the state of the art complementing section 2 in that case. It follows some steps already included in the book [35] but also ask for a careful check of the estimates at each level of approximation.
- We present in Section 11 some models occurring in other contexts where the new mathematical techniques presented here could be useful in the future.
- Section 12 is a list of some of the notations that we use.
- Section 13 is an appendix recalling basic facts on Besov spaces which are used in the article.

2 Classical theory by E. FEIREISL and P.–L. LIONS, open problems and physical considerations

We consider for the moment compressible fluid dynamics in a general domain Ω which can be the whole space \mathbb{R}^d , a periodic box or a bounded smooth domain with adequate boundary conditions. We do not precise the boundary conditions and instead leave those various choices open as they may depend on the problem and we want to insist in this section on the common difficulties and approaches. We will later present our precise estimates in the periodic setting for simplicity.

2.1 A priori estimates

We collect the main physical *a priori* estimates for very general barotropic systems on $\mathbb{R}_+ \times \Omega$

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mathcal{D}u + \nabla P(\rho) = \rho f, \end{cases} \quad (2.1)$$

where \mathcal{D} is only assumed to be a negative differential operator in divergence form on u s.t.

$$\int_{\Omega} u \cdot \mathcal{D} u \, dx \sim - \int_{\Omega} |\nabla u|^2 \, dx, \quad (2.2)$$

and for any ϕ and u

$$\int_{\Omega} \phi \cdot \mathcal{D} u \, dx \leq C \|\nabla \phi\|_{L^2} \|\nabla u\|_{L^2}. \quad (2.3)$$

The following estimates form the basis of the classical theory of existence of weak solutions and we will use them in our own proof. We only give the formal derivation of the estimates at the time being.

First of all, the total energy of the fluid is dissipated. This energy is the sum of the kinetic energy and the potential energy (due to the compressibility) namely

$$E(\rho, u) = \int_{\Omega} \left(\rho \frac{|u|^2}{2} + \rho e(\rho) \right) \, dx,$$

where

$$e(\rho) = \int_{\rho_{\text{ref}}}^{\rho} P(s)/s^2 \, ds$$

with ρ_{ref} a constant reference density. Observe that formally from (2.1)

$$\partial_t \left(\rho \frac{|u|^2}{2} \right) + \operatorname{div} \left(\rho u \frac{|u|^2}{2} \right) - u \cdot \mathcal{D} u + u \cdot \nabla P(\rho) = \rho f \cdot u,$$

and thus

$$\frac{d}{dt} \int_{\Omega} \rho \frac{|u|^2}{2} - \int_{\Omega} u \cdot \mathcal{D} u - \int_{\Omega} P(\rho) \operatorname{div} u = \int_{\Omega} \rho f \cdot u.$$

On the other hand, by the definition of e , the continuity equation on ρ implies that

$$\partial_t (\rho e(\rho)) + \operatorname{div} (\rho e(\rho) u) + P(\rho) \operatorname{div} u = 0.$$

Integrating and combining with the previous equality leads to the energy equality

$$\frac{d}{dt} E(\rho, u) - \int_{\Omega} u \cdot \mathcal{D} u = \int_{\Omega} \rho f \cdot u. \quad (2.4)$$

Let us quantify further the estimates which follow from (2.4). Assume that $P(\rho)$ behaves roughly like ρ^γ in the following weak sense

$$C^{-1} \rho^\gamma - C \leq P(\rho) \leq C \rho^\gamma + C, \quad (2.5)$$

then $\rho e(\rho)$ also behaves like ρ^γ . Note that (2.5) does not imply any monotonicity on P which could keep oscillating. One could also work with even more general assumption than (2.5): Different exponents γ on the left-hand side and the right-hand side for instance... But for simplicity we use (2.5).

Assuming that f is bounded (or with enough integrability), one now deduces from (2.4) the following uniform bounds

$$\begin{aligned} \sup_t \int_{\Omega} \rho |u|^2 dx &\leq C + E(\rho^0, u^0), \\ \sup_t \int_{\Omega} \rho^\gamma dx &\leq C, \\ \int_0^T \int_{\Omega} |\nabla u|^2 dx &\leq C. \end{aligned} \tag{2.6}$$

We can now improve on the integrability of ρ , as it was first observed by P.-L. LIONS. Choose any smooth, positive $\chi(t)$ with compact support, and test the momentum equation by $\chi g = \chi \mathcal{B} \rho^a$ where \mathcal{B} is a linear operator (in x) s.t.

$$\operatorname{div} g = (\rho^a - \bar{\rho}^a), \quad \|\nabla g\|_{L^p} \leq C_p \|\rho^a - \bar{\rho}^a\|_{L^p}, \quad \|\mathcal{B} \phi\|_{L^p} \leq C_p \|\phi\|_{L^p}, \quad \forall 1 < p < \infty,$$

where we denote by $\bar{\rho}^a$ the average of ρ^a over Ω . Finding g is straightforward in the whole space but more delicate in bounded domain as the right boundary conditions must also be imposed. This is where E. FEIREISL *et al.* introduce the BOGOVSKI operator. We obtain that

$$\begin{aligned} \int \chi(t) \int_{\Omega} \rho^a P(\rho) dx dt &\leq \int \chi(t) \int_{\Omega} g (\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mathcal{D}u - \rho f) dx dt \\ &\quad + \int \chi(t) \int_{\Omega} \bar{\rho}^a P(\rho). \end{aligned}$$

By (2.5), the left-hand side dominates

$$\int \chi(t) \int_{\Omega} \rho^{a+\gamma} dx dt.$$

It is possible to bound the terms in the right-hand side. For instance by (2.3)

$$\begin{aligned} - \int \chi(t) g \mathcal{D}u dx dt &\leq C \|\nabla u\|_{L^2([0, T], L^2(\Omega))} \|\chi \nabla g\|_{L^2([0, T], L^2(\Omega))} \\ &\leq C \|\nabla u\|_{L^2([0, T], L^2(\Omega))} \|\chi(\rho^a - \bar{\rho}^a)\|_{L^2([0, T], L^2(\Omega))}, \end{aligned}$$

by the choice of g . Given the bound (2.6) on ∇u , this term does not pose any problem if $2a < a + \gamma$. Next

$$\int \chi g \partial_t(\rho u) dx dt = - \int (g \chi'(t) + \chi(t) \mathcal{B}(\partial_t(\rho^a - \bar{\rho}^a))) \rho u dx dt. \tag{2.7}$$

The first term in the right-hand side is easy to bound; as for the second one, the continuity equation implies

$$\begin{aligned} \int \chi(t) \mathcal{B}(\partial_t(\rho^a - \bar{\rho}^a)) \rho u dx dt &= - \int \chi [\mathcal{B}(\operatorname{div}(u \rho^a))] \rho u \\ &\quad - \int \chi \left[(a-1) \mathcal{B}(\rho^a \operatorname{div} u - \overline{\operatorname{div}(u \rho^a)} + (a-1) \rho^a \operatorname{div} u) \right] \rho u. \end{aligned} \tag{2.8}$$

Using the properties of \mathcal{B} and the energy estimates (2.6), it is possible to control those terms as well as the last one in (2.7), provided $a \leq 2\gamma/d - 1$ and $\gamma > d/2$ which leads to

$$\int_0^T \int_{\Omega} \rho^{\gamma+a} dx dt \leq C(T, E(\rho^0, u^0)). \quad (2.9)$$

2.2 Heuristic presentation of the method by E. FEIREISL and P. -L. LIONS

Let us explain, briefly and only heuristically the main steps to prove global existence of weak solutions in the barotropic case with constant viscosities and power γ pressure law. Our purpose is to highlight why a specific form of the pressure or of the stress tensor is needed in the classical approaches. We also refer for such a general presentation of the theory to the book by A. NOVOTNY and I. STRASKRABA [53], the monograph *Etats de la Recherche* edited by D. BRESCH [13] or the book by P. PLOTNIKOV and J. SOKOLOWSKI [57].

Let us first consider the simplest model with constant viscosity coefficients μ and λ . In that case, the compressible Navier–Stokes equation reads on $\mathbb{R}_+ \times \Omega$

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mu \Delta u - (\lambda + \mu) \nabla \operatorname{div} u + \nabla P(\rho) = \rho f, \end{cases} \quad (2.10)$$

with $P(\rho) = a\rho^\gamma$. For simplicity, we work in a smooth, Lipschitz regular, bounded domain Ω with homogeneous Dirichlet boundary conditions on the velocity

$$u|_{\partial\Omega} = 0. \quad (2.11)$$

A key concept for the existence of weak solutions is the notion of renormalized solution to the continuity equation, as per the theory for linear transport equations by R.J. DIPERNA and P.-L. LIONS. Assuming ρ and u are smooth and satisfy the continuity equation, for all $b \in \mathcal{C}([0, +\infty))$, one may multiply the equation by $b'(\rho)$ to find that (ρ, u) also

$$\partial_t b(\rho) + \operatorname{div}(b(\rho)u) + (b'(\rho)\rho - b(\rho))\operatorname{div} u = 0. \quad (2.12)$$

This leads to the following definition

Definition 2.1 *For any $T \in (0, +\infty)$, f, ρ_0, m_0 satisfying some technical assumptions, we say that a couple (ρ, u) is a weak renormalized solution with bounded energy if it has the following properties*

- $\rho \in L^\infty(0, T; L^\gamma(\Omega)) \cap \mathcal{C}^0([0, T], L_{\text{weak}}^\gamma(\Omega))$, $\rho \geq 0$ a.e. in $(0, T) \times \Omega$, $\rho|_{t=0} = \rho_0$ a.e. in Ω ;
- $u \in L^2(0, T; H_0^1(\Omega))$, $\rho|u|^2 \in L^\infty(0, T; L^1(\Omega))$, ρu is continuous in time with value in the weak topology of $L_{\text{weak}}^{2\gamma/(\gamma+1)}(\Omega)$, $(\rho u)|_{t=0} = m_0$ a.e. Ω ;
- (ρ, u) extended by zero out of Ω solves the mass and momentum equations in \mathbb{R}^d , in $\mathcal{D}'((0, T) \times \mathbb{R}^d)$;

- For any smooth b with appropriate monotony properties, $b(\rho)$ solves the renormalized Eq. (2.12).
- For almost all $\tau \in (0, T)$, (ρ, u) satisfies the energy inequality

$$E(\rho, u)(\tau) + \int_0^\tau \int_\Omega (\mu |\nabla u|^2 + (\lambda + \mu) |\operatorname{div} u|^2) \leq E_0 + \int_0^\tau \int_\Omega \rho f \cdot u.$$

In this inequality, $E(\rho, u)(\tau) = \int_\Omega (\rho |u|^2/2 + \rho e(\rho))(\tau)$, with $e(\rho) = \int_{\rho_{\text{ref}}}^\rho P(s)/s^2 ds$ (ρ_{ref} being any constant reference density), denotes the total energy at time τ and $E_0 = \int_\Omega |m_0|^2/2\rho_0 + \rho_0 e(\rho_0)$ denotes the initial total energy.

Assuming $P(\rho) = a\rho^\gamma$ (in that case $e(\rho)$ may equal to $a\rho^{\gamma-1}/(\gamma-1)$), the theory developed by P.-L. LIONS to prove the global existence of renormalized weak solution with bounded energy asks for some limitation on the adiabatic constant γ namely $\gamma > 3d/(d+2)$. E. FEIREISL *et al.* have generalized this approach in order to cover the range $\gamma > 3/2$ in dimension 3 and more generally $\gamma > d/2$ where d is the space dimension.

We present the initial proof due to P.-L. LIONS and indicate quickly at the end how it was improved by E. FEIREISL *et al.*. The method relies on the construction of a sequence of approximate solution, derivation of *a priori* estimates and passage to the limit which requires delicate compactness estimate. We skip for the time being the construction of such an approximate sequence, see for instance the book by A. NOVOTNY and I. STRASKRABA for details.

The approximate sequence, denoted by (ρ_k, u_k) , should satisfy the energy inequality leading to a first uniform *a priori bound*, using that $\mu > 0$ and $\lambda + 2\mu/d > 0$

$$\sup_t \int_\Omega (\rho_k |u_k|^2/2 + a\rho_k^\gamma/(\gamma-1)) dx + \mu \int_0^t \int_\Omega |\nabla u_k|^2 dx dt \leq C,$$

for some constant independent of n .

For $\gamma > d/2$, we also have the final *a priori* estimate (2.9) explained in the previous subsection namely

$$\int_0^\infty \int_\Omega \rho_k^{\gamma+a} \leq C(R, T) \text{ for } a \leq \frac{2}{d}\gamma - 1.$$

When needed for clarification, we denote by \bar{U} the weak limit of a general sequence U_k (up to a subsequence). Using the energy estimate and the extra integrability property proved on the density, and by extracting subsequences, one obtains the following convergence

$$\begin{aligned} \rho_k &\rightharpoonup \rho \text{ in } \mathcal{C}^0([0, T]; L_{\text{weak}}^\gamma(\Omega)), \\ \rho_k^\gamma &\rightharpoonup \bar{\rho}^\gamma \text{ in } L^{(\gamma+a)/\gamma}((0, T) \times \Omega), \\ \rho_k u_k &\rightharpoonup \rho u \text{ in } \mathcal{C}^0([0, T]; L^{2\gamma/(\gamma+1)}(\Omega)), \\ \rho_k u_k^i u_k^j &\rightharpoonup \rho u^i u^j \text{ in } \mathcal{D}'((0, T) \times \Omega) \text{ for } i, j = 1, 2, 3. \end{aligned}$$

The convergence of the nonlinear terms $\rho_k u_k$ and $\rho_k u_k \otimes u_k$ uses the compactness in time of ρ_k deduced from the uniform estimate on $\partial_t \rho_k$ given by the continuity equation and the compactness in time of $\sqrt{\rho_k} u_k$ deduced from the uniform estimate on $\partial_t(\rho_k u_k)$ given by the momentum equation. This is combined with the L^2 estimate on ∇u_k .

Consequently, the extensions by zero to $(0, T) \times \mathbb{R}^3/\Omega$ of the functions $\rho, u, \overline{\rho^\gamma}$, denoted again $\rho, u, \overline{\rho^\gamma}$, satisfy the system in $\mathbb{R}_+ \times \mathbb{R}^d$

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mu \Delta u - (\lambda + \mu) \nabla \operatorname{div} u + a \nabla \overline{\rho^\gamma} = \rho f. \end{cases} \quad (2.13)$$

The difficulty consists in proving that (ρ, u) is a renormalized weak solution with bounded energy and the main point is showing that $\overline{\rho^\gamma} = \rho^\gamma$ *a.e.* in $(0, T) \times \Omega$.

This requires compactness on the density sequence which cannot follow from the previous *a priori* estimates only. Instead P.-L. LIONS uses a weak compactness of the sequence $\{a\rho_k^\gamma - \lambda + 2\mu \operatorname{div} u_k\}_{k \in \mathbb{N}^*}$ which is usually called the viscous effective flux. This property was previously identified in one space dimension by D. HOFF and D. SERRE. More precisely, we have the following property for all function $b \in \mathcal{C}^1([0, +\infty))$ satisfying some increasing properties at infinity

$$\begin{aligned} \lim_{k \rightarrow +\infty} \int_0^T \int_\Omega (a\rho_k^\gamma - (2\mu + \lambda) \operatorname{div} u_k) b(\rho_k) \varphi dx dt \\ = \int_0^T \int_\Omega (a\overline{\rho^\gamma} - (2\mu + \lambda) \operatorname{div} u) \overline{b(\rho)} \varphi dx dt \end{aligned} \quad (2.14)$$

where the over-line quantities design the weak limit of the corresponding quantities and $\varphi \in \mathcal{D}((0, T) \times \Omega)$. Note that such a property is reminiscent of compensated compactness as the weak limit of a product is shown to be the product of the weak limits. In particular the previous property implies that

$$\overline{\rho \operatorname{div} u} - \rho \operatorname{div} u = \frac{\overline{P(\rho)} \rho - \overline{P(\rho)} \rho}{2\mu + \lambda} \quad (2.15)$$

Taking the divergence of the momentum equation, we get the relation

$$\Delta[(2\mu + \lambda) \operatorname{div} u_k - P(\rho_k)] = \operatorname{div}[\partial_t(\rho_k u_k) + \operatorname{div}(\rho_k u_k \otimes u_k)].$$

Note that here the form of the stress tensor has been strongly used. From this identity, P.-L. LIONS proved the property (2.14) based on harmonic analysis due to R. COIFMAN and Y. MEYER (regularity properties of commutators) and takes the observations by D. SERRE made in the one-dimensional case into account. The proof by E. FEIREISL is based on div-curl Lemma introduced by F. MURAT and L. TARTAR.

To simplify the remaining calculations, we assume $\gamma \geq 3d/(d+2)$ and in that case due to the extra integrability on the density, we get that $\rho_k \in L^2((0, T) \times \Omega)$. This lets us choose $b(s) = s \ln s$ in the renormalized formulation for ρ_k and ρ and

take the difference of the two equations. Then pass to the limit $n \rightarrow +\infty$ and use the identity of weak compactness on the effective flux to replace terms with divergence of velocity by terms with density using (2.15), leading to

$$\partial_t(\overline{\rho \ln \rho} - \rho \ln \rho) + \operatorname{div}((\overline{\rho \ln \rho} - \rho \ln \rho)u) = \frac{1}{2\mu + \lambda}(\overline{P(\rho)}\rho - \overline{P(\rho)\rho}).$$

Observe that the monotonicity of the pressure $P(\rho) = a\rho^\gamma$ implies that

$$\overline{P(\rho)}\rho - \overline{P(\rho)\rho} \leq 0.$$

This is the one point where the monotonicity assumption is used. It allows to show that the defect measure for the sequence of density satisfies

$$\operatorname{dft}[\rho_k - \rho](t) = \int_{\Omega} \overline{\rho \ln \rho}(t) - \rho \ln \rho(t) dx \leq \operatorname{dft}[\rho_k - \rho](t = 0).$$

On the other hand, the strict-convexity of the function $s \mapsto s \ln s$, $s \geq 0$ implies that $\operatorname{dft}[\rho_k - \rho] \geq 0$. If initially this quantity vanishes, it then vanishes at every later time.

Finally the commutation of the weak convergence with a strictly convex function yields *the strong convergence* of the density ρ_k in L^1_{loc} . Combined with the uniform bound of ρ_k in $L^{\gamma+a}((0, T) \times \Omega)$, we get the strong convergence of the pressure term ρ_k^γ .

This concludes the proof in the case $\gamma \geq 3d/(d+2)$. The proof of E. FEIREISL works even if the density is not *a priori* square integrable. For that E. FEIREISL observes that it is possible to control the amplitude of the possible oscillations on the density in a norm L^p with $p > 2$ allowing to use an effective flux property with some truncature. Namely he introduced the following oscillation measure

$$\operatorname{osc}_p[\rho_k - \rho] = \sup_{n \geq 1} [\limsup_{k \rightarrow +\infty} \|T_n(\rho_k) - T_n(\rho)\|_{L^p((0, T) \times \Omega)}],$$

where T_n are cut-off functions defined as

$$T_n(z) = nT\left(\frac{z}{n}\right), \quad n \geq 1$$

with $T \in \mathcal{C}^2(\mathbb{R})$

$$T(z) = z \text{ for } z \leq 1, \quad T(z) = 2 \text{ for } z \geq 3, \quad T \text{ concave on } \mathbb{R}.$$

The existence result can then obtained up to $\gamma > d/2$: See again the review by A. NOVOTNY and I. STRASKRABA [53].

To the author's knowledge there exists few extension of the previous study to more general pressure laws or more general stress tensor. Concerning a generalization of the pressure law, as explained in the introduction there exist the works by B. DUCOMET, E. FEIREISL, H. PETZELTOVA, I. STRASKRABA [29] and E. FEIREISL [33] where the hypothesis imposed on the pressure P imply that

$$P(z) = r_3(z) - r_4(z)$$

where r_3 is non-decreasing in $[0, +\infty)$ with $r_4 \in \mathcal{C}^2([0, +\infty))$ satisfying $r_4 \geq 0$ and $r_4(z) \equiv 0$ when $z \geq Z$ for a certain $Z \geq 0$. The form is used to show that it is possible respectively to continue to control the amplitude of the oscillations $\text{osc}_p[\rho_k - \rho]$ and then to show that the defect measure vanishes if initially it vanishes. The two papers [33] and [29] we refer to allow to consider for instance the two important cases : Van der Waals equation of state and some cold nuclear equations of state with finite number of monomial (see the subsection on the physical discussion).

2.3 The limitations of the LIONS-FEIREISL theory

The previous heuristical part makes explicit the difficulty in extending the global existence result for more general non-monotone pressure law or for non-isotropic stress tensor. First of all the key point in the previous approach was

$$\overline{P(\rho)} \rho - \overline{P(\rho) \rho} \leq 0.$$

This property is intimately connected to the monotonicity of $P(\rho)$ or of $P(\rho)$ for $\rho \geq \rho_c$ with truncation operators as in [33] or [29]. Non-monotone pressure terms cannot satisfy such an inequality and are therefore completely outside the current theory.

The difficulty with anisotropic stress tensor is that we are losing the other key relation in the previous proof namely (2.15). For non-isotropic stress tensor with an additional vertical component and power pressure law for instance, we get instead the following relation

$$\overline{\rho \text{div} u} - \rho \text{div} u \leq a \frac{\overline{\rho A_\mu \rho^\gamma} - \overline{\rho A_\mu} \overline{\rho^\gamma}}{\mu_x + \lambda}$$

with some non-local anisotropic operator $A_\mu = (\Delta - (\mu_z - \mu_x) \partial_z^2)^{-1} \partial_z^2$ where Δ is the total Laplacian in terms of (x, z) with variables $x = (x_1, \dots, x_{d-1})$, $z = x_d$.

Unfortunately, we are again losing the structure and in particular the sign of the right-hand side, as observed in particular in [16]. Furthermore even small anisotropic perturbations of an isotropic stress tensor cannot be controlled in terms of the defect measure introduced by E. FEIREISL and collaborators: Remark the non-local behavior in the right-hand side due to the term A_μ . For this reason, the anisotropic case seems to fall completely out the theory developed by P.-L. LIONS and E. FEIREISL.

Those two open questions are the main objective of this monograph.

2.4 Physical discussions on pressure laws and stress tensors

The derivation of the compressible Navier–Stokes system from first principles is delicate and goes well beyond the scope of this manuscript. In several respects the system is only an approximation and this should be kept in mind in any discussion of the precise form of the equations which should allow for some uncertainty.

2.4.1 Equations of state

It is in general a non straightforward question to decide what kind of pressure law should be used depending on the many possible applications: mixtures of fluids, solids, and even the interior of stars. Among possible equation of state, one can find several well known laws such as Dalton's law of partial pressures (1801), ideal gas law (Clapeyron 1834), Van der Waals equation of state (Van De Waals 1873), virial equation of state (H. Hamerlingh Onnes 1901).

In general the pressure law $P(\rho, \vartheta)$ can depend on both the density ρ and the temperature ϑ . We present the temperature dependent system in full later but include the temperature already in the present discussion to emphasize its relevance and importance.

Let us give some important examples of equations of state

- State equations are barotropic if $P(\rho)$ depends only on the density. As explained in the book by E. FEIREISL [34] (see pages 8–10 and 13–15), the simplest example of a barotropic flow is an isothermal flow where the temperature is assumed to be constant. If both conduction of heat and its generation by dissipation of mechanical energy can be neglected then the temperature is uniquely determined as a function of the density (if initially the entropy is constant) yielding a barotropic state equation for the pressure $P(\rho) = a\rho^\gamma$ with $a > 0$ and $\gamma = (R + c_v)/c_v > 1$. Another barotropic flow was discussed in [29].
- The classical Van der Waals equation reads

$$(P + a\rho^2)(b - \rho) = c\rho\vartheta,$$

where a , b , c are constants. The pressure law is non-monotone if the temperature is below a critical value, $\vartheta < \vartheta_c$, but it satisfies (1.2). In compressible fluid dynamics, the Van der Waals equation of state is sometimes simplified by neglecting specific volume changes and becomes

$$(P + a)(b - \rho) = c\rho\vartheta,$$

with similar properties.

- Using finite-temperature Hartree-Fock theory, it is possible to obtain a temperature dependent equation of state of the following form

$$P(\rho, \vartheta) = a_3(1 + \sigma)\rho^{2+\sigma} - a_0\rho^2 + k\vartheta \sum_{n \geq 1} B_n \rho^n, \quad (2.16)$$

where k is the Boltzmann's constant, and where the last expansion (a simplified virial series) converges rapidly because of the rapid decrease of the B_n .

- Equations of state can include other physical mechanism. A good example is found in the article [29] where radiation comes into play: a photon assembly is superimposed to the nuclear matter background. If this radiation is in quasi-local thermodynamical equilibrium with the (nuclear) fluid, the resulting

mixture nucleons+photons can be described by a one-fluid heat-conducting Navier–Stokes system, provided one adds to the equation of state a Stefan-Boltzmann contribution of black-body type

$$P_R(\vartheta) = a\vartheta^4 \text{ with } a > 0,$$

and provided one adds a corresponding contribution to the energy equation. The corresponding models are more complex and do not satisfy (1.2) in general.

- In the context of the previous example, a further simplification can be introduced leading to the so-called Eddington’s standard model. This approximation assumes that the ratio between the total pressure $P(\rho, \vartheta) = P_G(\rho, \vartheta) + P_R(\vartheta)$ and the radiative pressure $P_R(\vartheta)$ is a pure constant

$$\frac{P_R(\vartheta)}{P_G(\rho, \vartheta) + P_R(\vartheta)} = 1 - \beta,$$

where $0 < \beta < 1$ and P_G is given for instance by (2.16). Although crude, this model is in good agreement with more sophisticated models, in particular for the sun.

One case where this model leads to a pressure law satisfying (1.2) is when one keeps only the low order term into the virial expansion. Suppose that $\sigma = 1$ and let us plug the expression of the two pressure laws in this relation,

$$2a_3\rho^3 - a_0\rho^2 + kB_1\vartheta\rho = \frac{\beta}{1-\beta} \frac{a}{3} \vartheta^4.$$

By solving this algebraic equation to leading order (high temperature), one gets

$$\vartheta \approx \left(\frac{6a_3(1-\beta)}{a\beta} \right) \rho^{3/4},$$

leading to the pressure law

$$P(\rho, \vartheta) = \frac{2a_3}{\beta} \rho^3 - a_0\rho^2 + kB_1 \left(\frac{6a_3(1-\beta)}{a\beta} \right) \rho^{7/4},$$

which satisfies (1.2) because of the constant coefficients.

However in this approximation, only the higher order terms were kept. Considering non-constant coefficient or keeping all the virial sum in the pressure law was out of the scope of [29] and leads to precisely the type of non-monotone pressure laws that we consider in the present work.

- The virial equation of state for heat conducting Navier–Stokes equations can be derived from statistical physics and reads

$$P(\rho, \vartheta) = \rho\vartheta \left(\sum_{n \geq 0} B_n(\vartheta) \rho^n \right)$$

with $B_0 = \text{cte}$ and the coefficients $B_n(\vartheta)$ have to be specified for $n \geq 1$.

While the full virial pressure law is beyond the scope of this article, we can handle truncated virial with appropriate assumptions or pressure laws of the type $P(\rho, \vartheta) = P_e(\rho) + P_{th}(\rho, \vartheta)$

- Pressure laws can also incorporate many other type of phenomena. Compressible fluids may include or model biological agents which have their own type of interactions. In addition, as explained later, our techniques also apply to other types of “momentum” equations. The range of possible pressure laws is then even wider.

Based on these examples, the possibilities of pressure laws are many. Most are not monotone and several do not satisfy (1.2), proving the need for a theory able to handle all sort of behaviors.

2.4.2 Stress tensors

One finds a similar variety of stress tensors as for pressure laws. We recall that we denote $D(u) = (\nabla u + (\nabla u)^T)/2$.

- The isotropic stress tensor with constant coefficients

$$\mathcal{D} = \mu \Delta u + (\lambda + \mu) \nabla \operatorname{div} u,$$

which is the classical example that can be handled by the LIONS-FEIREISL theory: See for instance [49], [53] and [57] with $\gamma > d/2$. See also the recent interesting work by P.I. PLOTNIKOV and W. WEIGANT (see [56]) in the two-dimensional in space case with $\gamma = 1$.

- Isotropic stress tensors with non constant coefficients better represent the physics of the fluid however. Those coefficients can be temperature ϑ dependent

$$\mathcal{D} = 2 \operatorname{div} (\mu(\vartheta) D(u)) + \nabla (\lambda(\vartheta) \operatorname{div} u).$$

Provided adequate non-degeneracy conditions are made on μ and λ , this case can still be efficiently treated by the LIONS-FEIREISL theory under some assumptions on the pressure law. See for instance [34] or [35].

- The coefficients of the isotropic stress tensors may also depend on the density

$$\mathcal{D} = 2 \operatorname{div} (\mu(\rho, \vartheta) D(u)) + \nabla (\lambda(\rho, \vartheta) \operatorname{div} u).$$

This is a very difficult problem in general. The almost only successful insight in this case can be found in [15, 16, 17, 62] with no dependency with respect to the temperature. Those articles require a very special form of $\mu(\rho)$ and $\lambda(\rho)$ and without such precise assumptions, almost nothing is known. Note also the very nice paper concerning global existence of strong solutions in two-dimension by A. KAZHIKHOV and V.A. VAIGANT where μ is constant but $\lambda = \rho^\beta$ with $\beta \geq 3$, see [44].

- Geophysical flow cannot in general be assumed to be isotropic but instead some directions have different behaviors; this can be due to gravity in large scale fluids for instance. A nice example is found in the Handbook written by R. TEMAM and M. ZIANE, where the eddy-viscous term \mathcal{D} is given by

$$\mathcal{D} = \mu_h \Delta_x u + \mu_z \partial_z^2 u + (\lambda + \mu) \nabla \operatorname{div} u,$$

with $\mu_h \neq \mu_z$. While such an anisotropy only requires minor modifications for the incompressible Navier–Stokes system, it is not compatible with the LIONS-FEIREISL approach, see for instance [16].

3 Equations and main results: The barotropic case

We will from now on work on the torus Π^d . This is only for simplicity in order to avoid discussing boundary conditions or the behavior at infinity. The proofs would easily extend to other cases as mentioned at the end of the paper.

3.1 Statements of the results: Theorem 3.1 and 3.2

We present in this section our main existence results. As usual for global existence of weak solutions to nonlinear PDEs, one has to prove stability estimates for sequences of approximate solutions and construct such approximate sequences. The main contribution in this paper and the major part of the proofs concern the stability procedure and more precisely the compactness of the density.

I) Isotropic compressible Navier–Stokes equations with general pressure.

Let us consider the isotropic compressible Navier–Stokes equations

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mu \Delta u - (\lambda + \mu) \nabla \operatorname{div} u + \nabla P(\rho) = \rho f, \end{cases} \quad (3.1)$$

with $2\mu/d + \lambda$, a pressure law P which is continuous on $[0, +\infty)$, P locally Lipschitz on $(0, +\infty)$ with $P(0) = 0$ such that there exists $C > 0$ with

$$C^{-1} \rho^\gamma - C \leq P(\rho) \leq C \rho^\gamma + C \quad (3.2)$$

and for all $s \geq 0$

$$|P'(s)| \leq \bar{P} s^{\tilde{\gamma}-1}. \quad (3.3)$$

One then has global existence

Theorem 3.1 *Assume that the initial data u_0 and $\rho_0 \geq 0$ with $\int_{\Pi^d} \rho_0 = M_0 > 0$ satisfies the bound*

$$E_0 = \int_{\Pi^d} \left(\frac{|(\rho u)_0|^2}{2\rho_0} + \rho_0 e(\rho_0) \right) dx < +\infty.$$

Let the pressure law P satisfies (3.2) and (3.3) with

$$\gamma > (\max(2, \tilde{\gamma}) + 1) \frac{d}{d+2}. \quad (3.4)$$

Then there exists a global weak solution of the compressible Navier–Stokes system (3.1) in the sense of Definition 2.1.

Remark. Let us note that the solution satisfies the explicit regularity estimate

$$\int_{\Pi^{2d}} \mathbb{I}_{\rho_k(x,t) \geq \eta} \mathbb{I}_{\rho_k(y,t) \geq \eta} K_h(x-y) \chi(\delta\rho_k)(t) \leq \frac{C \|K_h\|_{L^1}}{\eta^{1/2} |\log h|^{\theta/2}},$$

for some $\theta > 0$ where K_h is defined in proposition 4.1, $\delta\rho_k$ and χ are defined in Section 8, see (8.1).

II) A non-isotropic compressible Navier–Stokes equations. We consider an example of non-isotropic compressible Navier–Stokes equations

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(A(t) \nabla u) - (\mu + \lambda) \nabla \operatorname{div} u + \nabla P(\rho) = 0, \end{cases} \quad (3.5)$$

with $A(t)$ a given smooth and symmetric matrix, satisfying

$$A(t) = \mu \operatorname{Id} + \delta A(t), \quad \mu > 0, \quad \frac{2}{d} \mu + \lambda - \|\delta A(t)\|_{L^\infty} > 0. \quad (3.6)$$

where δA will be a perturbation around $\mu \operatorname{Id}$. We again take P locally Lipschitz on $[0, +\infty)$ with $P(0) = 0$ but require it to be monotone after a certain point

$$C^{-1} \rho^{\gamma-1} - C \leq P'(\rho) \leq C \rho^{\gamma-1} + C. \quad (3.7)$$

with $\gamma > d/2$. The second main result that we obtain is

Theorem 3.2 *Assume that the initial data u_0 and $\rho_0 \geq 0$ with $\int_{\Pi^d} \rho_0 = M_0 > 0$ satisfies the bound*

$$E_0 = \int_{\Pi^d} \left(\frac{|(\rho u)_0|^2}{2\rho_0} + \rho_0 e(\rho_0) \right) dx < +\infty.$$

Let the pressure P satisfy (3.7) with

$$\gamma > \frac{d}{2} \left[\left(1 + \frac{1}{d} \right) + \sqrt{1 + \frac{1}{d^2}} \right].$$

There exists a universal constant $C_\star > 0$ such that if

$$\|\delta A\|_\infty \leq C_\star (2\mu + \lambda),$$

then there exists a global weak solution of the compressible Navier–Stokes equation in the sense of Definition 2.1 replacing the isotropic energy inequality by the following anisotropic energy

$$E(\rho, u)(\tau) + \int_0^\tau \int_\Omega (\nabla_x u^T A(t) \nabla u + (\mu + \lambda) |\operatorname{div} u|^2) \leq E_0.$$

Remark. Let us note that the constraint on γ corresponds to the constraint on p : $p > \gamma + \gamma/(\gamma - 1)$ where p is the extra integrability property on ρ .

3.2 Important comments/comparison with previous results

The choice was made to focus on explaining the new method instead of trying to write results as general as possible but at the cost of further burdening the proofs. For this reason, Theorems 3.1 and 3.2 are only two examples of what can be done.

We explain how to apply our new method to the Navier–Stokes–Fourier system (with an additional equation for temperature) in section 10. The Navier–Stokes–Fourier system is physically more relevant than the barotropic case and as seen from the discussion in subsection 2.4, it exhibits even more examples of non-monotone pressure laws.

I) *Possible extensions.* In section 11, we also present applications to various other important models, in particular in the Bio-Sciences where the range of possible pressure laws (or what plays their role such as chemical attraction/repulsion) is wide. But there are many other possible extensions; for instance (3.2) could be replaced with a more general

$$C^{-1} \rho^{\gamma_1} - C \leq P(\rho) \leq C \rho^{\gamma_2} + C,$$

with different exponents $\gamma_1 \neq \gamma_2$. While the proofs would essentially remain the same, the assumption (3.4) would then have to be replaced and would involve γ_1 and γ_2 . Similarly, it is possible to consider spatially dependent stress tensor $A(t, x)$ in Theorem 3.2. This introduces additional terms in the proof but those can easily be handled as long as A is smooth by classical methods for pseudo differential operators.

II) *Comparison with previous results.*

II-1) *Non-monotone pressure laws.* Theorem 3.1 is the first result to allow for completely non-monotone pressure laws. Among many important previous contributions, we refer to [29, 33, 19, 49] and [34, 35, 53, 20] for the Navier–Stokes–Fourier system, which are our main point of comparison. All of those require $P' > 0$ after a certain point and in fact typically a condition like (3.7). The removal of the key assumption of monotonicity has important consequences:

- From the physical and modeling point of view, it opens the possibility of working with a wider range of equations of state as discussed in subsection 2.4 and it makes the current theory on viscous, compressible fluids more robust to perturbation of the model.

- Changes of monotonicity in P can create and develop oscillations in the density ρ (because some “regions” of large density become locally attractive). It was a major question whether such oscillations remain under control at least over bounded time intervals. This shows that the stability for bounded times is very different from uniform in time stability as $t \rightarrow +\infty$. Only the latter requires assumptions of thermodynamical nature such as the monotonicity of P .
- Obviously well posedness for non monotone P could not be obtained as is done here for the compressible Euler system. As can be seen from the proofs, the viscous stress tensor in the compressible Navier-Stokes system has precisely the critical scaling to control the oscillations created by the non-monotonicity. This implies for instance that in phase transition phenomena, the transition occurs smoothly precisely at the scale of the viscosity.
- Our results could have further consequences for instance to show convergence of numerical schemes (or for other approximate systems). Typical numerical schemes for compressible Navier–Stokes raise issues of oscillations in the density which are reminiscent of the ones faced in this article. The question of convergence of numerical schemes to compressible Navier–Stokes is an important and delicate subject in its own, going well beyond the scope of this short comment. We refer for instance to the works by R. EYMARD, T. GALLOUËT, R. HERBIN, J.–C. LATCHÉ and T.K. KARPER: See for instance [38, 31] for the simpler Stokes case, [30, 37, 39] for Navier–Stokes and more recently to the work [21, 45].

Concerning the requirement on the growth of the pressure at ∞ , that is on the coefficient γ in (3.4)

- In the typical case where $\tilde{\gamma} = \gamma$, (3.4) leads to the same constraint as in P.–L. LIONS [49] for a similar reason: The need to have $\rho \in L^2$ to make sense of $\rho \operatorname{div} u$. It is worse than the $\gamma > d/2$ required for instance in [34]. In $3d$, we hence need $\gamma > 9/5$ versus only $\gamma > 3/2$ in [34].
- It may be possible to improve on (3.4) while still using the method introduced here but propagating compactness on appropriate truncation of ρ ; for instance by writing an equivalent of Lemma 7.1 on $\phi(\rho(x)) - \phi(\rho(y))$ as in the multi-dimensional setting by E. FEIREISL. This possibility was left to future works. Note that the requirement on $\gamma > d/2$ comes from the need to gain integrability as per (2.9) along the strategy presented in subsection 2.1. Our new method still relies on this estimate and therefore has no hope, on its own, to improve on the condition $\gamma > d/2$.
- In the context of general pressure laws, and even more so for Navier–Stokes–Fourier, assumption (3.4) is not a strong limitation. Virial-type pressure laws, where $P(\rho)$ is a polynomial expansion, automatically satisfy it for instance as do many other examples discussed in subsection 2.4.

II-2) *Anisotropic stress tensor.* Theorem 3.2 is so far the only result of global existence of weak solutions which is able to handle anisotropy in the stress tensor. It

applies for instance to eddy-viscous tensor mentioned above for geophysical flows

$$\mathcal{D} = \mu_h \Delta_x u + \mu_z \partial_z^2 u + (\lambda + \mu) \nabla \operatorname{div} u,$$

where $\mu_h \neq \mu_z$ and corresponding to

$$A_{ij} = \mu_h \delta_{ij} \quad \text{for } i, j = 1, 2, \quad A_{33} = \mu_z, \quad A_{ij} = 0 \quad \text{otherwise.} \quad (3.8)$$

This satisfies the assumptions of Theorem 3.2 provided $|\mu_h - \mu_z|$ is not too large, which is usually the case in the context of geophysical flows.

Additional examples of applications are given in section 11 but we wish to emphasize here that it is also possible to have a fully symmetric anisotropy, namely $\operatorname{div}(A D u)$ with $D(u) = \nabla u + \nabla u^T$ in the momentum equation. This is the equivalent of the anisotropic case in linear elasticity and it is also an important case for compressible fluids. Note that it leads to a different form of the stress tensor. With the above choice of A , Eq. 3.8, one would instead obtain

$$\operatorname{div}(A D u) = \mu_h \Delta_x u + \mu_z \partial_z^2 u + \mu_z \nabla \partial_z u_z + (\lambda + \mu) \nabla \operatorname{div} u.$$

Accordingly we chose to write Theorem 3.2 with the non-symmetric anisotropy $\operatorname{div}(A \nabla u)$ as it corresponds to the eddy-viscous term by R. TEMAM and M. ZIANE mentioned above. But the extension to the symmetric anisotropy is possible although it introduces some minor complications. For instance one cannot simply obtain $\operatorname{div} u$ by solving a scalar elliptic system but one has to solve a vector valued one instead; please see the remark just after (5.10) and at the end of the proof of Theorem 3.2 in section 9.

Ideally one would like to obtain an equivalent of Theorem 3.2 assuming only uniform elliptic bounds on $A(t)$ and much lower bound on γ . Theorem 3.2 is a first attempt in that direction, which can hopefully later be improved.

However the reach of Theorem 3.2 should not be minimized because non isotropy in the stress tensor appears to be a level of difficulty above even non-monotone pressure laws. Losing the pointwise relation between $\operatorname{div} u$ and $P(\rho(x))$ is a *major hurdle*, as it can also be seen from the proofs later in the article. Instead one has to work with

$$\operatorname{div} u(t, x) = P(\rho(t, x)) + L P(\rho) + \text{effective pressure},$$

with L a non local operator of order 0. The difficulty is to control appropriately this non local term so that its contribution can eventually be bounded by the dissipation due to the local pressure term.

Notations. For simplicity, in the rest of the article, C will denote a numerical constant whose value may change from line to line. It may depend on some uniform estimates on the sequences of functions considered (as per bounds (5.5) or (5.4) for instance) but it will never depend on the sequence under consideration (denoted with index k) or the scaling parameters h or h_0 .

4 Sketch of the new compactness method

The standard compactness criteria used in the compressible Navier–Stokes framework is the Aubin–Lions–Simon Lemma to get compactness on the terms ρu and $\rho u \otimes u$. A more complex trick is used to get the strong convergence of the density. More precisely it combines extra integrability estimates on the density and the effective flux property (a kind of weak compactness) and then a convexity-monotonicity tool to conclude.

Here we present a tool which will be the cornerstone in our study to prove compactness on the density and which will be appropriate to cover more general equation of state or stress tensor form.

In order to give the main idea of the method, we present it first in this section for the well known case of linear transport equations, *i.e.* assuming that u is given. We then give a rough sketch of the main ideas we will use in the rest of the article. This presents the steps we will follow for proofs in the more general setting.

4.1 The compactness criterion

We start by a well known result providing compactness of a sequence

Proposition 4.1 *Let ρ_k be a sequence uniformly bounded in some $L^p((0, T) \times \Pi^d)$ with $1 \leq p < \infty$. Assume that \mathcal{K}_h is a sequence of positive, bounded functions s.t.*

- i.* $\forall \eta > 0, \quad \sup_h \int_{|x| \geq \eta} \mathcal{K}_h(x) dx < \infty, \quad \text{supp } \mathcal{K}_h \in B(0, R),$
- ii.* $\|\mathcal{K}_h\|_{L^1(\Pi^d)} \rightarrow +\infty.$

If $\partial_t \rho_k \in L^q([0, T] \times W^{-1,q}(\Pi^d))$ with $q \geq 1$ uniformly in k and

$$\limsup_k \left[\frac{1}{\|\mathcal{K}_h\|_{L^1}} \limsup_{t \in [0, T]} \int_{\Pi^{2d}} \mathcal{K}_h(x-y) |\rho_k(t, x) - \rho_k(t, y)|^p dx dy \right] \rightarrow 0, \quad \text{as } h \rightarrow 0$$

then ρ_k is compact in $L^p([0, T] \times \Pi^d)$. Conversely if ρ_k is compact in $L^p([0, T] \times \Pi^d)$ then the above quantity converges to 0 with h .

For the reader's convenience, we just quickly recall that the compactness in space is connected to the classical approximation by convolution. Denote $\bar{\mathcal{K}}_h$ the normalized kernel

$$\bar{\mathcal{K}}_h = \frac{\mathcal{K}_h}{\|\mathcal{K}_h\|_{L^1}}.$$

Write

$$\begin{aligned} \|\rho_k - \bar{\mathcal{K}}_h \star_x \rho_k\|_{L^p}^p &\leq \frac{1}{\|\mathcal{K}_h\|_{L^1}^p} \int_{\Pi^d} \left(\int_{\Pi^d} \mathcal{K}_h(x-y) |\rho_k(t, x) - \rho_k(t, y)| dx \right)^p dy \\ &\leq \frac{1}{\|\mathcal{K}_h\|_{L^1}} \int_{\Pi^{2d}} \mathcal{K}_h(x-y) |\rho_k(t, x) - \rho_k(t, y)|^p dx dy \end{aligned}$$

which converges to zero as $h \rightarrow 0$ uniformly in k by assumption. On the other-hand for a fixed h , the sequence $\overline{K}_h \star_x u_k$ in k is compact in x . This complete the compactness in space. Concerning the compactness in time, we just have to couple everything and use the uniform bound on $\partial_t \rho_k$ as per the usual AUBIN-LIONS-SIMON Lemma.

The \mathcal{K}_h functions. In this paper we choose

$$K_h(x) = \frac{1}{(h + |x|)^a}, \quad \text{for } |x| \leq 1/2.$$

with some $a > d$ and K_h non negative, independent of h for $|x| \geq 2/3$, K_h constant outside $B(0, 3/4)$ and periodized such as to belong in $C^\infty(\Pi^d \setminus B(0, 3/4))$. For convenience we denote

$$\overline{K}_h(x) = \frac{K_h(x)}{\|K_h\|_{L^1}}.$$

This is enough for linear transport equations but for compressible Navier-Stokes we also need

$$\mathcal{K}_{h_0}(x) = \int_{h_0}^1 \overline{K}_h(x) \frac{dh}{h}.$$

We will see that

$$\frac{C^{-1}}{(h_0 + |x|)^d} \leq \mathcal{K}_{h_0}(x) \leq \frac{C}{(h_0 + |x|)^d}.$$

4.2 Compactness for linear transport equation

Consider a sequence of solutions ρ_k , on the torus Π^d (so as to avoid any discussion of boundary conditions or behavior at infinity) to

$$\partial_t \rho_k + \operatorname{div}(\rho_k u_k) = 0, \tag{4.1}$$

where u_k is assumed to satisfy for some $1 < p \leq \infty$

$$\sup_k \|u_k\|_{L_t^p W_x^{1,p}} < \infty, \tag{4.2}$$

with $\operatorname{div} u_k$ compact in x , *i.e.*

$$\limsup_k \varepsilon_k(h) = \frac{1}{\|\overline{K}_h\|_{L^1}} \int_0^T \int_{\Pi^{2d}} \overline{K}_h(x-y) |\operatorname{div} u_k(t,x) - \operatorname{div} u_k(t,y)|^p \longrightarrow 0, \tag{4.3}$$

as $h \rightarrow 0$. The condition on the divergence is replaced by bounds on ρ_k

$$\frac{1}{C} \leq \inf_x \rho_k \leq \sup_x \rho_k \leq C, \quad \forall t \in [0, T]. \tag{4.4}$$

One then has the well known

Proposition 4.2 *Assume ρ_k solves (4.1) with the bounds (4.2), (4.3) and (4.4); assume moreover that the initial data ρ_k^0 is compact. Then ρ_k is locally compact and more precisely*

$$\int_{\Pi^{2d}} K_h(x-y) |\rho_k(t,x) - \rho_k(t,y)| dx dy \leq C \frac{\|K_h\|_{L^1}}{|\log(h + \varepsilon_k(h) + \tilde{\varepsilon}_k(h))|},$$

where

$$\tilde{\varepsilon}_k(h) = \frac{1}{\|K_h\|_{L^1}} \int_{\Pi^{2d}} K_h(x-y) |\rho_k^0(x) - \rho_k^0(y)| dx dy.$$

This type of results for *non Lipschitz* velocity fields u_k was first obtained by R.J. DI PERNA and P.-L. LIONS in [27] with the introduction of renormalized solutions for $u_k \in W^{1,1}$ and appropriate bounds on $\operatorname{div} u_k$. This was extended to $u_k \in BV$, first by F. BOUCHUT in [9] in the kinetic context (see also M. HAURAY in [41]) and then by L. AMBROSIO in [4] in the most general case. We also refer to C. LE BRIS and P.-L. LIONS in [46, 49], and to the nice lecture notes written by C. DE LELLIS in [25]. In general $u_k \in BV$ is the optimal regularity as shown by N. DEPAUW in [26]. This can only be improved with specific additional structure, such as provided by low dimension, see [3, 11, 22, 23, 40], Hamiltonian properties [18, 43], or as a singular integral [10].

Of more specific interest for us are the results which do not require bounds on $\operatorname{div} u_k$ (which are not available for compressible Navier–Stokes) but replace them by bounds on ρ_k , such as (4.4). The compactness in Prop. 4.2 was first obtained in [5].

Explicit regularity estimates of ρ_k have first been derived by G. CRIPPA and C. DE LELLIS in [24] (see also [42] for the $W^{1,1}$ case). These are based on explicit control on the characteristics. While it is quite convenient to work on the characteristics in many settings, this is not the case here, in particular due to the coupling between $\operatorname{div} u_k$ and $p(\rho_k)$.

In many respect the proof of Prop. 4.2 is an equivalent of the method of G. CRIPPA and C. DE LELLIS in [24] at the PDE level, instead of the ODE level. Its interest will be manifest later in the article when dealing with the full Navier–Stokes system. The idea of controlling the compactness of solutions to transport equations through estimates such as provided by Prop. 4.1 was first introduced in [7] but relied on a very different method.

Proof One does not try to propagate directly

$$\int_{\Pi^{2d}} K_h(x-y) |\rho_k(t,x) - \rho_k(t,y)| dx dy.$$

Instead one introduces the weight $w(t,x)$ solution to the auxiliary equations

$$\partial_t w + u_k \cdot \nabla w = -\lambda M |\nabla u_k|, \quad w|_{t=0} = 1, \quad (4.5)$$

where $M f$ denotes the maximal function of f (recalled in Section 6) and λ is constant to be chosen large enough.

First step: Propagation of a weighted regularity. We propagate

$$R(t) = \int_{\Pi^{2d}} K_h(x-y) |\rho_k(t,x) - \rho_k(t,y)| w(t,x) w(t,y) dx dy.$$

Using (4.1), we obtain that

$$\begin{aligned} & \partial_t |\rho_k(x) - \rho_k(y)| + \operatorname{div}_x (u_k(x) |\rho_k(x) - \rho_k(y)|) + \operatorname{div}_y (u_k(y) |\rho_k(x) - \rho_k(y)|) \\ & \leq \frac{1}{2} (\operatorname{div} u_k(x) + \operatorname{div} u_k(y)) |\rho_k(x) - \rho_k(y)| \\ & \quad - \frac{1}{2} (\operatorname{div} u_k(x) - \operatorname{div} u_k(y)) (\rho_k(x) + \rho_k(y)) s_k, \end{aligned}$$

where $s_k = \operatorname{sign}(\rho_k(x) - \rho_k(y))$. We refer to subsection 7.1 for the details of this calculation, which is rigorously justified for a fixed k through the theory of renormalized solutions in [27]. From this equation on $|\rho_k(x) - \rho_k(y)|$, we deduce

$$\begin{aligned} \frac{d}{dt} R(t) &= \int_{\Pi^{2d}} \nabla K_h(x-y) (u_k(x) - u_k(y)) |\rho_k(t,x) - \rho_k(t,y)| w(t,x) w(t,y) \\ & \quad - \frac{1}{2} \int_{\Pi^{2d}} K_h(x-y) (\operatorname{div} u_k(x) - \operatorname{div} u_k(y)) (\rho_k(x) - \rho_k(y)) s_k w(x) w(y) \\ & \quad + \int_{\Pi^{2d}} K_h(x-y) |\rho_k(x) - \rho_k(y)| \left(\partial_t w + u_k \cdot \nabla w + \frac{1}{2} \operatorname{div} u_k w \right) w(y) \\ & \quad + \text{symmetric}. \end{aligned}$$

Observe that by (4.5), $w \leq 1$ and therefore by (4.4) and the definition of ε_k in (4.3), the second term in the right-hand side is easily bounded

$$\int_{\Pi^{2d}} K_h(x-y) (\operatorname{div} u_k(x) - \operatorname{div} u_k(y)) (\rho_k(x) - \rho_k(y)) s_k w(x) w(y) \leq C \varepsilon_k(h).$$

For the first term, one uses the well known inequality (see [59, 60] or section 6)

$$|u_k(x) - u_k(y)| \leq C_d |x - y| (M |\nabla u_k|(x) + M |\nabla u_k|(y)),$$

combined with the remark that from the choice of K_h

$$|\nabla K_h(x-y)| |x-y| \leq C K_h(x-y).$$

Therefore

$$\begin{aligned} & \int_{\Pi^{2d}} \nabla K_h(x-y) (u_k(x) - u_k(y)) |\rho_k(t,x) - \rho_k(t,y)| w(t,x) w(t,y) \\ & \leq C \int_{\Pi^{2d}} K_h(x-y) (M |\nabla u_k|(x) + M |\nabla u_k|(y)) |\rho_k(x) - \rho_k(y)| w(x) w(y), \end{aligned}$$

and combining everything

$$\begin{aligned} \frac{d}{dt}R(t) &\leq C \|K_h\|_{L^1} \varepsilon_k(h) \\ &+ \int_{\Pi^{2d}} K_h |\rho_k(x) - \rho_k(y)| (\partial_t w + u_k \cdot \nabla w + C (\operatorname{div} u_k + M |\nabla u_k|) w) w(y) \\ &+ \text{symmetric}. \end{aligned}$$

Since $\operatorname{div} u_k \leq d |\nabla u_k| \leq d M |\nabla u_k|$, by taking the constant λ large enough in (4.5),

$$\partial_t w + u_k \cdot \nabla w + C (\operatorname{div} u_k + M |\nabla u_k|) w \leq 0,$$

and hence

$$R(t) \leq \int_{\Pi^{2d}} K_h |\rho_k^0(x) - \rho_k^0(y)| + C t \|K_h\|_{L^1} \varepsilon_k(h) \leq C \|K_h\|_{L^1} (\varepsilon_k(h) + \tilde{\varepsilon}_k(h)). \quad (4.6)$$

Second step: property of the weight. We need to control the measure of the set where the weight w is small. Obviously if w were to vanish everywhere then the control of $R(t)$ would be trivial but of very little interest. From Eq. (4.5)

$$\partial_t(\rho_k |\log w|) + \operatorname{div}(\rho_k \log w) = \lambda \rho_k M |\nabla u_k|.$$

And thus

$$\frac{d}{dt} \int_{\Pi^d} |\log w| \rho_k dx = \lambda \int_{\Pi^d} \rho_k M |\nabla u_k| dx \leq \lambda \|\rho_k\|_{L^{p^*}} \|M |\nabla u_k|\|_{L^p} \leq C, \quad (4.7)$$

by (4.2), (4.4) and the fact that the maximal function is bounded on L^p for $p > 1$.

Third step: Conclusion of the proof. Using again (4.4)

$$|\{x, w(t, x) \leq \eta\}| \leq \frac{C}{|\log \eta|} \int_{\Pi^d} |\log w| \rho_k dx \leq \frac{C}{|\log \eta|}.$$

Thus

$$\begin{aligned} &\int_{\Pi^{2d}} K_h(x-y) |\rho_k(t, x) - \rho_k(t, y)| \\ &= \int_{w(x) > \eta, w(y) > \eta} K_h |\rho_k(t, x) - \rho_k(t, y)| \\ &+ \int_{w(x) \leq \eta \text{ or } w(y) \leq \eta} K_h(x-y) |\rho_k(t, x) - \rho_k(t, y)|, \end{aligned}$$

and so

$$\begin{aligned} \int_{\Pi^{2d}} K_h(x-y) |\rho_k(t, x) - \rho_k(t, y)| &\leq \frac{1}{\eta^2} \int K_h |\rho_k(t, x) - \rho_k(t, y)| w(t, x) w(t, y) \\ &+ \frac{C}{|\log \eta|} \|K_h\|_{L^1} \\ &\leq C \|K_h\|_{L^1} \left(\frac{\varepsilon_k(h) + \tilde{\varepsilon}_k(h)}{\eta^2} + \frac{1}{|\log \eta|} \right), \end{aligned}$$

which by minimizing in η finishes the proof. \square

4.3 A rough sketch of the extension to compressible Navier–Stokes

The aim of this subsection is to provide a rough idea of how to extend the previous method. We will only consider the case of general pressure laws and assume that the stress tensor is isotropic. When now considering the compressible Navier–Stokes system, the divergence $\operatorname{div} u_k$ is not given anymore but has to be calculated from ρ_k through a relation of the kind

$$\operatorname{div} u_k = P(\rho_k) + R_k, \quad (4.8)$$

where R_k includes the force applied on the fluid, the effective stress tensor... To keep things as simple as possible here, we temporarily assume that

$$\begin{aligned} \sup_k \|R_k\|_{L^\infty} &< \infty, \\ \limsup_k \varepsilon_k(h) &= \frac{1}{\|K_h\|_{L^1}} \int_0^T \int_{\Pi^{2d}} K_h(x-y) |R_k(t,x) - R_k(t,y)|^p \longrightarrow 0. \end{aligned} \quad (4.9)$$

We do not assume monotonicity on the pressure P but simply the control

$$|P'(\rho)| \leq C \rho^{\gamma-1}. \quad (4.10)$$

A modification of the previous proof then yields

Proposition 4.3 *Assume ρ_k solves (4.1) and the bounds*

$$\sup_k \|\rho_k\|_{L_t^\infty L_x^1} < \infty, \quad \sup_k \|\rho_k\|_{L_{t,x}^p} < \infty \quad \text{with } p \geq \gamma + 1.$$

Assume moreover that $\sup_k \|u_k\|_{L_t^2 H_x^1} < \infty$, that (4.8) holds with the bounds (4.9) on R_k and (4.10) on P . Then ρ_k is locally compact away from vacuum and more precisely

$$\int_{\rho_k(x) \geq \eta, \rho_k(y) \geq \eta} K_h(x-y) |\rho_k(t,x) - \rho_k(t,y)| dx dy \leq C_\eta \frac{\|K_h\|_{L^1}}{|\log(h + (\varepsilon_k(h)) + \tilde{\varepsilon}_k(h))|},$$

where

$$\tilde{\varepsilon}_k(h) = \frac{1}{\|K_h\|_{L^1}} \int_{\Pi^{2d}} K_h(x-y) |\rho_k^0(x) - \rho_k^0(y)| dx dy.$$

Unfortunately Prop. 4.3 is only a rough and unsatisfactory attempt for the following reasons

- The main problem with Prop. 4.3 is that it does not imply compactness on the sequence ρ_k because it only controls oscillations of ρ_k for large enough values but we do not have any lower bounds on ρ_k . In fact not only can ρ_k vanish but for weak solutions, vacuum could even form: That is there may be a set of

non vanishing measure where $\rho_k = 0$. This comes from the fact that the proof only give an estimate on

$$\int_{\Pi^{2d}} K_h(x-y) |\rho_k(t,x) - \rho_k(t,y)| w(x) w(y) dx dy,$$

but since there is no lower bound on ρ_k anymore, estimates like (4.7) only control the set where $w(x) w(y)$ is small and both $\rho_k(x)$ and $\rho_k(y)$ are small. Unfortunately $|\rho_k(t,x) - \rho_k(t,y)|$ could be large while only one of $\rho_k(x)$ and $\rho_k(y)$ is small (and hence $w(x) w(y)$ is small as well).

- The solution is to work with $w(x) + w(y)$ instead of $w(x) w(y)$. Now the sum $w(x) + w(y)$ can only be small if $|\rho_k(t,x) - \rho_k(t,y)|$ is small as well, meaning that a bound on

$$\int_{\Pi^{2d}} K_h(x-y) |\rho_k(t,x) - \rho_k(t,y)| (w(x) + w(y)) dx dy,$$

together with estimates like (4.7) would control the compactness on ρ_k . Unfortunately this leads to various additional difficulties because some terms are now not localized at the right point. For instance one has problems estimating the commutator term in $\nabla K_h \cdot (u_k(x) - u_k(y))$ or one cannot directly control terms like $\operatorname{div} u_k(x) w(y)$ by the penalization which would now be of the form $M |\nabla u_k|(x) w(x)$. Some of these problems are solved by using more elaborate harmonic analysis tools, others require a more precise analysis of the structure of the equations. Those difficulties are even magnified for anisotropic stress tensor which add even trickier non-local terms.

- The integrability assumption on ρ_k , $p > \gamma + 1$ is not very realistic and too demanding. If $p = \gamma(1+2/d) - 1$ as for the compressible Navier-Stokes equations with power law $P(\rho) = a\rho^\gamma$, then this requires $\gamma > d$. Improving it creates important difficulty in the interaction with the penalization. It forces us to modify the penalization and prevent us from getting an inequality like (4.7) and in fact only modified inequalities can be obtained, of the type

$$\sup_k \int_{\Pi^d} |\log w(t,x)|^\theta \rho_k(t,x) dx < \infty.$$

- The bounds (4.9) that we have assumed for simplicity on R_k cannot be deduced from the equations. The effective pressure is not bounded in L^∞ and it is not *a priori* compact (it will only be so at the very end as a consequence of ρ_k being compact). Instead we will have to establish regularity bounds on the effective pressure when integrated against specific test functions; but in a manner more precise than the existing LIONS-FEIREISL theory, see Lemma 8.3 later.
- This is of course only a stability result, in order to get existence one has to work with appropriate approximate system. This will be the subject of Section 9.

Proof One now works with a different equation for the weight

$$\partial_t w + u_k \cdot \nabla w = -\lambda (M |\nabla u_k| + \rho_k^\gamma), \quad w|_{t=0} = 1, \quad (4.11)$$

where Mf is again the maximal function of f .

First step: Propagation of some weighted regularity The beginning of the first step essentially remains the same as in the proof of Prop. 4.1: One propagates

$$R(t) = \int_{\Pi^{2d}} K_h(x-y) |\rho_k(t,x) - \rho_k(t,y)| w(t,x) w(t,y) dx dy.$$

The initial calculations are nearly identical. The only difference is that we do not have (4.3) any more so we simply keep the term with $\operatorname{div} u_k(x) - \operatorname{div} u_k(y)$ for the time being. We thus obtain

$$\begin{aligned} \frac{d}{dt} R(t) &\leq \int_{\Pi^{2d}} K_h(x-y) (\operatorname{div} u_k(x) - \operatorname{div} u_k(y)) \\ &\quad (\rho_k(t,x) + \rho_k(t,y)) s_k w(t,x) w(t,y) \\ &\quad - \lambda \int_{\Pi^{2d}} (\rho_k^\gamma(x) + \rho_k^\gamma(y)) |\rho_k(t,x) - \rho_k(t,y)| w(t,x) w(t,y), \end{aligned} \quad (4.12)$$

by taking the additional term in Eq. (4.11) into account. This is of course where the coupling between u_k and ρ_k comes into play, here only through the simplified equation (4.8). Thus

$$\begin{aligned} &\int_{\Pi^{2d}} K_h(x-y) (\operatorname{div} u_k(x) - \operatorname{div} u_k(y)) (\rho_k(t,x) + \rho_k(t,y)) s_k w(t,x) w(t,y) \\ &= \int_{\Pi^{2d}} K_h(x-y) (P(\rho_k(x)) - P(\rho_k(y))) \\ &\quad (\rho_k(t,x) + \rho_k(t,y)) s_k w(t,x) w(t,y) \\ &+ \int_{\Pi^{2d}} K_h(x-y) (R_k(x) - R_k(y)) (\rho_k(t,x) + \rho_k(t,y)) s_k w(t,x) w(t,y). \end{aligned} \quad (4.13)$$

By the uniform L^p bound on ρ_k and the estimate (4.9), one has

$$\begin{aligned} &\int_0^t \int_{\Pi^{2d}} K_h(x-y) (R_k(x) - R_k(y)) (\rho_k(t,x) + \rho_k(t,y)) s_k w(t,x) w(t,y) \\ &\leq C \|K_h\|_{L^1} (\varepsilon_k(h))^{1-1/p}. \end{aligned} \quad (4.14)$$

Using now (5.9), it is possible to bound

$$\begin{aligned} |P(\rho_k(x)) - P(\rho_k(y))| &\leq |\rho_k(t,x) - \rho_k(t,y)| \int_0^1 |P'(s\rho_k(x) + (1-s)\rho_k(y))| ds \\ &\leq C (\rho_k^{\gamma-1}(x) + \rho_k^{\gamma-1}(y)) |\rho_k(t,x) - \rho_k(t,y)|, \end{aligned}$$

leading to

$$\begin{aligned}
& \int_{\Pi^{2d}} K_h(x-y) (P(\rho_k(x)) - P(\rho_k(y))) (\rho_k(t,x) + \rho_k(t,y)) s_k w(t,x) w(t,y) \\
& \leq C \int_{\Pi^{2d}} K_h(x-y) (\rho_k^{\gamma-1}(x) + \rho_k^{\gamma-1}(y)) (\rho_k(t,x) + \rho_k(t,y)) \\
& \quad |\rho_k(t,x) - \rho_k(t,y)| w(t,x) w(t,y) \\
& \leq C \int_{\Pi^{2d}} K_h(x-y) (\rho_k^{\tilde{\gamma}}(x) + \rho_k^{\tilde{\gamma}}(y)) |\rho_k(t,x) - \rho_k(t,y)| w(t,x) w(t,y).
\end{aligned}$$

Using now this estimate, the Equality (4.13), the compactness (4.14) and by taking λ large enough one finds from (4.12)

$$R(t) \leq R(0) + C \|K_h\|_{L^1} (\varepsilon_k(h))^{1-1/p}.$$

Second step and third steps: Property of the weight and conclusion. The starting point is again the same and gives

$$\int_{\Pi^d} |\log w(t,x)| \rho_k(t,x) dx \leq C,$$

with C independent of k but where we now need $\rho_k \in L^p$ with $p \geq \gamma + 1$ because of the additional term in Eq. (4.11). Now

$$\begin{aligned}
& \int_{\rho_k(x) \geq \eta, \rho_k(y) \geq \eta} K_h(x-y) |\rho_k(t,x) - \rho_k(t,y)| \\
& = \int_{\rho_k(x) \geq \eta, \rho_k(y) \geq \eta, w(x) \geq \eta', w(y) \geq \eta'} K_h(x-y) |\rho_k(t,x) - \rho_k(t,y)| \\
& \quad + \|K\|_{L^1} \int_{\rho_k(x) \geq \eta, w(x) \leq \eta'} (1 + \rho_k(x)).
\end{aligned}$$

On the one hand,

$$\begin{aligned}
& \int_{\rho_k(x) \geq \eta, w(x) \leq \eta'} (1 + \rho_k(x)) \leq \left(\frac{1}{\eta} + 1\right) \int_{w(x) \leq \eta'} \rho_k(x) \\
& \leq \left(\frac{1}{\eta} + 1\right) \frac{1}{|\log \eta'|} \int |\log w(x)| \rho_k(x) \leq \left(\frac{1}{\eta} + 1\right) \frac{C}{|\log \eta'|}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
& \int_{\rho_k(x) \geq \eta, \rho_k(y) \geq \eta, w(x) \geq \eta', w(y) \geq \eta'} K_h(x-y) |\rho_k(t,x) - \rho_k(t,y)| \\
& \leq \frac{1}{(\eta')^2} \int_{\Pi^{2d}} K_h(x-y) |\rho_k(t,x) - \rho_k(t,y)| w(t,x) w(t,y).
\end{aligned}$$

Therefore

$$\int_{\rho_k(x) \geq \eta, \rho_k(y) \geq \eta} K_h(x-y) |\rho_k(t, x) - \rho_k(t, y)| \leq \left(\frac{1}{\eta} + 1\right) \frac{C}{|\log \eta'|} \\ + \frac{C}{(\eta')^2} \|K_h\|_{L^1} \left(\tilde{\varepsilon}_k(h) + (\varepsilon_k(h))^{1-1/p} \right),$$

which concludes the proof by optimizing in η' . \square

5 Stability results

5.1 The equations

5.1.1 General pressure law.

Let (ρ_k, u_k) solve

$$\partial_t \rho_k + \operatorname{div}(\rho_k u_k) = \alpha_k \Delta \rho_k. \quad (5.1)$$

and

$$\mu_k(t, x) \operatorname{div} u_k = P_k(\rho_k) + \Delta^{-1} \operatorname{div}(\partial_t(\rho_k u_k) + \operatorname{div}(\rho_k u_k \otimes u_k)) + F_k. \quad (5.2)$$

The viscosity of the fluid is assumed to be bounded from below and above

$$\exists \bar{\mu}, \quad \frac{1}{\bar{\mu}} \leq \mu_k(t, x) \leq \bar{\mu} \quad (5.3)$$

We consider the following control on the density (for $p > 1$)

$$\sup_k \left[\|\rho_k\|_{L^\infty([0, T], L^1(\Pi^d) \cap L^\gamma(\Pi^d))} + \|\rho_k\|_{L^p([0, T] \times \Pi^d)} \right] < \infty. \quad (5.4)$$

and the following control for u_k

$$\sup_k \left[\|\rho_k |u_k|^2\|_{L^\infty([0, T], L^1(\Pi^d))} + \|\nabla u_k\|_{L^2(0, T; L^2(\Pi^d))} \right] < +\infty. \quad (5.5)$$

We also need some control on the time derivative of $\rho_k u_k$ through

$$\exists \bar{p} > 1, \quad \sup_k \|\partial_t(\rho_k u_k)\|_{L_t^2 W_x^{-1, \bar{p}}} < \infty, \quad (5.6)$$

and on the time derivative of ρ_k namely

$$\exists q > 1, \quad \sup_k \|\partial_t \rho_k\|_{L_t^q W_x^{-1, q}} < \infty. \quad (5.7)$$

Remark. Note that usually (see for instance [34]–[49]), 5.6 and 5.7 are consequences of the momentum equation and the mass equation using the uniform estimates given by the energy estimates and the extra integrability on the density.

Concerning the equation of state, we will consider P_k continuous on $[0, +\infty)$, $P_k(0) = 0$, P_k locally Lipschitz on $(0, +\infty)$ with one of the two following cases:

- i) Pressure laws with a quasi-monotone property: There exists ρ_0 s.t.

$$(P_k(s)/s)' \geq 0 \text{ for all } s \geq \rho_0, \quad \lim_{s \rightarrow +\infty} P_k(s) = +\infty. \quad (5.8)$$

- ii) Non-monotone pressure laws, where we only assume for all $s \geq 0$

$$|P'_k(s)| \leq \bar{P} s^{\tilde{\gamma}-1}. \quad (5.9)$$

Remark Note that i) with the lower bound $P(\rho) \geq C^{-1}\rho^\gamma - C$ provides the same assumptions than in the article [33] by E. FEIREISL. Point i) will be used to construct approximate solutions in the non-monotone case.

5.1.2 A non-isotropic stress tensor.

In that case, assume (ρ_k, u_k) solve (5.1) with $\alpha_k = 0$

$$\partial_t \rho_k + \operatorname{div}(\rho_k u_k) = 0.$$

and

$$\begin{aligned} \operatorname{div} u_k &= \nu_k P_k(\rho_k) + \nu_k a_\mu A_\mu P_k(\rho_k) \\ &\quad + \nu_k (\Delta - a_\mu E_k)^{-1} \operatorname{div}(\partial_t(\rho_k u_k) + \operatorname{div}(\rho_k u_k \otimes u_k)) \end{aligned} \quad (5.10)$$

where $A_\mu = (\Delta - a_\mu E_k)^{-1} E_k$.

Remark. If one considers a symmetric anisotropy, $\operatorname{div}(A D u)$ in Theorem 3.2, then instead of (5.10) we have the more complicated formula

$$\begin{aligned} \operatorname{div} u_k &= \nu_k P_k(\rho_k) + \nu_k a_\mu A_\mu P_k(\rho_k) \\ &\quad + \nu_k \operatorname{div}(\Delta I - a_\mu E_k)^{-1} (\partial_t(\rho_k u_k) + \operatorname{div}(\rho_k u_k \otimes u_k)), \end{aligned} \quad (5.11)$$

where $A_\mu = (\Delta I - a_\mu E_k)^{-1} \cdot \tilde{E}_k$. But now E_k and \tilde{E}_k may be different and are vector-valued operators, so that in particular $(\Delta I - a_\mu E_k)^{-1}$ means inverting a vector valued elliptic system. Except for the formulation there would however be no actual difference in the rest of the proof.

Coming back to (5.10), we assume ellipticity on ν_k

$$0 < \underline{\nu} \leq \nu_k \leq \bar{\nu} < \infty. \quad (5.12)$$

We assume that E_k is a given operator (differential or integral) s.t.

- $(\Delta - a_\mu E_k)^{-1} \Delta$ is bounded on every L^p space,
- $A_\mu = (\Delta - a_\mu E_k)^{-1} E_k$ is bounded of norm less than 1 on every L^p space and can be represented by a convolution with a singular integral denoted by A_μ still

$$A_\mu f = A_\mu \star_x f, \quad |A_\mu(x)| \leq \frac{C}{|x|^d}, \quad \int A_\mu(x) dx = 0.$$

Note here that to make more apparent the smallness of the non isotropic part, we explicitly scale it with a_μ . We consider again the control (5.4) on the density but for $p > \gamma^2/(\gamma - 1)$, and the bound (5.5) for u_k . We also need the same controls: (5.6) on the time derivative of $\rho_k u_k$ and (5.7) on the time of the ρ_k .

The main idea in this part is to investigate the compactness for an anisotropic viscous stress obtained as the perturbation of the usual isotropic viscous stress tensor, namely $-\operatorname{div}(A \nabla u) + (\lambda + \mu) \nabla \operatorname{div} u$ assuming $A = \mu Id + \delta A$ and $a_\mu = \|\delta A\| \leq \varepsilon$ for some small enough ε .

5.2 The main stability results: Theorems 5.1, 5.2 and 5.4

5.2.1 General pressure laws

The main step in that case is to prove the two compactness results

Theorem 5.1 *Assume that ρ_k solves (5.1), u_k solves (5.2) with the bounds (5.3), (5.5), (5.6), (5.7), and that μ_k and F_k are compact in L^1 . Moreover*

- i) If $\alpha_k > 0$, we assume the estimate (5.4) on ρ_k with $\gamma > 3/2$ and $p > 2$ and quasi-monotonicity on P_k through (5.8).*
- ii) if $\alpha_k = 0$, it is enough to assume (5.4) with $\gamma > 3/2$ and $p > \max(2, \tilde{\gamma})$ and only (5.9) on P_k .*

Then the sequence ρ_k is compact in L^1_{loc} .

We also provide a complementary result which is a more precise rate of compactness away from the vacuum namely

Theorem 5.2 *Assume again that ρ_k solves (5.1) with $\alpha_k = 0$, u_k solves (5.2) with the bounds (5.3), (5.5), (5.6), (5.7) and that μ_k and F_k are compact in L^1 . Assume that (5.4) holds with $\gamma > d/2$ and $p > \max(2, \tilde{\gamma})$ and that P_k satisfies (5.9). Then there exists $\theta > 0$ and a continuous function ε with $\varepsilon(0) = 0$, depending only on μ_k and F_k s.t.*

$$\int_{\Pi^{2d}} \mathbb{I}_{\rho_k(x) \geq \eta} \mathbb{I}_{\rho_k(y) \geq \eta} K_h(x - y) \chi(\delta \rho_k) \leq \frac{C \|K_h\|_{L^1}}{\eta^{1/2} |\log(\varepsilon(h) + h^\theta)|^{\theta/2}}.$$

For instance if μ_k and F_k are uniformly in $W^{s,1}$ for $s > 0$, then for some constant $C > 0$

$$\int_{\Pi^{2d}} \mathbb{I}_{\rho_k(x) \geq \eta} \mathbb{I}_{\rho_k(y) \geq \eta} K_h(x - y) \chi(\delta \rho_k) \leq C \frac{\|K_h\|_{L^1}}{\eta^{1/2} |\log h|^{\theta/2}}.$$

Since those results depend on the regularity of μ_k and F_k , we denote $\varepsilon_0(h)$ a continuous function with $\varepsilon_0(0) = 0$ s.t.

$$\int_0^T \int_{\Pi^{2d}} K_h(x - y) (|F_k(t, x) - F_k(t, y)| + |\mu_k(t, x) - \mu_k(t, y)|) \leq \varepsilon_0(h). \quad (5.13)$$

Remark 5.3 Note, for use in the Navier-Stokes-Fourier system, that in Theorems 5.1 or 5.2, it is possible to consider instead a pressure law $P_k(\rho, t, x)$ with explicit dependence on time and space. One still assumes (5.8) or (5.9) with a constant C independent of t and x but one needs in addition to assume compactness in x , that is for any $L > 0$

$$\int_0^T \int_{\Pi^{2d}} K_h(x-y) \left(\sup_{\rho \leq L} |P_k(\rho, t, x) - P_k(\rho, t, y)| \right) = \varepsilon_L(h) \longrightarrow 0, \quad (5.14)$$

as $h \rightarrow 0$.

5.2.2 Non isotropic stress tensor

In that case our result reads

Theorem 5.4 Assume that ρ_k solves (5.1), u_k solves (5.10) with the bounds (5.5), (5.6), (5.7) and (5.12) together with all the assumptions on E_k below (5.10). Assume as well that P_k satisfies (5.8) and that (5.4) with $\gamma > d/2$ and $p > \gamma^2/(\gamma-1)$. There exists a universal constant $C_* > 0$ s.t. if

$$a_\mu \leq C_*,$$

then ρ_k is compact in L^1_{loc} .

Remarks. Theorems 5.1, 5.2, 5.4 are really the main contributions of this article. For instance, deducing Theorems 3.1 and 3.2 follows usual and straightforward approximation procedures.

As such the main improvements with respect to the existing theory can be seen in the fact that point ii) in Theorem 5.1 does not require monotonicity on P_k and in the fact that Theorem 5.4 does not require isotropy on the stress tensor.

Our starting approximate system involves diffusion, $\alpha_k \neq 0$, in the continuity equation (5.1). As can be seen from point i) of Theorem 5.1, our compactness result in that case requires an isotropic stress tensor and a pressure P_k which is monotone after a certain point by (5.8). This limitation is the reason why we have to consider also approximations \tilde{P}_k and \tilde{E}_k of the pressure and the stress tensor. While it may superficially appear that we did not improve the existing theory in that case with diffusion, we want to point out that

- We could not have used P.-L. LIONS' approach because this requires strict monotonicity: $P'_k > 0$ everywhere. Instead any non-monotone pressure P satisfying (5.9) can be approximated by \tilde{P}_k satisfying (5.8) simply by considering $\tilde{P}_k = P + \varepsilon_k \rho^{\tilde{\gamma}}$ as long as $\tilde{\gamma} > \tilde{\gamma}$ and thus without changing the requirements on γ .
- E. FEIREISL *et al* can handle “quasi-monotone” pressure laws satisfying (5.8) together with diffusion but they require higher integrability on ρ_k for this: $p \geq 4$ in (5.4). This in turn leads to a more complex approximation procedure.

6 Useful technical lemmas

Let us recall the well known inequality, which we used in subsection 4.2 and will use several times in the following (see [59] for instance)

$$|\Phi(x) - \Phi(y)| \leq C |x - y| (M|\nabla\Phi|(x) + M|\nabla\Phi|(y)), \quad (6.1)$$

where M is the localized maximal operator

$$M f(x) = \sup_{r \leq 1} \frac{1}{|B(0, r)|} \int_{B(0, r)} f(x + z) dz. \quad (6.2)$$

As it will be seen later, there is a technical difficulty in the proof, which would lead us to try (and fail) to control $M|\nabla u_k|(y)$ by $M|\nabla u_k|(x)$. Instead we have to be more precise than (6.1) in order to avoid this. To deal with such problems, we use more sophisticated tools. First one has

Lemma 6.1 *There exists $C > 0$ s.t. for any $u \in W^{1,1}(\Pi^d)$, one has*

$$|u(x) - u(y)| \leq C |x - y| (D_{|x-y|}u(x) + D_{|x-y|}u(y)),$$

where we denote

$$D_h u(x) = \frac{1}{h} \int_{|z| \leq h} \frac{|\nabla u(x + z)|}{|z|^{d-1}} dz.$$

Proof A full proof of such well known result can for instance be found in [42] in a more general setting namely $u \in BV$. The idea is simply to consider trajectories $\gamma(t)$ from x to y which stays within the ball of diameter $|x - y|$ to control

$$|u(x) - u(y)| \leq \int_0^1 \gamma'(t) \cdot \nabla u(\gamma(t)) dt.$$

And then to average over all such trajectories with length of order $|x - y|$. Similar calculations are also present for instance in [32]. \square

Note that this result implies the estimate (6.1) as

Lemma 6.2 *There exists $C > 0$, for any $u \in W^{1,p}(\Pi^d)$ with $p \geq 1$*

$$D_h u(x) \leq C M|\nabla u|(x).$$

Proof Do a dyadic decomposition and define i_0 s.t. $2^{-i_0-1} < h \leq 2^{-i_0}$

$$\begin{aligned}
D_h u(x) &\leq \frac{1}{h} \sum_{i \geq i_0} \int_{2^{-i-1} < |z| \leq 2^{-i}} \frac{|\nabla u(x+z)|}{|z|^{d-1}} dz \\
&\leq \sum_{i \geq i_0} \frac{2^{(i+1)(d-1)}}{h} \int_{2^{-i-1} < |z| \leq 2^{-i}} |\nabla u(x+z)| dz \\
&\leq 2^{d-1} \sum_{i \geq i_0} |B(0,1)| \frac{2^{-i}}{h} M |\nabla u|(x) \leq C M |\nabla u|(x).
\end{aligned}$$

□

The key improvement in using D_h is that small translations of the operator D_h are actually easy to control

Lemma 6.3 *For any $1 < p < \infty$, there exists $C > 0$ s.t. for any $u \in H^1(\Pi^d)$*

$$\int_{h_0}^1 \int_{\Pi^d} \bar{K}_h(z) \|D_{|z|} u(\cdot) - D_{|z|} u(\cdot + z)\|_{L^p} dz \frac{dh}{h} \leq C \|u\|_{B_{p,1}^1}, \quad (6.3)$$

where the definition and basic properties of the Besov space $B_{p,1}^1$ are recalled in section 13. As a consequence

$$\int_{h_0}^1 \int_{\Pi^d} \bar{K}_h(z) \|D_{|z|} u(\cdot) - D_{|z|} u(\cdot + z)\|_{L^2} dz \frac{dh}{h} \leq C |\log h_0|^{1/2} \|u\|_{H^1}. \quad (6.4)$$

It is also possible to disconnect the shift from the radius in $D_r u$ and obtain for instance

$$\int_{h_0}^1 \int_{\Pi^{2d}} \bar{K}_h(z) \bar{K}_h(w) \|D_{|z|} u(\cdot) - D_{|z|} u(\cdot + w)\|_{L^2} dz dw \frac{dh}{h} \leq C |\log h_0|^{1/2} \|u\|_{H^1}. \quad (6.5)$$

We can in fact write a more general version of Lemma 6.3 for any kernel

Lemma 6.4 *For any $1 < p < \infty$, any family $N_r \in W^{s,1}(\Pi^d)$ for some $s > 0$ s.t.*

$$\begin{aligned}
&\sup_{|\omega| \leq 1} \sup_r r^{-s} \int_{\Pi^d} |z|^s |N_r(z) - N_r(z + r\omega)| dz < \infty, \\
&\sup_r (\|N_r\|_{L^1} + r^s \|N_r\|_{W^{s,1}}) < \infty,
\end{aligned}$$

there exists $C > 0$ s.t. for any $u \in L^p(\Pi^d)$

$$\int_{h_0}^1 \int_{\Pi^d} \bar{K}_h(z) \|N_h \star u(\cdot) - N_h \star u(\cdot + z)\|_{L^p} dz \frac{dh}{h} \leq C |\log h_0|^{1/2} \|u\|_{L^p}. \quad (6.6)$$

We will mostly use the specific version in Lemma 6.3 but will need the more general Lemma 6.4 to handle the anisotropic case in Lemma 8.6. Both lemmas are in fact a corollary of a classical result

Lemma 6.5 *For any $1 < p < \infty$, any family L_r of kernels satisfying for some $s > 0$*

$$\int L_r = 0, \quad \sup_r (\|L_r\|_{L^1} + r^s \|L_r\|_{W^{s,1}}) \leq C_L, \quad \sup_r r^{-s} \int |z|^s |L_r(z)| dz \leq C_L. \quad (6.7)$$

Then there exists $C > 0$ depending only on C_L above s.t. for any $u \in L^p(\Pi^d)$

$$\int_{h_0}^1 \|L_r \star u\|_{L^p} \frac{dr}{r} \leq C \|u\|_{B_{p,1}^0}. \quad (6.8)$$

As a consequence for $p \leq 2$

$$\int_{h_0}^1 \|L_r \star u\|_{L^p} \frac{dr}{r} \leq C |\log h_0|^{1/2} \|u\|_{L^p}. \quad (6.9)$$

Note that by a simple change of variables in r , one has for instance for any fixed power l

$$\int_{h_0}^1 \|L_{r^l} \star u\|_{L^p} \frac{dr}{r} \leq C_l |\log h_0|^{1/2} \|u\|_{L^p}.$$

Remark. The bounds (6.4) and (6.9) could also be obtained by straightforward application of the so-called square function, see the book written by E.M. STEIN [59]. We instead use Besov spaces as this yields the interesting and optimal inequalities (6.3)-(6.8) as an intermediary step.

Proof of Lemma 6.3 and Lemma 6.4 assuming Lemma 6.5. First of all observe that $D_h u = N_h \star u$ with

$$N_h = \frac{1}{h |z|^{d-1}} \mathbb{I}_{|z| \leq h}$$

which satisfies all the assumptions of Lemma 6.4. Therefore the proofs of Lemmas 6.3 and 6.4 are identical, just by replacing D_h by $N_h \star$. Hence we only give the proof of Lemma 6.3, Calculate

$$\int_{h_0}^1 \overline{K}_h(z) \frac{dh}{h} \leq \int_{h_0}^1 \frac{C h^{\nu-d}}{(h+|z|)^\nu} \frac{dh}{h} \leq \frac{C}{(|z|+h_0)^d}.$$

Note also for future use that the same calculation provides

$$\int_{h_0}^1 \overline{K}_h(z) \frac{dh}{h} \geq \frac{1}{C(|z|+h_0)^d}. \quad (6.10)$$

Therefore, using spherical coordinates

$$\begin{aligned} & \int_{h_0}^1 \int_{\Pi^d} \overline{K}_h(z) \|D_{|z|} u(\cdot) - D_{|z|} u(\cdot + z)\|_{L^p} dz dh \\ & \leq C \int_{S^{d-1}} \int_{h_0}^1 \|D_r u(\cdot) - D_r u(\cdot + r\omega)\|_{L^p} \frac{dr}{r + h_0} d\omega. \end{aligned}$$

Denote

$$L_\omega(x) = \frac{\mathbb{1}_{|x| \leq 1}}{|x|^{d-1}} - \frac{\mathbb{1}_{|x-\omega| \leq 1}}{|x-\omega|^{d-1}}, \quad L_{\omega,r}(x) = r^{-d} L_\omega(x/r),$$

and remark that $L_\omega \in W^{s,1}$ with a norm uniform in ω and with support in the unit ball. Moreover

$$D_r u(x) - D_r u(x + r\omega) = \int |\nabla u|(x - rz) L_\omega(z) dz = L_{\omega,r} \star |\nabla u|.$$

We hence apply Lemma 6.5 since the family $L_{\omega,r}$ satisfies the required hypothesis and we get

$$\int_{h_0}^1 \|L_{\omega,r} \star \nabla u\|_{L^p} \frac{dr}{r} \leq C \|u\|_{B_{1,p}^1},$$

with a constant C independent of ω and so

$$\begin{aligned} & \int_{h_0}^1 \int_{\Pi^d} \overline{K}_h(z) \|D_{|z|} u(\cdot) - D_{|z|} u(\cdot + z)\|_{L^p} dz dh \\ & \leq C \int_{S^{d-1}} \int_{h_0}^1 \|L_{\omega,r} \star \nabla u\|_{L^p} \frac{dr}{r} d\omega \leq C \int_{S^{d-1}} \|u\|_{B_{1,p}^1} d\omega, \end{aligned}$$

yielding (6.3). The bound (6.4) is deduced in the same manner. The proof of the bound (6.5) follows the same steps; the only difference is that the average over the sphere is replaced by a smoother integration against the weight $1/(1+|w|)^a$. \square

Proof of Lemma 6.5. First remark that L_r is not smooth enough to be used as the basic kernels Ψ_k in the classical Littlewood-Paley decomposition (see section 13) as in particular the Fourier transform of L_r is not necessarily compactly supported. We use instead the Littlewood-Paley decomposition of u . Denote

$$U_k = \Psi_k \star u.$$

The kernel L_r has 0 average and so

$$L_r \star U_k = \int_{\Pi^d} L_r(x-y) (U_k(y) - U_k(x)) dy.$$

Therefore

$$\begin{aligned} \|L_r \star U_k\|_{L^p} & \leq \int_{\Pi^d} L_r(z) \|U_k(\cdot) - U_k(\cdot + z)\|_{L^p} dz \\ & \leq \int_{\Pi^d} L_r(z) |z|^s \|U_k\|_{W^{s,p}} dz, \end{aligned}$$

yielding by the assumption on L_r , for $k < |\log_2 r|$

$$\|L_r \star U_k\|_{L^p} \leq C r^s 2^{ks} \|U_k\|_{L^p}, \quad (6.11)$$

by Prop. 13.2. Note that C only depends on $\int |z|^s L_r(z) dz$.

We now use similarly that $L_r \in W^{s,1}$ and deduce for $k \geq |\log_2 r|$ by Prop. 13.2,

$$\|L_r \star U_k\|_{L^p} \leq \|L_r\|_{W^{s,1}} \|U_k\|_{W^{-s,p}} \leq C r^{-s} 2^{-ks} \|U_k\|_{L^p}, \quad (6.12)$$

where C only depends on $\sup_r r^s \|L_r\|_{W^{s,1}}$. From the decomposition of f

$$\begin{aligned} \int_{h_0}^1 \|L_r \star u\|_{L^p} \frac{dr}{r} &= \sum_{k=0}^{\infty} \int_{h_0}^1 \|L_r \star U_k\|_{L^p} \frac{dr}{r} \\ &\leq C \sum_{k=0}^{\infty} \|U_k\|_{L^p} \left(\mathbb{I}_{k \leq |\log_2 h_0|} \int_{h_0}^{2^{-k}} r^s 2^{ks} \frac{dr}{r} + \int_{\max(h_0, 2^{-k})}^1 r^{-s} 2^{-ks} \frac{dr}{r} \right), \end{aligned}$$

by using (6.11) and (6.12). This shows that

$$\int_{h_0}^1 \|L_r \star u\|_{L^p} \frac{dr}{r} \leq C \sum_{k \leq |\log_2 h_0|} \|U_k\|_{L^p} + C \sum_{k > |\log_2 h_0|} \frac{2^{-ks}}{h_0^s} \|U_k\|_{L^p}. \quad (6.13)$$

Now simply bound

$$\begin{aligned} \sum_{k \leq |\log_2 h_0|} \|U_k\|_{L^p} + \sum_{k > |\log_2 h_0|} \frac{2^{-ks}}{h_0^s} \|U_k\|_{L^p} &\leq C \sum_{k=0}^{\infty} 2^k \|U_k\|_{L^p} \\ &= C \|u\|_{B_{p,1}^0}, \end{aligned}$$

which gives (6.8).

Next remark that

$$\sum_{k > |\log_2 h_0|} \frac{2^{-ks}}{h_0^s} \|U_k\|_{L^p} \leq C \sup_k \|U_k\|_{L^p} \leq C \|u\|_{B_{p,\infty}^0}.$$

Therefore (6.13) combined with Lemma 13.3 yields

$$\int_{h_0}^1 \|L_r \star u\|_{L^p} \frac{dr}{r} \leq C \sqrt{|\log_2 h_0|} \|u\|_{L^p} + C \|u\|_{B_{p,\infty}^0},$$

which gives (6.9) by Prop. 13.2. \square

Finally we emphasize that

Lemma 6.6 *The kernel*

$$\mathcal{K}_{h_0}(z) = \int_{h_0}^1 \bar{K}_h(z) \frac{dh}{h}$$

also satisfies i) and ii) of Prop. 4.1.

Proof This is a straightforward consequence of using (6.10). \square

7 Renormalized equation and weights

We explain here the various renormalizations of the transport equation satisfied by ρ_k , We then define the weights we will consider and give their properties.

7.1 Renormalized equation

We explain in this subsection how to obtain the equation satisfied by various quantities that we will need and of the form $Z_k^{x,y} \chi(\rho_k^x - \rho_k^y)$ where $Z_k^{x,y}$ is chosen as $Z_k^{x,y} = K_h(x-y)W_{k,h}^{x,y}$ with $W_{k,h}^{x,y} = W_{k,h}(t,x,y)$ appropriate weights.

Lemma 7.1 *Assume ρ_k solves (5.1) with (5.4) for $p > 2$ and that u_k satisfies (5.5). Denote for convenience*

$$\begin{aligned} \rho_k^x &= \rho_k(t,x), & \rho_k^y &= \rho(t,y), & u_k^x &= u_k(t,x), & u_k^y &= u_k(t,y), \\ \delta\rho_k &= \rho_k^x - \rho_k^y, & \bar{\rho}_k &= \rho_k^x + \rho_k^y. \end{aligned}$$

Then for any $\chi \in W^{2,\infty}$

$$\begin{aligned} & \int_{\Pi^{2d}} [K_h(x-y)W_{k,h}^{x,y} \chi(\delta\rho_k)](t) - \int_{\Pi^{2d}} [K_h(x-y)W_{k,h}^{x,y} \chi(\delta\rho_k)](0) \\ & + \int_0^t \int_{\Pi^{2d}} (\chi'(\delta\rho_k)\delta\rho_k - \chi(\delta\rho_k)) (\operatorname{div}_x u_k^x + \operatorname{div}_y u_k^y) K_h(x-y)W_{k,h}^{x,y} \\ & - \int_0^t \int_{\Pi^{2d}} \chi(\delta\rho_k) [u_k^x \cdot \nabla_x K_h(x-y) + u_k^y \cdot \nabla_y K_h(x-y) \\ & \qquad \qquad \qquad + \alpha_k(\Delta_x + \Delta_y)K_h(x-y)] W_{k,h}^{x,y} \\ & - 2 \int_0^t \int_{\Pi^{2d}} \alpha_k \nabla K_h(x-y) \chi(\delta\rho_k) [\nabla_x W_{k,h}^{x,y} - \nabla_y W_{k,h}^{x,y}] \\ & - 2 \alpha_k \int_0^t \int_{\Pi^{2d}} K_h(x-y) \chi(\delta\rho_k) [\Delta_x W_{k,h}^{x,y} + \Delta_y W_{k,h}^{x,y}] \\ & = \int_0^t \int_{\Pi^{2d}} \chi(\delta\rho_k) [\partial_t W_{k,h}^{x,y} + u_k^x \cdot \nabla_x W_{k,h}^{x,y} + u_k^y \cdot \nabla_y W_{k,h}^{x,y} \\ & \qquad \qquad \qquad - \alpha_k(\Delta_x + \Delta_y)W_{k,h}^{x,y}] K_h(x-y) \\ & - \frac{1}{2} \int_0^t \int_{\Pi^{2d}} \chi'(\delta\rho_k) K_h(x-y) W_{k,h}^{x,y} (\operatorname{div}_x u_k^x - \operatorname{div}_y u_k^y) \bar{\rho}_k \\ & + \frac{1}{2} \int_0^t \int_{\Pi^{2d}} \chi'(\delta\rho_k) K_h(x-y) W_{k,h}^{x,y} (\operatorname{div}_x u_k^x + \operatorname{div}_y u_k^y) \delta\rho_k \\ & - \alpha_k \int_0^t \int_{\Pi^{2d}} \chi''(\rho_k) K_h(x-y) W_{k,h}^{x,y} (|\nabla_x \delta\rho_k|^2 + |\nabla_y \delta\rho_k|^2). \end{aligned}$$

Proof The result essentially relies on a doubling of variable argument and straight forward algebraic calculations.

Since ρ_k solves (5.1), one has that

$$\partial_t \rho_k^x + \operatorname{div}_x(\rho_k^x u_k^x) = \alpha_k \Delta_x \rho_k^x,$$

$$\partial_t \rho_k^y + \operatorname{div}_y(\rho_k^y u_k^y) = \alpha_k \Delta_y \rho_k^y.$$

Recalling $\delta \rho_k = \rho_k^x - \rho_k^y$, and using that $\rho_k \in L_{t,x}^p$ with $p > 2$ and hence $\rho_k \operatorname{div} u_k$ is well defined, one can check that

$$\partial_t \delta \rho_k + \operatorname{div}_x(u_k^x \delta \rho_k) + \operatorname{div}_y(u_k^y \delta \rho_k) = \alpha_k (\Delta_x + \Delta_y) \delta \rho_k - \rho_k^y \operatorname{div}_x u_k^x + \rho_k^x \operatorname{div}_y u_k^y.$$

Then, recalling the notation $\bar{\rho}_k = \rho_k^x + \rho_k^y$, we observe that

$$\begin{aligned} & -\rho_k^y \operatorname{div}_x u_k^x + \rho_k^x \operatorname{div}_y u_k^y = \frac{1}{2} \left(\operatorname{div}_x u_k^x \rho_k^x - \operatorname{div}_x u_k^x \rho_k^y \operatorname{div}_y u_k^y \rho_k^x - \operatorname{div}_y u_k^y \rho_k^y \right. \\ & \quad \left. - \operatorname{div}_x u_k^x \rho_k^x - \operatorname{div}_x u_k^x \rho_k^y + \operatorname{div}_y u_k^y \rho_k^x + \operatorname{div}_y u_k^y \rho_k^y \right) \\ & = \frac{1}{2} (\operatorname{div}_x u_k^x + \operatorname{div}_y u_k^y) \delta \rho_k - \frac{1}{2} (\operatorname{div}_x u_k^x - \operatorname{div}_y u_k^y) \bar{\rho}_k. \end{aligned}$$

Consequently, we can write

$$\begin{aligned} \partial_t \delta \rho_k + \operatorname{div}_x(u_k^x \delta \rho_k) + \operatorname{div}_y(u_k^y \delta \rho_k) &= \alpha_k (\Delta_x + \Delta_y) \delta \rho_k \\ &+ \frac{1}{2} (\operatorname{div}_x u_k^x + \operatorname{div}_y u_k^y) \delta \rho_k - \frac{1}{2} (\operatorname{div}_x u_k^x - \operatorname{div}_y u_k^y) \bar{\rho}_k, \end{aligned} \quad (7.1)$$

We now turn to the renormalized equation that means the equation satisfied by $\chi(\delta_k)$ for a nonlinear function $s \mapsto \chi(s)$. Formally the equation can be obtained by multiplying (7.1) by $\chi'(\delta \rho_k)$. If $\alpha_k = 0$ and ρ_k is not smooth then the formal calculation can be justified following DI PERNA–LIONS techniques using regularizing by convolution and the estimate (5.5), *i.e.* $u_k \in L_t^2 H_x^1$. Then

$$\begin{aligned} & \partial_t \chi(\delta \rho_k) + \operatorname{div}_x(u_k^x \chi(\delta \rho_k)) + \operatorname{div}_y(u_k^y \chi(\delta \rho_k)) \\ & \quad + (\chi'(\delta \rho_k) \delta \rho_k - \chi(\delta \rho_k)) (\operatorname{div}_x u_k^x + \operatorname{div}_y u_k^y) \\ & = \alpha_k (\Delta_x + \Delta_y) \chi(\delta \rho_k) - \frac{1}{2} \chi'(\delta \rho_k) (\operatorname{div}_x u_k^x - \operatorname{div}_y u_k^y) \bar{\rho}_k \\ & \quad + \frac{1}{2} \chi'(\delta \rho_k) (\operatorname{div}_x u_k^x + \operatorname{div}_y u_k^y) \delta \rho_k - \alpha_k \chi''(\rho_k) (|\nabla_x \delta \rho_k|^2 + |\nabla_y \delta \rho_k|^2). \end{aligned}$$

For any V_k^x, V_k^y and smooth enough $Z_{k,h}^{x,y}$, one has in the sense of distributions

$$Z_{k,h}^{x,y} \Delta_x V_k^x = \operatorname{div}_x (Z_{k,h}^{x,y} \nabla_x V_k^x) - \operatorname{div}_x (V_k^x \nabla_x Z_{k,h}^{x,y}) + V_k^x \Delta_x Z_{k,h}^{x,y}.$$

$$Z_{k,h}^{x,y} \Delta_y V_k^y = \operatorname{div}_y (Z_{k,h}^{x,y} \nabla_y V_k^y) - \operatorname{div}_y (V_k^y \nabla_y Z_{k,h}^{x,y}) + V_k^y \Delta_y Z_{k,h}^{x,y}.$$

Consequently, we get the following equation for $Z_{k,h}^{x,y}\chi(\delta\rho_k)$:

$$\begin{aligned} & \partial_t[Z_{k,h}^{x,y}\chi(\delta\rho_k)] + \operatorname{div}_x(u_k^x\chi(\delta\rho_k)Z_{k,h}^{x,y}) + \operatorname{div}_y(u_k^y\chi(\delta\rho_k)Z_{k,h}^{x,y}) \\ & + (\chi'(\delta\rho_k)\delta\rho_k - \chi(\delta\rho_k))(\operatorname{div}_x u_k^x + \operatorname{div}_y u_k^y)Z_{k,h}^{x,y} \\ & - \chi(\delta\rho_k)[\partial_t Z_{k,h}^{x,y} + u_k^x \cdot \nabla_x Z_{k,h}^{x,y} + u_k^y \cdot \nabla_y Z_{k,h}^{x,y} - \alpha_k(\Delta_x + \Delta_y)Z_{k,h}^{x,y}] = r.h.s., \end{aligned}$$

with

$$\begin{aligned} r.h.s. = & -\frac{1}{2}\chi'(\delta\rho_k)Z_{k,h}^{x,y}(\operatorname{div}_x u_k^x - \operatorname{div}_y u_k^y)\bar{\rho}_k + \frac{1}{2}\chi'(\delta\rho_k)Z_{k,h}^{x,y}(\operatorname{div}_x u_k^x + \operatorname{div}_y u_k^y)\delta\rho_k \\ & - \alpha_k\chi''(\rho_k)Z_{k,h}^{x,y}(|\nabla_x \delta\rho_k|^2 + |\nabla_y \delta\rho_k|^2) \\ & + 2\alpha_k Z_{k,h}^{x,y}(\Delta_x + \Delta_y)\chi(\delta\rho_k) + \alpha_k[\operatorname{div}_x(Z_{k,h}^{x,y}\nabla_x\chi(\delta\rho_k)) - \operatorname{div}_x(\chi(\delta\rho_k)\nabla_x Z_{k,h}^{x,y}) \\ & + \operatorname{div}_y(Z_{k,h}^{x,y}\nabla_y\chi(\delta\rho_k)) - \operatorname{div}_y(\chi(\delta\rho_k)\nabla_y Z_{k,h}^{x,y})]. \end{aligned}$$

Integrating in time and space and performing the required integration by parts, we get the desired equality writing $Z_{k,h}^{x,y} = K_h(x-y)W_{k,h}^{x,y}$. \square

7.2 The weights: Choice and properties

In this subsection, we choose the PDEs satisfied by the weights, we state and then prove some of their properties.

7.2.1 Basic considerations

We define weights $w(t, x)$ which satisfy

$$\partial_t w + u_k(t, x) \cdot \nabla_x w = -Dw + \alpha_k \Delta_x w, \quad w(t=0, x) = w^0(x) \quad (7.2)$$

for some appropriate penalization D depending on the case under consideration. Note that w depends on k , but we do not precise the index to keep notations simple. The choice of D will be based on the need to control “bad” terms when looking at the propagation of the weighted quantity. The choice will also have to ensure that the weights are not too small, too often.

7.2.2 Isotropic viscosity, general pressure laws.

The case with $\alpha_k > 0$ and monotone pressure. The simplest choice for the penalization D to define w_0 is

$$D_0 = \lambda M |\nabla u_k|, \quad (7.3)$$

with λ a fixed constant (chosen later on) and M the localized maximal operator as defined by (6.2). We choose accordingly in that case

$$w_0|_{t=0} = w_0^0 \equiv 1.$$

The case $\alpha_k = 0$ and non-monotone pressure. In the absence of diffusion in (5.1) ($\alpha_k = 0$) and when the pressure term P_k is non-monotone for instance, one needs to add a term $\rho_k^{\tilde{\gamma}}$ in the penalization. This would lead to very strong assumptions, in particular on the exponent p in (5.4) (and hence γ) as explained after Prop. 4.3. It is possible to obtain better results by taking the more refined

$$\frac{D_1}{\lambda} = \rho_k^{p-1} (p-1) |\operatorname{div} u_k| + |\operatorname{div} u_k| + M |\nabla u_k| + \rho_k^{\tilde{\gamma}}, \quad (7.4)$$

for the general compactness result. We take for simplicity

$$w_1|_{t=0} = w_1^0 \equiv \exp(-\lambda(\sup \rho_k^0)^{p-1}).$$

The reason for the first term in D_1 compared to D_0 is to ensure that $w_1 \leq e^{-\lambda \rho_k^{p-1}}$ which helps compensates the penalization in $\rho_k^{\tilde{\gamma}}$ to get the property on $\rho_k |\log w_1|^\theta$ for some $\theta > 0$. The three last terms are needed to respectively counterbalance: Additional divergence terms in the propagation quantity compared to w_0 , the same $M |\nabla u_k|$ as for w_0 and the $\rho_k^{\tilde{\gamma}}$ for terms coming from the pressure.

7.2.3 Anisotropic stress tensor.

The choice for the penalization, denoted D_a in this case and leading to the weight w_a , is now

$$\frac{D_a}{\lambda} = M |\nabla u_k| + \overline{K_h} \star (|\operatorname{div} u_k| + |A_\mu \rho_k^\gamma|). \quad (7.5)$$

Note that the second term in the penalization is used to control the non local part of the pressure terms. As initial condition, we choose accordingly

$$w_a|_{t=0} = w_a^0 \equiv 1.$$

7.2.4 The forms of the weights.

Two types of weights W are used

$$W(t, x, y) = w(t, x) + w(t, y), \quad \text{or} \quad W(t, x, y) = w(t, x) w(t, y).$$

The first one will provide compactness and will be used with (7.3) or (7.4). The second, used with (7.4) gives better explicit regularity estimates but far from vacuum and is considered for the sake of completeness. Therefore one defines

$$\begin{aligned} W_0(t, x, y) &= w_0(t, x) + w_0(t, y), & W_1(t, x, y) &= w_1(t, x) + w_1(t, y), \\ W_2(t, x, y) &= w_1(t, x) w_1(t, y), & W_a(t, x, y) &= w_a(t, x) + w_a(t, y). \end{aligned} \quad (7.6)$$

As for the penalization, we use the notation W when the particular choice is not relevant and W_i , $i = 0, 1, 2$ or a otherwise. For all choices, one has

$$\partial_t W + u_k(x) \cdot \nabla_x W + u_k(y) \cdot \nabla_y W = -Q + \alpha_k \Delta_{x,y} W. \quad (7.7)$$

The term Q depends on the choices of penalizations and weights with the four possibilities

$$\begin{aligned} Q_0 &= D_0 w_0(t, x) + D_0 w_0(t, y), & Q_1 &= D_1 w_1(t, x) + D_1 w_1(t, y), \\ Q_2 &= (D_1(t, x) + D_1(t, y)) w_1(t, x) w_1(t, y), & Q_a &= D_a w_a(t, x) + D_a w_a(t, y). \end{aligned}$$

7.2.5 The weight properties

We summarize the main estimates on the weights previously defined

Proposition 7.2 *Assume that ρ_k solves (5.1) with the bounds (5.4) for $p \geq 2$ and (5.5) on u_k . Then*

i) *For any t, x :*

$$0 \leq w_0(t, x) \leq 1, \quad 0 \leq w_a(t, x) \leq 1, \quad 0 \leq w_1(t, x) \leq e^{-\lambda} \rho_k(t, x)^{p-1}.$$

ii) *One has*

$$\int \rho_k(t, x) |\log w_0(t, x)| dx \leq C(1 + \lambda),$$

if $\alpha_k = 0$ and $p > \max(2, \tilde{\gamma})$ then similarly there exists $\theta > 0$ s.t.

$$\int \rho_k(t, x) |\log w_1(t, x)|^\theta dx \leq C_\lambda.$$

while finally if $p \geq \gamma + 1$

$$\int \rho_k(t, x) |\log w_a(t, x)| dx \leq C(1 + \lambda). \quad (7.8)$$

iii) *For any η , we have the two estimates*

$$\int_{\Pi^d} \rho_k \mathbb{I}_{\overline{K}_h} \star w_0 \leq \eta dx \leq C \frac{1 + \lambda}{|\log \eta|},$$

and if $p \geq \gamma + 1$

$$\int_{\Pi^d} \rho_k \mathbb{I}_{\overline{K}_h} \star w_a \leq \eta dx \leq C \frac{1 + \lambda}{|\log \eta|}.$$

iv) *Denoting $w_{a,h} = \overline{K}_h \star w_a$, if $p > \gamma$, we have for some $0 < \theta < 1$*

$$\begin{aligned} & \int_{h_0}^1 \int_0^t \left\| \overline{K}_h \star (\overline{K}_h \star (|\operatorname{div} u_k| + |A_\mu \rho_k^\gamma|) w_a) - \right. \\ & \left. (\overline{K}_h \star (|\operatorname{div} u_k| + |A_\mu \rho_k^\gamma|)) \omega_{a,h} \right\|_{L^q} dt \frac{dh}{h} \leq C |\log h_0|^\theta, \end{aligned}$$

with $q = \min(2, p/\gamma)$.

Remark 7.3 Part i) tells us that w_i is small at the right points (in particular when ρ_k is large). On the other hand we want w_i to be small only on a set of small mass otherwise one obviously does not control much. This is the role of part ii). We use part iv) to regularize weights in the anisotropic case. Part iii) is also used to get control under the form given in 4.1 from the estimates with weights.

Remark 7.4 Even when $\alpha_k > 0$, it would be possible to define D_1 in order to have a bound like $w_1 \leq e^{-\lambda \rho_k(t,x)^{q-1}}$. For instance take

$$\begin{aligned} \frac{D_1}{\lambda} &= \rho_k^{q-1} (q-1) |\operatorname{div} u_k| + \frac{\alpha_{k,\lambda}}{\lambda} |\nabla \log w_1|^2 \\ &\quad + |\operatorname{div} u_k| + M |\nabla u_k| + L \rho_k^\gamma, \end{aligned}$$

with

$$\alpha_{k,\lambda} = \alpha_k \left(1 - \frac{\bar{\alpha}_\lambda}{1 + \lambda \rho_k^{q-1}} \right), \quad \bar{\alpha}_\lambda = \frac{q-2}{(q-1)}.$$

However one needs $q \leq p/2$ and $q > 2$ which already forces $p > 4$. Moreover the main difficulty when $\alpha_k > 0$ comes from the proof of Lemma 8.1 which forces us to work with $\bar{K}_h \star w_0$ and not w_0 . Because of that any pointwise inequality between $w(x)$ and $\rho_k(x)$ is mostly useless.

Proof Point i). The penalizations D_0 and D_a are positive hence, since w_0 and w_a solve (7.2), $0 \leq w_0 \leq 1$ and $0 \leq w_a \leq 1$. Similarly $0 \leq w_1 \leq 1$. Now for the upper bound on w_1 in terms of ρ_k , assume that $\alpha_k = 0$. Note that since ρ_k solves (5.1)

$$\partial_t e^{-\lambda \rho_k^{p-1}} + u_k \cdot \nabla e^{-\lambda \rho_k^{p-1}} = \lambda (p-1) \rho_k^{p-1} \operatorname{div} u_k e^{\lambda \rho_k^{p-1}}.$$

Remark that

$$D_1 \geq \lambda (p-1) \rho_k^{p-1} |\operatorname{div} u_k| \geq -\lambda (p-1) \rho_k^{p-1} \operatorname{div} u_k w_1,$$

and hence by (7.2)

$$\partial_t w_1 + u_k \cdot \nabla w_1 \leq \lambda (p-1) \rho_k^{p-1} \operatorname{div} u_k w_1.$$

Denoting $g = e^{-\lambda \rho_k^{p-1}} - w_1$, we obtain

$$\partial_t g + u_k \cdot \nabla g \geq (\lambda (p-1) \rho_k^{p-1} \operatorname{div} u_k) g.$$

By the maximum principle, $g \geq 0$ which finishes the proof of i). Note that no assumption on p was needed here but if ρ_k is not smooth the theory of renormalized solutions is used.

Point ii). By point i), $w \leq 1$, hence $|\log w| = -\log w$ and from (7.2), denoting $|\log w_i| = A_i$,

$$\begin{aligned} \partial_t(A_i) + u_k \cdot \nabla_x(A_i) - D_i &= -\frac{\alpha_k}{w_i} \Delta w_i \\ &= \alpha_k \Delta A_i - \alpha_k |\nabla A_i|^2. \end{aligned} \tag{7.9}$$

For A_0 , we directly apply

$$\partial_t(\rho_k A_0) + \nabla_x \cdot (\rho_k A_0) = D_0 \rho_k + \alpha_k \Delta(\rho_k A_0) - 2\alpha_k \nabla \rho_k \cdot \nabla A_0 - \alpha_k \rho_k |\nabla A_0|^2,$$

and integrate to find

$$\begin{aligned} \int \rho_k A_0 dx = & C + \int_0^t \int D_0(t, x) \rho_k dx ds - \alpha_k \int_0^t \int \rho_k |\nabla A_0|^2 dx ds \\ & - 2\alpha_k \int \nabla \rho_k \cdot \nabla A_0 dx ds. \end{aligned}$$

Simply bound

$$\int \nabla \rho_k \cdot \nabla A_0 \leq \frac{1}{4} \int \rho_k |\nabla A_0|^2 + \int \frac{|\nabla \rho_k|^2}{\rho_k}.$$

On the other hand, using renormalization techniques,

$$\frac{d}{dt} \int \rho_k \log \rho_k dx = - \int \rho_k \operatorname{div} u_k dx - \alpha_k \int \frac{|\nabla \rho_k|^2}{\rho_k} dx.$$

However as $p \geq 2$ then $\rho_k \operatorname{div} u_k$ is bounded uniformly in $L^1_{t,x}$ and $\rho_k \log \rho_k$ in $L^\infty_t(L^1_x)$. This implies, from the previous equality, that

$$\alpha_k \int_0^t \int \frac{|\nabla \rho_k|^2}{\rho_k} dx ds \leq C,$$

and consequently

$$-2\alpha_k \int_0^t \int \nabla \rho_k \cdot \nabla A_0 dx ds \leq \frac{\alpha_k}{2} \int_0^t \int \rho_k |\nabla A_0|^2 dx ds + C.$$

Using this in the equality on $\int \rho_k A_0$ given previously, we get

$$\int \rho_k A_0 dx = C + \int_0^t \int D_0(t, x) \rho_k dx ds - \frac{\alpha_k}{2} \int_0^t \int \rho_k |\nabla A_0|^2 dx ds.$$

In that case, we know that $\|D_0\|_{L^2} \leq C\lambda$ and since $p \geq 2$ we get

$$\int \rho_k |\log w_0| dx + \alpha_k \int_0^t \int \rho_k |\nabla A_0|^2 dx ds \leq C(1 + \lambda). \quad (7.10)$$

Concerning w_a , the estimate is similar, even simpler as $\alpha_k = 0$, to get

$$\int \rho_k |\log w_a| dx \leq C(1 + \lambda)$$

assuming $p \geq \gamma + 1$. Indeed $M|\nabla u_k|$ and $\bar{K}_h \star |\operatorname{div} u_k|$ are bounded in L^2 by (5.5). Finally

$$\begin{aligned} \int_0^T \int_{\Pi^d} \rho_k \bar{K}_h \star (|A_\mu \rho_k^\gamma|) &\leq \left(\int_0^T \int_{\Pi^d} \rho_k^{\gamma+1} \right)^{1/(\gamma+1)} \left(\int_0^T \int_{\Pi^d} |A_\mu \rho_k^\gamma|^{(\gamma+1)/\gamma} \right)^{\gamma/(\gamma+1)} \\ &\leq C \int_0^T \int_{\Pi^d} \rho_k^{\gamma+1}, \end{aligned}$$

since A_μ is continuous on any L^q space for $1 < q < +\infty$. The right-hand side is bounded assuming $p > \gamma + 1$.

For ω_1 , the estimate is as before a bit different. We now assume that $\alpha_k = 0$, define $\tilde{A}_1 = (1 + A_1)^\theta$ and obtain from (7.9)

$$\partial_t \tilde{A}_1 + u_k \cdot \nabla \tilde{A}_1 = \theta \frac{D_1}{(1 + A_1)^{1-\theta}}.$$

Integrating and recalling $A_1 \geq \lambda \rho_k^{p-1}$

$$\begin{aligned} \int \rho_k \tilde{A}_1 dx &\leq C + C \int_0^t \int \frac{1 + \rho_k^p}{1 + \rho_k^{(p-1)(1-\theta)}} |\operatorname{div} u_k| dx ds \\ &\quad + C \int_0^t \int \frac{\rho_k}{1 + \rho_k^{(p-1)(1-\theta)}} M |\nabla u_k| dx ds \\ &\quad + C \int_0^t \int \frac{\rho_k^{\tilde{\gamma}+1}}{1 + \rho_k^{(p-1)(1-\theta)}} dx ds \leq C, \end{aligned}$$

for some $\theta > 0$ provided $p > \max(2, \tilde{\gamma})$. If $p > 2$ and $p > \tilde{\gamma}$ then $\tilde{\gamma} + 1 - p + 1 < p$. This gives the desired control regarding $\rho |\log w_1|^\theta$ for an exponent θ small enough.

Point iii) Estimate (7.10) will not be enough in the proof and we will need to control the mass of ρ_k where $\bar{K}_h \star w_0$ is small. Denote

$$\Omega_{h,\eta} = \{x \in \Pi^d, \bar{K}_h \star w_0(x) \leq \eta\}, \quad \tilde{\Omega}_{h,\eta} = \{x \in \Omega_{h,\eta}, w_0(x) \geq \sqrt{\eta}\}.$$

The time t is fixed during this argument and for simplicity we omit it.

One cannot easily estimate $|\Omega_{h,\eta}|$ directly but it is straightforward to bound $|\tilde{\Omega}_{h,\eta}|$. Assume $x \in \Omega_{h,\eta}$ i.e. $\bar{K}_h \star w_0(x) \leq \eta$. From the expression of K_h , if $|\delta| \leq h$

$$\bar{K}_h(z + \delta) \leq \frac{\|K_h\|_{L^1}^{-1}}{(h + |z + \delta|)^a} \leq C \bar{K}_h(z),$$

we deduce that for any $y \in B(x, h)$

$$\bar{K}_h \star w_0(y) \leq C \eta.$$

Now cover $\tilde{\Omega}_{\eta,h}$ by $\bigcup_i C_i^h$ with C_i^h disjoint hyper-cubes of diameter h/C . For any i , denote $\tilde{\Omega}_{\eta,h}^i = \tilde{\Omega}_{\eta,h} \cap C_i^h$.

If $\tilde{\Omega}_{\eta,h}^i \neq \emptyset$ then $\bar{K}_h \star w_0(x) \leq C\eta$ on the whole C_i^h . In that case

$$\begin{aligned} C\eta h^d &\geq \int_{C_i^h} \bar{K}_h \star w_0 \geq \int_{C_i^h} \int_{\tilde{\Omega}_{\eta,h}^i} \bar{K}_h(x-y) w_0(y) dy dx \\ &\geq \frac{\sqrt{\eta}}{C} |\tilde{\Omega}_{\eta,h}^i|. \end{aligned}$$

We conclude that $|\tilde{\Omega}_{\eta,h}^i| \leq C\sqrt{\eta} h^d$. Summing over the cubes, we deduce that one has

$$|\tilde{\Omega}_{\eta,h}| \leq C\sqrt{\eta}.$$

Finally

$$\int_{\Omega_{\eta,h}} \rho_k dx \leq \int_{\tilde{\Omega}_{\eta,h}} \rho_k dx + \frac{2}{|\log \eta|} \int_{\Pi^d} \rho_k |\log w_0| dx \leq C\eta^{1/2-1/2\gamma} + \frac{C}{|\log \eta|},$$

since $\rho \in L_t^\infty(L_x^\gamma)$ for some $\gamma > 1$. This is the desired bound.

The same bound may be obtained on the quantity $\rho_k \mathbb{1}_{\bar{K}_h \star w_a \leq \eta}$ in a similar way when $p > \gamma + 1$ because of bound (7.8) on $\rho_k |\log w_a|$.

Point iv) Denote to simplify $f = |\operatorname{div} u_k| + |A_\mu \rho_k^\gamma|$, then by the definition of $w_{a,h}$

$$\begin{aligned} &\int_{h_0}^1 \int_0^t \left\| \bar{K}_h \star (\bar{K}_h \star f w_a) - (\bar{K}_h \star f) \omega_{a,h} \right\|_{L^q} dt \frac{dh}{h} \\ &\leq \int_{h_0}^1 \int_0^t \bar{K}_h(z) \left\| (\bar{K}_h \star f(\cdot + z) - \bar{K}_h \star f(\cdot)) \omega_a(\cdot + z) \right\|_{L^q} dt \frac{dh}{h} \\ &\leq \int_0^t \int_{h_0}^1 \bar{K}_h(z) \left\| (\bar{K}_h \star f(t, \cdot + z) - \bar{K}_h \star f(t, \cdot)) \right\|_{L^q} \frac{dh}{h} dt \\ &\leq C |\log h_0|^{1/2} \int_0^t \|f(t, \cdot)\|_{L_x^q} dt \leq C |\log h_0|^{1/2}, \end{aligned}$$

by a direct application of Lemma 6.4 with $N_h = \bar{K}_h$ provided f is uniformly bounded in $L_t^1 L_x^q$ which is guaranteed by $q \leq \min(2, p/\gamma)$. \square

8 Proof of Theorems 5.1, 5.2 and 5.4

We start with the propagation of regularity on the transport equation in terms of the regularity of $\operatorname{div} u_k$, more precisely $\operatorname{div}_x u_k^x - \operatorname{div}_y u_k^y$. We prove in the second subsection some estimates on the effective pressure. This allows to write a Lemma in the third subsection controlling $\operatorname{div}_x u_k^x - \operatorname{div}_y u_k^y$ and then to close the loop in the fourth subsection thus concluding the proof.

8.1 The propagation of regularity on the transport equation

This subsection uses only Eq. (5.1) on ρ_k without yet specifying the coupling between $\operatorname{div} u_k$ and ρ_k (for instance through (5.2)). Recall that we denote

$$\delta\rho_k = \rho_k(t, x) - \rho_k(t, y), \quad \bar{\rho}_k = \rho_k(t, x) + \rho_k(t, y).$$

Choose any C^2 function χ s.t.

$$\left| \chi(\xi) - \frac{1}{2}\chi'(\xi)\xi \right| \leq \frac{1}{2}\chi'(\xi)\xi, \quad \chi'(\xi)\xi \leq C\chi(\xi) \leq C|\xi|. \quad (8.1)$$

It is for instance possible to take $\chi(\xi) = \xi^2$ for $|\xi| \leq 1/2$ and $\chi(\xi) = |\xi|$ for $|\xi| \geq 1/2$. Note however that the first inequality on the right forces $\chi(\xi) \geq |\xi|/C$.

Similarly for the anisotropic viscous term, for some $\ell > 0$, choose any $\chi_a \in C^2$

$$\begin{aligned} \left| \chi_a(\xi) - \frac{1}{2}\chi'_a(\xi)\xi \right| &\leq \frac{1-\ell}{2}\chi'_a(\xi)\xi, \quad \chi'_a(\xi)\xi \leq C\chi_a(\xi) \leq C|\xi|^{1+\ell}, \\ (\xi^\gamma + \tilde{\xi}^\gamma) (-\chi'_a(\xi - \tilde{\xi})(\xi - \tilde{\xi}) + 2\chi_a(\xi - \tilde{\xi})) & \\ &\geq -(\xi^\gamma - \tilde{\xi}^\gamma) \frac{\ell-1}{\ell} \chi'_a(\xi - \tilde{\xi})(\xi + \tilde{\xi}). \end{aligned} \quad (8.2)$$

Note that it is possible to simply choose $\chi_a = |\xi|^{1+\ell}$. But to unify the notations and the calculations with the other terms involving χ , we use the abstract χ_a .

The properties on this non-linear function will be strongly used to characterize the effect of the pressure law in the contribution of $\operatorname{div}_x u_k(x) - \operatorname{div}_y u_k(y)$ in the anisotropic case. The form of χ_a and the choice of ℓ will have to be determined very precisely so that the corresponding which will be exactly counterbalanced by the λ terms in Lemma 8.2.

We can write two distinct Lemmas respectively concerning the non monotone pressure law case and the anisotropic tensor case.

Lemma 8.1 *Assume that ρ_k solves (5.1) with estimates (5.4) and (5.5) on u_k .*

i) *With diffusion, $\alpha_k > 0$, if $p > 2$, $\exists \varepsilon_{h_0}(k) \rightarrow 0$ as $k \rightarrow \infty$ for a fixed h_0*

$$\begin{aligned} &\int_{h_0}^1 \int_{\Pi^{4d}} \bar{K}_h(x-z) \bar{K}_h(y-w) W_0(t, z, w) K_h(x-y) \chi(\delta\rho_k) dx dy dz dw \frac{dh}{h} \\ &\leq C(\varepsilon_{h_0}(k) + |\log h_0|^{1/2}) \\ &- \frac{1}{2} \int_{h_0}^1 \int_0^t \int_{\Pi^{4d}} K_h(x-y) (\operatorname{div} u_k(x) - \operatorname{div} u_k(y)) \chi'_0 \bar{\rho}_k W_0 \bar{K}_h(x-z) \bar{K}_h(y-w) \frac{dh}{h} \\ &- \frac{1}{2} \int_{h_0}^1 \int_0^t \int_{\Pi^{4d}} K_h(x-y) (\operatorname{div} u_k(x) + \operatorname{div} u_k(y)) \\ &\quad (\chi'(\delta\rho_k) \delta\rho_k - 2\chi(\delta\rho_k)) W_0 \bar{K}_h(x-z) \bar{K}_h(y-w) \frac{dh}{h} \\ &- \frac{\lambda}{2} \int_{h_0}^1 \int_0^t \int_{\Pi^{3d}} K_h(x-y) \chi(\delta\rho_k) \bar{K}_h(x-z) M |\nabla u_k|(z) w_0(z) \frac{dh}{h}, \end{aligned}$$

where we recall that $W_0 = w_0(t, x) + w_0(t, y)$.

ii) Without diffusion, $\alpha_k = 0$, if $p \geq 2$,

$$\begin{aligned} & \int_{h_0}^1 \int_{\Pi^{2d}} \bar{K}_h(x-y) \chi(\delta\rho_k) (w_1(t, x) + w_1(t, y)) dx dy \frac{dh}{h} \\ & \leq C |\log h_0|^{1/2} \|u_k\|_{H^1} - 2\lambda \int_{h_0}^1 \int_0^t \int_{\Pi^{2d}} \bar{K}_h(x-y) \rho_k^{\tilde{\gamma}}(x) w_1(x) \chi(\delta\rho_k) \frac{dh}{h} \\ & - 2 \int_0^t \int_{h_0}^1 \int_{\Pi^{2d}} \bar{K}_h(x-y) (\operatorname{div} u_k(x) - \operatorname{div} u_k(y)) \left(\frac{1}{2} \chi'(\delta\rho_k) \bar{\rho}_k \right. \\ & \left. + \chi(\delta\rho_k) - \frac{1}{2} \chi'(\delta\rho_k) \delta\rho_k \right) w_1(x) \frac{dh}{h}. \end{aligned}$$

For the derivation of explicit regularity estimates, we have as well the version with the product weight, namely

$$\begin{aligned} & \int_{\Pi^{2d}} K_h(x-y) \chi(\delta\rho_k) w_1(t, x) w_1(t, y) dx dy \leq C \\ & - \int_0^t \int_{\Pi^{2d}} K_h(x-y) (\operatorname{div} u_k(x) - \operatorname{div} u_k(y)) \chi'(\delta\rho_k) \bar{\rho}_k w_1(x) w_1(y) \\ & - \lambda \int_0^t \int_{\Pi^{2d}} K_h(x-y) \left(\rho_k^{\tilde{\gamma}}(x) + \rho_k^{\tilde{\gamma}}(y) \right) \chi(\delta\rho_k) w_1(x) w_1(y). \end{aligned}$$

For convenience, we write separately the result we will use in the anisotropic case

Lemma 8.2 *Assume that ρ_k solves (5.1) with estimates (5.5)-(5.4). Without diffusion, $\alpha_k = 0$, assume (8.2) on χ_a with $p > \gamma + \ell + 1$, and denote $w_{a,h} = \bar{K}_h \star w_a$,*

$$\begin{aligned} & \int_{h_0}^1 \int_{\Pi^{2d}} \frac{\bar{K}_h(x-y)}{h} (w_{a,h}(x) + w_{a,h}(y)) \chi_a(\delta\rho_k)(t) \\ & \leq \int_{h_0}^1 \int_{\Pi^{2d}} \frac{\bar{K}_h(x-y)}{h} (w_{a,h}(x) + w_{a,h}(y)) \chi_a(\delta\rho_k)|_{t=0} + C |\log h_0|^\theta + I + II - \Pi_a, \end{aligned}$$

with the dissipation term

$$\Pi_a = \lambda \int_0^t \int_{h_0}^1 \int_{\Pi^{2d}} w_{a,h}(x) \chi_a(\delta\rho_k) \bar{K}_h \star (|\operatorname{div} u_k| + |A_{\mu\rho^\gamma}|)(x) \bar{K}_h \frac{dh}{h},$$

while

$$\begin{aligned} I = -\frac{1}{2} \int_{h_0}^1 \int_0^t \int_{\Pi^{2d}} \frac{\bar{K}_h(x-y)}{h} (\operatorname{div} u_k(x) - \operatorname{div} u_k(y)) \chi_a'(\delta\rho_k) \bar{\rho}_k \\ (w_{a,h}(x) + w_{a,h}(y)), \end{aligned}$$

and

$$II = -\frac{1}{2} \int_{h_0}^1 \int_0^t \int_{\Pi^{2d}} \frac{\overline{K}_h(x-y)}{h} (\operatorname{div} u_k(x) + \operatorname{div} u_k(y)) (\chi'_a(\delta\rho_k)\delta\rho_k - 2\chi_a(\delta\rho_k)) (w_{a,h}(x) + w_{a,h}(y)).$$

Remark. We emphasize that the λ terms in relations i) and ii) of Lemma 8.1 come from the penalization in the definition of the weights w_0 and w_1 . They will help to counterbalance terms coming from the contribution by $\operatorname{div}_x u_k(x) - \operatorname{div}_y u_k(y)$. Similarly the nonlocal term Π_a follows from the definition of the weights w_a . bad terms coming from the viscosity anisotropy.

Proof

Case i). Denote

$$W_{0,h}(t, x, y) = \int_{\Pi^{2d}} \overline{K}_h(x-z) \overline{K}_h(y-w) W_0(t, z, w) dz dw,$$

and let us use χ in the renormalized equation from Lemma 7.1. We get

$$\begin{aligned} & \int_{\Pi^d} W_{0,h}(t, x, y) K_h(x-y) \chi(\delta\rho_k) dx dy = A + B + D + E \\ & - \frac{1}{2} \int_0^t \int K_h(x-y) (\operatorname{div} u_k(x) - \operatorname{div} u_k(y)) \chi'(\delta\rho_k) \bar{\rho}_k W_{0,h} dx dy, \\ & - \frac{1}{2} \int_0^t \int K_h(x-y) (\operatorname{div} u_k(x) + \operatorname{div} u_k(y)) (\chi'(\delta\rho_k) \delta\rho_k - 2\chi(\delta\rho_k)) W_{0,h} dx dy, \end{aligned}$$

with, by the symmetry of K_h , \overline{K}_h and $W_{0,h}$, and in particular since $\nabla_x W_{0,h}(x, y) = \nabla_y W_{0,h}(y, x)$

$$\begin{aligned} A &= \int_0^t \int_{\Pi^{2d}} (u_k(t, x) - u_k(t, y)) \cdot \nabla K_h(x-y) \chi(\delta\rho_k) W_{0,h} dx dy dt, \\ B &= 2 \int_0^t \int_{\Pi^{2d}} K_h(x-y) (\partial_t W_{0,h}(x) + u_k \cdot \nabla_x W_{0,h} - \alpha_k \Delta_x W_{0,h}) \chi(\delta\rho_k) dx dy dt, \\ D &= 2\alpha_k \int_0^t \int_{\Pi^{2d}} \chi(\delta\rho_k) [\Delta_x K_h(x-y) W_{0,h} + 2K_h(x-y) \Delta_x W_{0,h}] dx dy dt, \\ E &= -2\alpha_k \int_0^t \int_{\Pi^{2d}} K_h W_{0,h} \chi''(\delta\rho_k) |\nabla_x \delta\rho_k|^2 dx dy dt. \end{aligned}$$

Now note that by the convexity of χ , $E \leq 0$. Then simply bound using (8.1),

$$D \leq 8\alpha_k h^{-2} \|K_h\|_{L^1} \|\rho\|_{L^1} \leq C\alpha_k h^{-2} \|K_h\|_{L^1},$$

leading us to choose $\varepsilon_{h_0}(k) = \alpha_k \int_{h_0}^1 h^{-2} \frac{dh}{h}$.

As for B , using Eq. (7.2)

$$B = B_1 - 2 \int K_h(x-y) \chi(\delta\rho_k) \tilde{K}_h \star_{x,y} R_0 dx dy dt,$$

with

$$B_1 = 2 \int K_h(x-y) \chi(\delta\rho_k) (u_k(x) - u_k(z)) \cdot \nabla_x \bar{K}_h(x-z) \bar{K}_h(y-w) W_0(t, z, w) dx dy dz dw dt.$$

We recall that $R_0(x, y) = D_0(x) w_0(x) + D_0(y) w_0(y)$ with $D_0 = \lambda \bar{K}_h \star (M |\nabla u_k|)$ and we thus only have to bound B_1 . By Lemma 6.1, we have

$$\begin{aligned} B_1 &\leq C \int K_h(x-y) \chi(\delta\rho_k) (D_{|x-y|} u_k(x) + D_{|x-y|} u_k(z)) \\ &\quad |x-z| |\nabla \bar{K}_h(x-z)| \bar{K}_h(y-w) W_0(t, z, w) \\ &\leq C \int K_h(x-y) \chi(\delta\rho_k) (D_{|x-y|} u_k(x) + D_{|x-y|} u_k(z)) \bar{K}_h(x-z) \bar{K}_h(y-w) W_0, \end{aligned}$$

as $|x| |\nabla K_h| \leq C K_h$.

Next recalling that $W_0(t, z, w) = w_0(t, z) + w_0(t, w)$, by symmetry

$$\begin{aligned} B_1 &\leq C \int K_h(x-y) \chi(\delta\rho_k) D_{|x-y|} u_k(z) \bar{K}_h(x-z) \bar{K}_h(y-w) w_0(t, z) \\ &\quad + C \int K_h(x-y) \chi(\delta\rho_k) (D_{|x-y|} u_k(x) + D_{|x-y|} u_k(w) - 2D_{|x-y|} u_k(z)) \\ &\quad \bar{K}_h(x-z) \bar{K}_h(y-w) w_0(t, z). \end{aligned}$$

Since $D_{|x-y|} u_k(z) \leq C M |\nabla u_k|(z)$, for λ large enough, the first term may be bounded by

$$-\frac{\lambda}{2} \int K_h(x-y) \chi(\delta\rho_k) \bar{K}_h \star_x (M |\nabla u_k| w_0) dx dy dt.$$

Use the uniform bound on $\|\rho_k\|_{L^p}$ with $p > 2$, to find

$$\begin{aligned} &\int K_h(x-y) \chi(\delta\rho_k) (D_{|x-y|} u_k(x) + D_{|x-y|} u_k(w) - 2D_{|x-y|} u_k(z)) \tilde{K}_h w_0(t, z) \\ &\leq C \int \|D_{|r|} u_k(\cdot) + D_{|r|} u_k(\cdot + r + u) - 2D_{|r|} u_k(\cdot + v)\|_{L^2} K_h(r) \bar{K}_h(u) \bar{K}_h(v), \end{aligned}$$

where we used that $w = x + (y-x) + (w-y)$. We now use Lemma 6.3 and more precisely the inequality (6.5) to obtain

$$\begin{aligned} &\int_{h_0}^1 \int \bar{K}_h(x-y) \chi(\delta\rho_k) (D_{|x-y|} u_k(x) + D_{|x-y|} u_k(w) - 2D_{|x-y|} u_k(z)) \tilde{K}_h w_0 \frac{dh}{h} \\ &\leq C |\log h_0|^{1/2} \int_0^t \|u_k(t, \cdot)\|_{H^1} dt \leq C |\log h_0|^{1/2}. \end{aligned}$$

Therefore we have that

$$\begin{aligned} \int_{h_0}^1 B \frac{\|K_h\|_{L^1}^{-1}}{h} dh &\leq C \varepsilon_{h_0}(k) + C |\log h_0|^{1/2} \\ &- \frac{3\lambda}{4} \int_{h_0}^1 \int_0^t \int \bar{K}_h(x-y) \chi(\delta\rho_k) M |\nabla u_k|(z) \bar{K}_h(x-z) w_0(z) dz dx dy dt \frac{dh}{h}. \end{aligned}$$

The computations are similar for A and we only give the main steps. We have, using again Lemma 6.1

$$\begin{aligned} &\int_{\Pi^{2d}} \nabla K_h(x-y) \cdot (u_k(t,x) - u_k(t,y)) \chi(\delta\rho_k) W_{0,h} \\ &\leq C \int_{\Pi^{2d}} K_h(x-y) (D_{|x-y|} u_k(x) + D_{|x-y|} u_k(y)) \chi(\delta\rho_k) W_{0,h}. \end{aligned}$$

By decomposing $W_{0,h}$, we can write just as for B_1

$$\begin{aligned} &\int_{\Pi^{2d}} \nabla K_h(x-y) \cdot (u_k(t,x) - u_k(t,y)) \chi(\delta\rho_k) W_{0,h} \\ &\leq C \int_{\Pi^{2d}} K_h(x-y) M |\nabla u_k|(z) \chi(\delta\rho_k) \bar{K}_h(x-z) w_0(z) \\ &+ C \int_{\Pi^{2d}} K_h(x-y) (D_{|x-y|} u_k(x) + D_{|x-y|} u_k(y) + D_{|x-y|} u_k(w) - 3D_{|x-y|} u_k(z)) \\ &\quad \chi(\delta\rho_k) \bar{K}_h(x-z) \bar{K}_h(y-w) w_0(z). \end{aligned}$$

The first term in the r.h.s. can again be bounded by

$$-\frac{\lambda}{2} \int K_h(x-y) \chi(\delta\rho_k) \bar{K}_h \star_x (M |\nabla u_k| w_0) dx dy dt.$$

The second term in the r.h.s. is now integrated in h and controlled as before thanks to the bound (6.5) in Lemma 6.3 and the uniform L^p bound on ρ_k and H^1 on u_k . This leads to

$$\begin{aligned} \int_{h_0^1} A \|K_h\|_{L^1}^{-1} \frac{dh}{h} &\leq C |\log h_0|^{1/2} \\ &+ \frac{\lambda}{4} \int_{h_0}^1 \int_0^t \int_{\Pi^{2d}} \bar{K}_h(x-y) M |\nabla u_k|(z) \chi(\delta\rho_k) \bar{K}_h(x-z) w_0(z) dx dy dz dt \frac{dh}{h}. \end{aligned}$$

Now summing all the contributions we get

$$\begin{aligned} \int_{h_0}^1 (A + B + D + E) \|K_h\|_{L^1}^{-1} \frac{dh}{h} &\leq C \varepsilon_{h_0}(k) + C |\log h_0|^{1/2} \\ &- \frac{\lambda}{2} \int_{h_0}^1 \int_0^t \int K_h(x-y) \chi(\delta\rho_k) \bar{K}_h \star (M |\nabla u_k| w_0). \end{aligned}$$

Note that indeed $\varepsilon_{h_0}(k) \rightarrow 0$ as $k \rightarrow \infty$ for a fixed h_0 . This concludes the proof in that first case.

Case ii). In this part, we assume $\alpha_k = 0$. We may not assume that ρ_k is smooth anymore. However by [27] since $\rho_k \in L^2$ and so is ∇u_k , one may use the renormalized relation with $\varphi = \chi$ and choose $W_{k,h}^{x,y} = W_i$. We then can use the identity given in Lemma 7.1. Denoting

$$\tilde{\chi}(\xi) = \frac{1}{\chi(\xi)} \left(\chi(\xi) - \frac{1}{2} \chi'(\xi) \xi \right),$$

we get for $i = 1, 2$

$$\int_{\Pi^{2d}} K_h(x-y) \chi(\delta \rho_k) W_i(t, x, y) dx dy \leq A_i + B_i + D_i,$$

where by the symmetry in x and y

$$\begin{aligned} A_1 &= \int_0^t \int_{\Pi^{2d}} (u_k(t, x) - u_k(t, y)) \cdot \nabla K_h(x-y) \chi(\delta \rho_k) W_1 dx dy dt \\ &\quad - \lambda \int_0^t \int_{\Pi^{2d}} K_h(x-y) (M |\nabla u_k|(x) w_1(x) + M |\nabla u_k|(y) w_1(y)) \chi(\delta \rho_k), \end{aligned}$$

while

$$\begin{aligned} A_2 &= \int_0^t \int_{\Pi^{2d}} (u_k(t, x) - u_k(t, y)) \cdot \nabla K_h(x-y) \chi(\delta \rho_k) W_2 dx dy dt \\ &\quad - \lambda \int_0^t \int_{\Pi^{2d}} K_h(x-y) (M |\nabla u_k|(x) + M |\nabla u_k|(y)) w_1(x) w_1(y) \chi(\delta \rho_k). \end{aligned}$$

And

$$\begin{aligned} B_1 &= 2 \int_0^t \int_{\Pi^{2d}} K_h(x-y) (\partial_t w_1(x) + u_k(x) \cdot \nabla_x w_1 + 2 \operatorname{div}_x u_k(x) \tilde{\chi}(\delta \rho_k) w_1 \\ &\quad + \lambda M |\nabla u_k|(x) w_1) \chi(\delta \rho_k) dx dy dt, \end{aligned}$$

while

$$\begin{aligned} B_2 &= 2 \int_0^t \int_{\Pi^{2d}} K_h(x-y) (\partial_t w_1(x) + u_k(x) \cdot \nabla_x w_1 + \operatorname{div}_x u_k(x) \tilde{\chi}(\delta \rho_k) w_1 \\ &\quad + \lambda M |\nabla u_k|(x) w_1) w_1(y) \chi(\delta \rho_k) dx dy dt. \end{aligned}$$

Finally

$$\begin{aligned} D_1 &= -2 \int_0^t \int_{\Pi^{2d}} K_h(x-y) (\operatorname{div} u_k(x) - \operatorname{div} u_k(y)) \left(\frac{1}{2} \chi'(\delta \rho_k) \bar{\rho}_k \right. \\ &\quad \left. + \chi(\delta \rho_k) - \frac{1}{2} \chi'(\delta \rho_k) \delta \rho_k \right) w_1(x) dx dy dt, \end{aligned}$$

and

$$D_2 = - \int_0^t \int_{\Pi^{2d}} K_h(x-y) (\operatorname{div} u_k(x) - \operatorname{div} u_k(y)) \chi'(\delta \rho_k) \bar{\rho}_k w_1(x) w_1(y) dx dy dt.$$

Note that we have split in several parts a null contribution in terms of the maximal function namely the ones with $M|\nabla u_k|^2$. Notice also the additional terms in D_1 that come from cross products such as $\operatorname{div} u_k(y) \tilde{\chi} \chi w_1(x)$ which would pose problems in B_1 .

The contributions D_1 and D_2 are already under the right form. Using Eq. (7.2) with (7.4), one may directly bound

$$B_1 \leq -2\lambda \int_0^t \int_{\Pi^{2d}} K_h(x-y) \rho_k^{\tilde{\gamma}}(t,x) w_1(t,x) \chi(\delta \rho_k) dx dy dt,$$

and

$$B_2 \leq -2\lambda \int_0^t \int_{\Pi^{2d}} K_h(x-y) \rho_k^{\tilde{\gamma}}(t,x) w_1(t,x) w_1(t,y) \chi(\delta \rho_k) dx dy dt.$$

The term A_2 is straightforward to handle as well. Use (6.1) to get

$$\begin{aligned} A_2 &\leq \int_0^t \int_{\Pi^{2d}} |\nabla K_h(x-y)| |x-y| (M|\nabla u_k|(x) + M|\nabla u_k|(y)) \chi(\delta \rho_k) W_2 \\ &\quad - \lambda \int_0^t \int_{\Pi^{2d}} K_h(x-y) (M|\nabla u_k|(x) + M|\nabla u_k|(y)) w_1(x) w_1(y) \chi(\delta \rho_k). \end{aligned}$$

Since $|x| |\nabla K_h| \leq CK_h$, by taking λ large enough, one obtains

$$A_2 \leq 0.$$

The term A_1 is more complex because it has no symmetry. By Lemma 6.1

$$\begin{aligned} A_1 &\leq C \int_0^t \int_{\Pi^{2d}} |\nabla K_h(x-y)| |x-y| (D_{|x-y|} u_k(x) + D_{|x-y|} u_k(y)) \chi(\delta \rho_k) \\ &\quad w_1(t,x) dx dy dt \\ &\quad - \lambda \int_0^t \int_{\Pi^{2d}} K_h(x-y) M|\nabla u_k|(x) w_1(t,x) \chi(\delta \rho_k) + \text{similar terms in } w_1(t,y). \end{aligned}$$

The *key problem* here is the $D_h u(y) w_1(x)$ term which one has to control by the term $M|\nabla u|(x) w_1(x)$. This is where integration over h and the use of Lemma 6.3 is needed (the other term in $w_1(y)$ is dealt with in a symmetric manner). For that we will add and subtract an appropriate quantity to see the quantity $D_{|x-y|} u_k(x) - D_{|x-y|} u_k(y)$.

By the definition of K_h , $|z| |\nabla K_h(z)| \leq C K_h(z)$ and by (8.1), $\chi(\delta\rho_k) \leq C(\rho_k(x) + \rho_k(y))$ with $\rho_k \in L^2$ uniformly moreover w_1 in uniformly bounded. Hence using Cauchy-Schwartz and denoting $z = x - y$

$$\begin{aligned} \int_{h_0}^1 \frac{A_1}{\|K_h\|_{L^1}} \frac{dh}{h} &\leq 2C \int_{h_0}^1 \int_0^t \int_{\Pi^d} \bar{K}_h(z) \|D_{|z|} u_k(\cdot) - D_{|z|} u_k(\cdot + z)\|_{L^2} \frac{dh}{h} \\ &\quad + 4C \int_0^t \int_{\Pi^{2d}} \mathcal{K}_{h_0}(x-y) D_{|x-y|} u_k(x) \chi(\delta\rho_k) W_1 \\ &\quad - 2\lambda \int_0^t \int_{\Pi^{2d}} \mathcal{K}_{h_0}(x-y) M |\nabla u_k|(x) w_1(x) \chi(\delta\rho_k) \\ &\leq 2C \int_{h_0}^1 \int_{\Pi^d} \bar{K}_h(z) \|D_{|z|} u_k(\cdot) - D_{|z|} u_k(\cdot + z)\|_{L^2} \frac{dh}{h}, \end{aligned}$$

by taking λ large enough since Lemma 6.2 bounds $D_{|x-y|} u_k(x)$ by $M |\nabla u_k|(x)$.

Finally using Lemma 6.3

$$\int_{h_0}^1 \frac{A_1}{\|K_h\|_{L^1}} \frac{dh}{h} \leq C |\log h_0|^{1/2} \|u_k\|_{H^1}.$$

Summing up $A_i + B_i + D_i$, and integrating against $\frac{dh}{\|K_h\|_{L^1} h}$ for $i = 1$, concludes the proof. \square

Proof of Lemma 8.2. In this part, we again assume $\alpha_k = 0$ and still use [27] to obtain the renormalized relation of Lemma 7.1 with $\varphi = \chi_a$ and $W_{k,h}^{x,y} = \bar{K}_h \star W_a = w_{a,h} + w_{a,h}$. With this exception the proof follows the lines of point i) in the previous Lemma 8.1, so we only sketch it here.

From Lemma 7.1, we get

$$\begin{aligned} &\int_{h_0}^1 \int_{\Pi^{2d}} \frac{\bar{K}_h(x-y)}{h} (w_{a,h}(x) + w_{a,h}(y)) \chi_a(\delta\rho_k)(t) \\ &\leq \int_{h_0}^1 \int_{\Pi^{2d}} \frac{\bar{K}_h(x-y)}{h} (w_{a,h}(x) + w_{a,h}(y)) \chi_a(\delta\rho_k)|_{t=0} + A + B + D + I + II + \Pi_a, \end{aligned}$$

with the terms

$$A = \int_0^t \int_{\Pi^{2d}} (u_k(t,x) - u_k(t,y)) \cdot \nabla \bar{K}_h(x-y) \chi_a(\delta\rho_k) (w_{a,h}(x) + w_{a,h}(y)),$$

while

$$\begin{aligned} B = 2 \left[\int_0^t \int_{\Pi^{3d}} K_h(x-y) \chi_a(\delta\rho_k) (u_k(x) - u_k(z)) \cdot \nabla_x \bar{K}_h(x-z) w_a(t,z) \right. \\ \left. - \lambda \int_0^t \int_{\Pi^{2d}} K_h(x-y) \chi_a(\delta\rho_k) \bar{K}_h \star (M |\nabla u_k| w_a) \right], \end{aligned}$$

and

$$D = -\lambda \left[\int_0^t \int_{h_0}^1 \int_{\Pi^{2d}} \chi_a(\delta\rho_k) \bar{K}_h \star (|\operatorname{div} u_k| + |A_\mu \rho^\gamma|) w_a(x) \bar{K}_h(x-y) \frac{dh}{h} \right. \\ \left. - \int_0^t \int_{h_0}^1 \int_{\Pi^{2d}} w_{a,h}(x) \chi_a(\delta\rho_k) \bar{K}_h \star (|\operatorname{div} u_k| + |A_\mu \rho^\gamma|)(x) \bar{K}_h(x-y) \frac{dh}{h} \right].$$

The dissipation term is under the right form

$$\Pi_a = -\lambda \int_0^t \int_{h_0}^1 \int_{\Pi^{2d}} w_{a,h}(x) \chi_a(\delta\rho_k) \bar{K}_h \star (|\operatorname{div} u_k| + |A_\mu \rho^\gamma|)(x) \bar{K}_h(x-y) \frac{dh}{h},$$

and so are by symmetry

$$I = -\frac{1}{2} \int_{h_0}^1 \int_0^t \int_{\Pi^{2d}} \frac{\bar{K}_h(x-y)}{h} (\operatorname{div} u_k(x) - \operatorname{div} u_k(y)) \chi'_a(\delta\rho_k) \bar{\rho}_k w_{a,h}(x),$$

and

$$II = -\frac{1}{2} \int_{h_0}^1 \int_0^t \int_{\Pi^{2d}} \frac{\bar{K}_h(x-y)}{h} (\operatorname{div} u_k(x) + \operatorname{div} u_k(y)) \\ (\chi'_a(\delta\rho_k) \delta\rho_k - 2\chi_a(\delta\rho_k)) w_{a,h}(x).$$

The terms A and B are treated exactly as in case i) of Lemma 8.1; they only require the higher integrability $p > \gamma + 1 + \ell$.

The only additional term is hence D which is required in order to write the dissipation term Π_a in the right form. D is bounded directly by point iv) in Lemma 7.2. Thus

$$A + B + D \leq C |\log h_0|^\theta,$$

for some $0 < \theta < 1$ which concludes the proof. \square

8.2 The control on the effective flux

Before coupling the previous estimate with the equation on $\operatorname{div} u_k$, we start with a lemma which will be used in every situation as it controls the regularity properties of

$$D\rho u_k = \Delta^{-1} \operatorname{div} (\partial_t(\rho_k u_k) + \operatorname{div}(\rho_k u_k \otimes u_k)),$$

per

Lemma 8.3 *Assume that ρ_k solves (5.1), that (5.5)-(5.6) hold and (5.4) with $\gamma > d/2$. Assume moreover that $\Phi \in L^\infty([0, T] \times \Pi^{2d})$ and that*

$$C_\Phi := \left\| \int_{\Pi^d} \bar{K}_h(x-y) \Phi(t, x, y) dy \right\|_{W_t^{1,1} W_x^{-1,1}} \\ + \left\| \int_{\Pi^d} \bar{K}_h(x-y) \Phi(t, x, y) dx \right\|_{W_t^{1,1} W_y^{-1,1}} < \infty,$$

then there exists $\theta > 0$ s.t.

$$\begin{aligned} & \int_0^t \int_{\Pi^d} K_h(x-y) \Phi(t, x, y) (D\rho u_k(t, y) - D\rho u_k(t, x)) dx dy dt \\ & \leq C \|K_h\|_{L^1} h^\theta \left(\|\Phi\|_{L_{t,x}^\infty} + C_\Phi \right). \end{aligned}$$

Proof This proof is divided in four steps: The first one concerns a control on $\rho_k |u_k - u_{k,\eta}|$ where $u_{k,\eta}$ is a regularization of u_k defined later-on; The second step concerns the proof of an estimate for $\Phi_x = (\overline{K}_h \star \Phi)(t, x)$ and $\Phi_y = (\overline{K}_h \star \Phi)(t, y)$ in $L_t^2 L_x^{\bar{p}'}$ with $\bar{p}' = \bar{p}/(\bar{p} - 1)$; The third step concerns a control with respect to h when Φ_x, Φ_y in $L_t^2 L_x^{\bar{p}'} \cap W_t^{1,+\infty} W^{-1,\infty-0}$ with $\bar{p}' = \bar{p}/(\bar{p} - 1)$; The last term is the end of the proof obtained by interpolation.

i) A control on $\rho_k |u_k - u_{k,\eta}|$ where $u_{k,\eta}$ is a regularization in space and time defined later-on. Choose a kernel $\mathcal{K}_\eta \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$ s.t. $\mathcal{K}_\eta(t, s) = 0$ if $|t - s| \geq \eta$, for smoothing in time. We still denote, with a slight abuse of notation

$$\mathcal{K}_\eta \star_t u_k(t) = \int_{\mathbb{R}_+} \mathcal{K}_\eta(t, s) u_k(s) ds.$$

Now since u_k is uniformly bounded in $L_t^2 H_x^1 \subset L_t^2 L_x^6$ in dimension $d = 3$ and in $L_t^2 L_x^q$ for any $q < \infty$ in dimension $d = 2$, one has

$$\begin{aligned} & \int \rho_k(t, x) \frac{(u_k(t, x) - u_k(s, x))^2}{1 + |u_k(t, x)| + |u_k(s, x)|} \mathcal{K}_\eta(t, s) dt ds dx \\ & \leq \int \rho_k(t, x) (u_k(t, x) - u_k(s, x)) \mathcal{K}_\eta(t, s) \overline{K}_{\eta'} \star_x \frac{u_k(t, \cdot) - u_k(s, \cdot)}{1 + |u_k(t, \cdot)| + |u_k(s, \cdot)|} \\ & \quad + C(\eta')^\theta \|\rho_k\|_{L_t^\infty L_x^\gamma}, \end{aligned}$$

with $\theta > 0$ if $\gamma > d/2$ for $d = 2, 3$. Note that

$$\|\partial_t \rho_k\|_{L_t^1 W_x^{-1,1}} \leq C,$$

and by interpolation, as $\gamma > d/2$ and thus $\gamma > 2d/(d+2)$, there exists $\theta > 0$ s.t.

$$\|\partial_t^\theta \rho_k\|_{L_t^2 H_x^{-1}} \leq C.$$

Thus

$$\begin{aligned} & \int \rho_k(t, x) \frac{(u_k(t, x) - u_k(s, x))^2}{1 + |u_k(t, x)| + |u_k(s, x)|} \mathcal{K}_\eta(t, s) dt ds dx \\ & \leq \int (\rho_k(t) u_k(t, x) - \rho_k(s) u_k(s, x)) \mathcal{K}_\eta(t, s) \overline{K}_{\eta'} \star \frac{u_k(t, \cdot) - u_k(s, \cdot)}{1 + |u_k(t, \cdot)| + |u_k(s, \cdot)|} \\ & \quad + C(\eta')^\theta + C \frac{\eta^\theta}{\eta'^d} \|u_k\|_{L_t^2 H_x^1}. \end{aligned}$$

Using (5.6), one deduces that

$$\begin{aligned} & \int \rho_k(t, x) \frac{(u_k(t, x) - u_k(s, x))^2}{1 + |u_k(t, x)| + |u_k(s, x)|} \mathcal{K}_\eta(t, s) dt ds dx \\ & \leq C\eta'^\theta + C \frac{\eta^\theta}{\eta'^d} + C \frac{\eta}{\eta'^d}. \end{aligned}$$

Optimizing in η' and interpolating (using again $\gamma > d/2$), one finally gets that for some $\theta > 0$

$$\int \rho_k(t, x) (u_k(t, x) - u_k(s, x))^2 \mathcal{K}_\eta(t, s) dt ds dx \leq C\eta^\theta.$$

Define for some $\nu > 0$

$$u_{k,\eta} = \overline{K}_{\eta^\nu} \star_x \mathcal{K}_\eta \star_t u_k.$$

Using the regularity in x of $u_{k,\eta}$, one has that

$$\int \rho_k(t, x) |u_k(t, x) - u_{k,\eta}(t, x)|^2 dx dt \leq C\eta^\theta,$$

and finally by (5.5), there exists $\theta > 0$ s.t.

$$\|\rho_k(u_k - u_{k,\eta})\|_{L_t^{1+\theta} L_x^{1+\theta}} \leq C\eta^\theta. \quad (8.3)$$

ii) *The case where Φ_x, Φ_y is only in $L_t^2 L_x^{\bar{p}'}$ with $\bar{p}' = \bar{p}/(\bar{p} - 1)$.* We recall that \bar{p} is the exponent in (5.6). Denote

$$I\Phi = \int_0^t \int_{\Pi^{2d}} K_h(x - y) \Phi(t, x, y) (D\rho u_k(t, y) - D\rho u_k(t, x)) dx dy dt$$

which can be seen as a linear form on Φ . Recall as well

$$\Phi_x = \int_{\Pi^d} \overline{K}_h(x - y) \Phi(t, x, y) dy, \quad \Phi_y = \int_{\Pi^d} \overline{K}_h(x - y) \Phi(t, x, y) dx.$$

By (5.6), $D\rho u_k$ is uniformly bounded in $L_t^2 L_x^{\bar{p}}$. Therefore

$$|I\Phi| \leq C \|K_h\|_{L^1} \left(\|\Phi_x\|_{L_t^2 L_x^{\bar{p}'}} + \|\Phi_y\|_{L_t^2 L_x^{\bar{p}'}} \right), \quad \frac{1}{\bar{p}'} = 1 - \frac{1}{\bar{p}} > 0. \quad (8.4)$$

iii) *The case Φ_x, Φ_y in $L_t^2 L_x^{\bar{p}'} \cap W_t^{1,+\infty} W^{-1,\infty-0}$ with $\bar{p}' = \bar{p}/(\bar{p} - 1)$.* Denote

$$\tilde{C}_\Phi = \|\Phi_x\|_{L_t^2 L_x^{\bar{p}'}} + \|\Phi_y\|_{L_t^2 L_x^{\bar{p}'}} + \|\Phi_x\|_{W_t^{1,\infty} W^{-1,\infty-0}} + \|\Phi_y\|_{W_t^{1,\infty} W^{-1,\infty-0}},$$

and

$$R_1 = \Delta^{-1} \operatorname{div} \rho_k (u_k - u_{k,\eta}).$$

Observe that by (8.3) and integration by part in time

$$\int_0^t \int_{\Pi^d} \Phi_x \partial_t R_1 dx dt \leq \tilde{C}_\Phi \eta^\theta \|K_h\|_{L^1}.$$

The same procedure can be performed with $\operatorname{div}(\rho_k u_k \otimes u_k)$. Denoting

$$D\rho u_{k,\eta} = \Delta^{-1} \operatorname{div}(\partial_t(\rho_k u_{k,\eta}) + \operatorname{div}(\rho_k u_k \otimes u_{k,\eta})),$$

one then has

$$\begin{aligned} I\Phi &\leq \tilde{C}_\Phi \|K_h\|_{L^1} \eta^\theta \\ &\quad + \int_0^t \int_{\Pi^{2d}} K_h(x-y) \Phi(t,x,y) (D\rho u_{k,\eta}(t,y) - D\rho u_{k,\eta}(t,x)) dx dy dt. \end{aligned}$$

However using (5.1)

$$\partial_t(\rho_k u_{k,\eta}) + \operatorname{div}(\rho_k u_k \otimes u_{k,\eta}) = \rho_k(\partial_t u_{k,\eta} + u_k \cdot \nabla u_{k,\eta}) + \alpha_k u_{k,\eta} \Delta \rho_k.$$

For some exponent κ

$$\|\Delta^{-1} \operatorname{div}(\rho_k(\partial_t u_{k,\eta} + u_k \cdot \nabla u_{k,\eta}))\|_{L_t^1 W_x^{1,1}} \leq C \eta^{-\kappa},$$

and

$$\alpha_k \|\Delta^{-1} \operatorname{div}(\alpha_k u_{k,\eta} \Delta \rho_k)\|_{L_t^2 L_x^2} \leq C \eta^{-\kappa} \sqrt{\alpha_k}.$$

Therefore

$$\begin{aligned} &\int_0^t \int_{\Pi^d} K_h(x-y) \Phi(t,x,y) (D\rho u_{k,\eta}(t,y) - D\rho u_{k,\eta}(t,x)) dx dy dt \\ &\leq \tilde{C}_\Phi \eta^{-\kappa} \|K_h\|_{L^1} (h + \sqrt{\alpha_k})^{1-0}. \end{aligned}$$

Finally

$$I\Phi \leq \tilde{C}_\Phi \|K_h\|_{L^1} (\eta^\theta + \eta^{-\kappa} (h + \sqrt{\alpha_k})),$$

and by optimizing in η , there exists $\theta > 0$ s.t.

$$I\Phi \leq C \|K_h\|_{L^1} h^\theta \left(\|\Phi_x\|_{W_t^1, \infty W^{-1, \infty-0}} + \|\Phi_y\|_{W_t^1, \infty W^{-1, \infty-0}} \right). \quad (8.5)$$

iv) *Interpolation between the two inequalities (8.4) and (8.5).* For any $s \in (0, 1)$ there exists $\theta > 0$ s.t.

$$\begin{aligned} I\Phi &\leq C \|K_h\|_{L^1} h^\theta \left(\|\Phi_x\|_{L_t^2 L_x^{\bar{p}'}} + \|\Phi_y\|_{L_t^2 L_x^{\bar{p}'}} \right. \\ &\quad \left. + \|\Phi_x\|_{W_t^{s, q+0} W_x^{-s, r+0}} + \|\Phi_y\|_{W_t^{1, q+0} W_x^{-1, r+0}} \right), \end{aligned}$$

with

$$\frac{1}{q} = \frac{1-s}{2}, \quad \frac{1}{r} = \frac{1-s}{\bar{p}'}$$

On the other hand, if for example Φ_x belongs to $L_{t,x}^\infty$ and to $W_t^{1,1}W_x^{-1,1}$ then by interpolation Φ_x is in $W_t^{s,1/s-0}W_x^{-s,1/s-0}$. Hence C_ϕ controls the $W_t^{s,q+0}W_x^{-s,r+0}$ norm provided

$$s < 1/q = \frac{1-s}{2}, \quad s < 1/r = \frac{1-s}{\bar{p}'}$$

This is always possible by taking s small enough (but strictly positive) as $\bar{p} > 1$ and hence $\bar{p}' < \infty$. This concludes the proof.

Note finally that the interpolations between Sobolev spaces are not exact which is the reason for the $1/s - 0$ or $q + 0$ and $r + 0$ (one would have to use Besov spaces instead, see for instance [47]). \square

8.3 The coupling with the pressure law

We handle all weights at the same time. For convenience, we denote

$$\chi_1(t, x, y) = \frac{1}{2}\chi'(\delta\rho_k)\bar{\rho}_k + \chi(\rho_k) - \frac{1}{2}\chi'(\delta\rho_k)\delta\rho_k.$$

In the case without viscosity, one has

Lemma 8.4 *Assume that ρ_k solves (5.1) with $\alpha_k = 0$, that (5.6), (5.5), (5.4) with $\gamma > d/2$ and $p > 2$ hold. Assume moreover that u solves (5.2) with μ_k compact in L^1 and satisfying (5.3), F_k compact in L^1 , P_k satisfying (5.9).*

i) Then there exists a continuous function $\varepsilon(\cdot)$ with $\varepsilon(0) = 0$ s.t.

$$\begin{aligned} & - \int_0^t \int_{\Pi^{2d}} K_h(x-y) (\operatorname{div} u_k(x) - \operatorname{div} u_k(y)) \chi_1 w_1(x) \\ & \leq C \|K_h\|_{L^1} \varepsilon(h) + C \int_0^t \int_{\Pi^{4d}} K_h(x-y) \left(1 + \rho_k^{\tilde{\gamma}}(x)\right) \chi(\delta\rho_k) w_1(x). \end{aligned}$$

ii) There exist $\theta > 0$ and a continuous function ε with $\varepsilon(0) = 0$, depending only on p and the smoothness of μ_k and F_k , s.t.

$$\begin{aligned} & - \int_0^t \int_{\Pi^{2d}} K_h(x-y) (\operatorname{div} u_k(x) - \operatorname{div} u_k(y)) \chi'(\delta\rho_k) \bar{\rho}_k w_1(x) w_1(y) \\ & \leq C \|K_h\|_{L^1} (\varepsilon(h) + h^\theta) \\ & + C \int_0^t \int_{\Pi^{4d}} K_h(x-y) \left(1 + \rho_k^{\tilde{\gamma}}(x) + \rho_k^{\tilde{\gamma}}(y)\right) \chi(\delta\rho_k) w_1(x) w_1(y). \end{aligned}$$

For instance if μ_k and F_k belong to $W^{s,1}$ for some $s > 0$ then one may take $\varepsilon(h) = h^\theta$ for some $\theta > 0$.

Lemma 8.5 *Assume that ρ_k solves (5.1), that (5.6), (5.5), (5.4) with $\gamma > d/2$ and $p > 2$ hold. Assume moreover that u solves (5.2) with μ_k compact in L^1 and satisfying (5.3), F_k compact in L^1 , P_k satisfying (5.8). Then there exists a continuous function $\varepsilon(\cdot)$ with $\varepsilon(0) = 0$ and depending only on the smoothness of μ_k and F_k s.t.*

$$\begin{aligned}
& -\frac{1}{2} \int_0^t \int_{\Pi^{4d}} K_h(x-y) (\operatorname{div} u_k(x) - \operatorname{div} u_k(y)) \chi'(\delta\rho_k) \bar{\rho}_k W_0 \bar{K}_h(x-z) \bar{K}_h(y-w) \\
& -\frac{1}{2} \int_0^t \int_{\Pi^{4d}} K_h(x-y) (\operatorname{div} u_k(x) + \operatorname{div} u_k(y)) (\chi'(\delta\rho_k) \delta\rho_k - 2\chi(\delta\rho_k)) \\
& \quad W_0(z,w) \bar{K}_h(x-z) \bar{K}_h(y-w) \\
& -\frac{\lambda}{2} \int_0^t \int_{\Pi^{3d}} K_h(x-y) \chi(\delta\rho_k) \bar{K}_h(x-z) M |\nabla u_k| w_0(z) \\
& \leq C \|K_h\|_{L^1} \varepsilon(h) + C \int_0^t \int_{\Pi^{4d}} K_h(x-y) \chi(\delta\rho_k) W_0 \bar{K}_h(x-z) \bar{K}_h(y-w).
\end{aligned}$$

Proof The computations are very similar for i) and ii) in Lemma 8.4 and for Lemma 8.5. For simplicity, in order to treat the proofs together as much as possible, we denote

$$\begin{aligned}
G_1(t, x, y) &= \chi_1(t, x, y) w_1(x), \quad G_2(t, x, y) = \chi'(\delta\rho_k) \bar{\rho}_k w_1(t, x) w_1(t, y), \\
G_0(t, x, y) &= \frac{1}{2} \chi_0'(\delta\rho_k) \bar{\rho}_k \int_{\Pi^{2d}} W_0(t, z, w) \bar{K}_h(x-z) \bar{K}_h(y-w) dz dw.
\end{aligned}$$

The first step is to truncate: Denote $I_k^L(t, x, y) = \phi(\rho_k(t, x)/L) \phi(\rho_k(t, y)/L)$ for some smooth and compactly supported ϕ ,

$$\begin{aligned}
& -\int_0^t \int_{\Pi^{2d}} K_h(x-y) (\operatorname{div} u_k(x) - \operatorname{div} u_k(y)) G_i \\
& \leq C \|K_h\|_{L^1} L^{-\theta_0} - \int_0^t \int_{\Pi^{2d}} K_h(x-y) (\operatorname{div} u_k(x) - \operatorname{div} u_k(y)) G_i I_k^L.
\end{aligned}$$

Here for $i = 0, 1, 2$, $G_i \leq C(\rho_k(t, x) + \rho_k(t, y))$ (even $G_2 \leq 2$) and consequently, as $\operatorname{div} u_k \in L^2$ uniformly, only $p > 2$ is required with $\theta_0 = (p-2)/2 > 0$.

Introduce an approximation $\mu_{k,\eta}$ of μ_k , satisfying (5.3) and s.t.

$$\begin{aligned}
\|\mu_{k,\eta}\|_{W_{t,x}^{2,\infty}} &\leq C \eta^{-2}, \quad \|\mu_{k,\eta} - \mu_k\|_{L^1} \leq \varepsilon_0(\eta), \\
\int_0^T \int_{\Pi^{2d}} K_h(x-y) |\mu_{k,\eta}(t, x) - \mu_{k,\eta}(t, y)| dx dy dt &\leq \|K_h\|_{L^1} \varepsilon_0(h),
\end{aligned} \tag{8.6}$$

from (5.13). Use (5.2) to decompose

$$\begin{aligned}
& -\int K_h(x-y) (\operatorname{div} u_k(x) - \operatorname{div} u_k(y)) G_i I_k^L dx dy \\
& = 2A_i + 2B_i + 2E_i,
\end{aligned}$$

with

$$A_i = - \int K_h(x-y) (P_k(\rho_k(x)) - P_k(\rho_k(y))) G_i \frac{I_k^L}{\mu_{k,\eta}(x)} dx dy,$$

and

$$B_i = \int K_h(x-y) \tilde{F}_k(t, x, y) G_i \frac{I_k^L}{\mu_{k,\eta}(x)} dx dy,$$

where

$$\begin{aligned} \tilde{F}_k(t, x, y) = & F_k(t, x) - F_k(t, y) + \mu_k(y) \mu_{k,\eta}(x) \operatorname{div} u_k(t, y) \left(\frac{1}{\mu_{k,\eta}(x)} - \frac{1}{\mu_k(y)} \right) \\ & - \mu_k(x) \mu_{k,\eta}(x) \operatorname{div} u_k(t, x) \left(\frac{1}{\mu_{k,\eta}(x)} - \frac{1}{\mu_k(x)} \right). \end{aligned}$$

Finally

$$E_i = \int K_h(x-y) (D\rho u_k(t, y) - D\rho u_k(t, x)) G_i \frac{I_k^L}{\mu_{k,\eta}(x)} dx dy,$$

with as before

$$D\rho u_k = \Delta^{-1} \operatorname{div} (\partial_t(\rho_k u_k) + \operatorname{div}(\rho_k u_k \otimes u_k)).$$

For B_i by the compactness of F_k , μ_k , estimates (8.6) and (5.13), and by (5.3)

$$B_i \leq C L \int_0^t \int_{\Pi^{2d}} K_h(x-y) |\tilde{F}_k(t, x, y)| \leq C L (\varepsilon_0(h) + \varepsilon_0(\eta)) \|K_h\|_{L^1}. \quad (8.7)$$

Note that again $|G_i| \leq C(\rho_k(x) + \rho_k(y))$ for $i = 0, 1, 2$.

For E_i , we use Lemma 8.3 by defining simply

$$\Phi_i(t, x, y) = G_i I_k^L(t, x, y) \frac{1}{\mu_{k,\eta}(t, x)}.$$

By (5.3),

$$\|\Phi_i\|_{L_{t,x}^\infty} \leq C L.$$

As for the time derivative of Φ , for $i = 1, 2$, G_i is a combination of functions of $\rho_k(t, x)$, $\rho_k(t, y)$ and w_i which all satisfy the same transport equation (with different right-hand sides). By (5.1),

$$\begin{aligned} \partial_t G_i + \operatorname{div}_x (u_k(x) G_i) + \operatorname{div}_y (u_k(y) G_i) = & f_{1,i} \operatorname{div}_x u_k(x) + f_{2,i} \operatorname{div}_y u_k(y) \\ & + f_{3,i} D_i(x) + f_{4,i} D_i(y), \end{aligned}$$

where the D_i are the penalizations introduced in Section 7.2 and the $f_{n,i}$ are again combinations of functions of $\rho_k(t, x)$, $\rho_k(t, y)$ and w_i . Every $f_{n,i}$ contains as a factor $\phi(\rho_k(x)/L)$ or a derivative of ϕ and thus

$$\|f_{n,i}\|_{L^\infty} \leq C L, \quad \forall n, i.$$

Finally by the smoothness of $\mu_{k,\eta}$

$$\begin{aligned} \partial_t \Phi_i + \operatorname{div}_x (u_k(x) \Phi_i) + \operatorname{div}_y (u_k(y) \Phi_i) &= \tilde{f}_{1,i} \operatorname{div}_x u_k(x) + \tilde{f}_{2,i} \operatorname{div}_y u_k(y) \\ &+ \tilde{f}_{3,i} D_i(x) + \tilde{f}_{4,i} D_i(y) + \Phi_i g_\eta, \end{aligned}$$

and it is easy to check that the constant C_{Φ_i} as defined in Lemma 8.3 is bounded by $C L \eta^{-1}$.

The case $i = 0$ is slightly more complicated as W_0 is integrated against \bar{K}_h so the equation on Φ_0 involves non local terms and we have to take into account extra terms as mentioned in the statement of Lemma 8.4. By (7.2), denoting $w_{0,h} = \bar{K}_h \star w_0$

$$\partial_t w_{0,h} + u_k(t,x) \cdot \nabla w_{0,h} - \alpha_k \Delta_x w_{0,h} = -\bar{K}_h \star (D_0 w_0) + R_h - \bar{K}_h \star (\operatorname{div} u_k w_0),$$

with

$$R_h = \int_{\Pi^d} \nabla \bar{K}_h(x-z) \cdot (u_k(t,x) - u_k(t,z)) w_0(t,z) dz.$$

Remark that R_h is uniformly bounded in $L^2_{t,x}$ by usual commutator estimates.

Finally as $\mu_{k,\eta}$ is smooth in time, one has

$$\begin{aligned} \partial_t \Phi_0 + \operatorname{div}_x (u_k(x) \Phi_0) + \operatorname{div}_y (u_k(y) \Phi_0) - \alpha_k (\Delta_x + \Delta_y) \Phi_0 \\ = f_{1,0} \operatorname{div}_x u_k(t,x) + f_{2,0} \operatorname{div}_y u_k(t,y) + \alpha_k (f_{3,0} |\nabla_x \rho_k(x)|^2 + f_{4,0} |\nabla_x \rho_k(y)|^2) \\ - \frac{\Phi_\rho}{\mu_{k,\eta}} (\bar{K}_h \star (D_0 w_0) + R_h - \bar{K}_h \star (\operatorname{div} u_k w_0)) - 2 \frac{\alpha_k}{\mu_{k,\eta}} \nabla_x \Phi_\rho \cdot \nabla_x w_{0,h} \\ + \Phi_0 g_\eta, \end{aligned}$$

where $\Phi_\rho = \delta \rho_k \bar{\rho}_k I_k^L(t,x,y)$, g_η is a function involving first and second derivatives of $\mu_{k,\eta}$ in t and x and ∇u_k . The $f_{j,0}$ are combinations of functions of $\rho_k(t,x)$ and $\rho_k(t,y)$, multiplied by $w_{0,h}$, and involving $\phi(\rho_k(x)/L)$, $\phi'(\rho_k(x)/L)$, or $\phi''(\rho_k(x)/L)$ and the corresponding term with $\rho_k(y)$.

By the L^∞ bounds on Φ_ρ , w_0 , each $f_{j,0}$ and by (5.5), one obtains

$$\left\| \partial_t \int \bar{K}_h(x-y) \Phi_0(t,x,y) dy \right\|_{L^1_t W_x^{-1,1}} \leq C L \eta^{-1}.$$

Therefore $C_\phi \leq C L \eta^{-1}$. Thus for all three cases, Lemma 8.3 yields

$$E_i \leq C L \eta^{-1} \|K_h\|_{L^1} h^\theta, \quad (8.8)$$

for some $\theta > 0$.

Proof of Lemma 8.5: The term A_0 . The terms A_i are where lies the main difference between Lemmas 8.4 and 8.5 as P_k is not monotone in the first case and monotone after a certain threshold in the second. For this reason we now proceed

separately for Lemma 8.5 and Lemma 8.4. In the case with diffusion for Lemma 8.5, there exist extra terms to handle in , namely $J + I$ with

$$J = -\frac{\lambda}{2} \int_0^t \int_{\Pi^{3d}} K_h(x-y) \chi(\delta\rho_k) \bar{K}_h(x-z) M |\nabla u_k|(z) w_0(z),$$

and

$$I = -\frac{1}{2} \int_0^t \int_{\Pi^{4d}} K_h(x-y) (\operatorname{div} u_k(x) + \operatorname{div} u_k(y)) \\ (\chi'(\delta\rho_k) \delta\rho_k - 2\chi(\delta\rho_k)) W_0 \bar{K}_h(x-z) \bar{K}_h(y-w).$$

We decompose this last term in a manner similar to what we have just done, first of all by introducing the truncation of ρ_k

$$I \leq C \|K_h\|_{L^1} L^{-\theta_0} \\ -\frac{1}{2} \int_0^t \int_{\Pi^{4d}} K_h(x-y) (\operatorname{div} u_k(x) + \operatorname{div} u_k(y)) \\ (\chi'(\delta\rho_k) \delta\rho_k - 2\chi(\delta\rho_k)) I_k^L W_0 \bar{K}_h(x-z) \bar{K}_h(y-w),$$

with again $\theta_0 = (p-2)/2$. Now introduce the μ_k

$$I \leq C \|K_h\|_{L^1} L^\theta - \frac{1}{2} \int_0^t \int_{\Pi^{4d}} K_h(x-y) (\mu_k(t,x) \operatorname{div} u_k(x) + \mu_k(t,y) \operatorname{div} u_k(y)) \\ \frac{I_k^L}{\mu_{k,\eta}(t,x)} (\chi'(\delta\rho_k) \delta\rho_k - 2\chi(\delta\rho_k)) W_0 \bar{K}_h(x-z) \bar{K}_h(y-w) \\ + \frac{1}{2} \int_0^t \int_{\Pi^{4d}} K_h(x-y) H_k(x,y) (\chi' \delta\rho_k - 2\chi(\delta\rho_k)) I_k^L W_0 \bar{K}_h(x-z) \bar{K}_h(y-w)$$

where

$$H_k(t,x,y) = \mu_k(x) \operatorname{div} u_k(x) \left(\frac{1}{\mu_{k,\eta}(x)} - \frac{1}{\mu_k(x)} \right) - \mu_k(y) \operatorname{div} u_k(y) \left(\frac{1}{\mu_k(y)} - \frac{1}{\mu_{k,\eta}(x)} \right).$$

By the compactness of μ_k , one has that

$$\int_0^t \int_{\Pi^{4d}} K_h(x-y) H_k(x,y) (\chi' \delta\rho_k - 2\chi(\delta\rho_k)) W_0 \bar{K}_h(x-z) \bar{K}_h(y-w) \\ \leq \|K_h\|_{L^1} \varepsilon_0(h) \|u_k\|_{L_t^2 H_x^1} \|\rho_k\|_{L_{t,x}^2} \leq C \varepsilon_0(h) \|K_h\|_{L^1}.$$

This implies that

$$I \leq -\frac{1}{2} \int_0^t \int_{\Pi^{4d}} K_h(x-y) (\mu_k(t,x) \operatorname{div} u_k(x) + \mu_k(t,y) \operatorname{div} u_k(y)) \\ \frac{I_k^L}{\mu_{k,\eta}(t,x)} (\chi'(\delta\rho_k) \delta\rho_k - 2\chi(\delta\rho_k)) W_0 \bar{K}_h(x-z) \bar{K}_h(y-w) \\ + C \|K_h\|_{L^1} (L^{-\theta} + \varepsilon_0(h)).$$

Using (5.2) or namely that $\mu_k \operatorname{div} u_k = D\rho u_k + F_k + P_k(\rho_k)$, the quantity $A_0 + I + J$ may be written

$$A_0 + I + J \leq C \|K_h\|_{L^1} (L^{-\theta} + \varepsilon_0(h)) + I_1 + I_2, \quad (8.9)$$

with

$$I_1 = A_0 - \frac{1}{2} \int_0^t \int_{\Pi^{4d}} K_h(x-y) (P_k(\rho_k(x)) + P_k(\rho_k(y))) \frac{I_k^L}{\mu_{k,\eta}(t,x)} (\chi'(\delta\rho_k) \delta\rho_k - 2\chi(\delta\rho_k)) W_0 \bar{K}_h(x-z) \bar{K}_h(y-w),$$

and

$$I_2 = -\frac{1}{2} \int_0^t \int_{\Pi^{4d}} K_h(x-y) (D\rho u_k(x) + F_k(x) + D\rho u_k(y) + F_k(y)) \frac{I_k^L}{\mu_{k,\eta}(t,x)} (\chi'(\delta\rho_k) \delta\rho_k - 2\chi(\delta\rho_k)) W_0 \bar{K}_h(x-z) \bar{K}_h(y-w) - \frac{\lambda}{2} \int_0^t \int_{\Pi^{3d}} K_h(x-y) \chi(\delta\rho_k) \bar{K}_h(x-z) M |\nabla u_k|(z) w_0(z).$$

In this case with diffusion, because P_k is essentially monotone, the term A_0 is mostly dissipative and helps control the rest. More precisely

$$I_1 = -\frac{1}{2} \int_0^t \int_{\Pi^{2d}} \left[K_h(x-y) [(P_k(\rho_k(x)) - P_k(\rho_k(y))) \chi'(\delta\rho_k) \bar{\rho}_k + (P_k(\rho_k(x)) + P_k(\rho_k(y))) (\chi'(\delta\rho_k) \delta\rho_k - 2\chi(\delta\rho_k))] \frac{I_k^L}{\mu_{k,\eta}(t,x)} \int_{\Pi^{2d}} W_0 \bar{K}_h(x-z) \bar{K}_h(y-w) dz dw \right] dx dy.$$

As $P_k \geq 0$ and by (8.1), $\chi'(\delta\rho_k) \delta\rho_k - 2\chi(\delta\rho_k) \geq -\chi'(\delta\rho_k) \delta\rho_k$, thus

$$(P_k(\rho_k(x)) - P_k(\rho_k(y))) \chi' \bar{\rho}_k + (P_k(\rho_k(x)) + P_k(\rho_k(y))) (\chi' \delta\rho_k - 2\chi(\delta\rho_k)) \geq \chi'(\delta\rho_k) [(P_k(\rho_k(x)) - P_k(\rho_k(y))) \bar{\rho}_k - (P_k(\rho_k(x)) + P_k(\rho_k(y))) \delta\rho_k].$$

Without loss of generality, we may assume that $\rho_k(x) \geq \rho_k(y)$ and hence $\chi'(\delta\rho_k) \geq 0$. Developing

$$(P_k(\rho_k(x)) - P_k(\rho_k(y))) \bar{\rho}_k - (P_k(\rho_k(x)) + P_k(\rho_k(y))) \delta\rho_k = 2 P_k(\rho_k(x)) \rho_k(y) - 2 P_k(\rho_k(y)) \rho_k(x).$$

As a consequence, by the quasi-monotonicity (5.8) of $P_k(s)/s$, for some $\bar{\rho}$ large enough with respect to ρ_0 , if $\rho_k(x) \geq \bar{\rho}$, then $P_k(\rho_k(x)) \rho_k(y) - P_k(\rho_k(y)) \rho_k(x) \geq 0$ (recall that we assumed $\rho_k(x) \geq \rho_k(y)$).

The only case where one does not have the right sign is hence where both $\rho_k(x)$ and $\rho_k(y)$ are bounded. Therefore using the regularity of P_k

$$(P_k(\rho_k(x)) - P_k(\rho_k(y))) \bar{\rho}_k - (P_k(\rho_k(x)) + P_k(\rho_k(y))) \delta\rho_k \geq -C |\delta\rho_k|.$$

Introducing this estimate in I_1 yields

$$\begin{aligned}
I_1 &\leq C \int_0^t \int_{\Pi^{2d}} K_h(x-y) |\delta\rho_k| |\chi'(\delta\rho_k)| \frac{I_k^L}{\mu_{k,\eta}(t,x)} \int_{\Pi^{2d}} W_0 \bar{K}_h(x-z) \bar{K}_h(y-w) \\
&\leq C \int_0^t \int_{\Pi^{4d}} K_h(x-y) \chi(\delta\rho_k) W_0 \bar{K}_h(x-z) \bar{K}_h(y-w).
\end{aligned} \tag{8.10}$$

Turning now to I_2 , we observe that $\mu_k M |\nabla u_k| \geq \mu_k \operatorname{div} u_k \geq D\rho u_k + F_k$ and that $\chi(\delta_k) \geq (2\chi(\delta_k) - \chi'(\delta_k) \delta_k)/C$. Therefore for λ large enough, using $W_0(z, w) = w_0(z) + w_0(w)$ and the symmetry

$$\begin{aligned}
I_2 &\leq -\frac{1}{2} \int_0^t \int_{\Pi^{4d}} K_h(x-y) (D\rho u_k(x) - D\rho u_k(z) + F_k(x) - F_k(z) \\
&\quad + D\rho u_k(y) - D\rho u_k(z) + F_k(y) - F_k(z)) \frac{I_k^L}{\mu_{k,\eta}(t,x)} (\chi'(\delta\rho_k) \delta\rho_k - 2\chi(\delta\rho_k)) \\
&\quad W_0 \bar{K}_h(x-z) \bar{K}_h(y-w).
\end{aligned}$$

The differences in the F_k are controlled by the compactness of F_k and the differences in the $D\rho u_k$ by Lemma 8.3 as for the terms E_i . Hence, finally

$$I_2 \leq C \|K_h\|_{L^1} (\varepsilon_0(h) + L\eta^{-1} h^\theta). \tag{8.11}$$

Conclusion for Lemma 8.5. We sum up the contributions from B_0 in (8.7), E_0 in (8.8), $A_0 + I + J$ in (8.9) together with the bounds on I_1 in (8.10) and I_2 in (8.11) to obtain

$$\begin{aligned}
&-\frac{1}{2} \int_0^t \int_{\Pi^{2d}} K_h(x-y) (\operatorname{div} u_k(x) - \operatorname{div} u_k(y)) G_0 \\
&-\frac{1}{2} \int_0^t \int_{\Pi^{4d}} K_h(x-y) (\operatorname{div} u_k(x) + \operatorname{div} u_k(y)) (\chi'(\delta\rho_k) \delta\rho_k - 2\chi(\delta\rho_k)) \\
&\quad W_0(z, w) \bar{K}_h(x-z) \bar{K}_h(y-w) \\
&-\frac{\lambda}{2} \int_0^t \int_{\Pi^{3d}} K_h(x-y) \chi(\delta\rho_k) \bar{K}_h(x-z) M |\nabla u_k|(z) w_0(z) \\
&\leq C \|K_h\|_{L^1} (L^{-\theta_0} + L(\varepsilon_0(h) + \varepsilon_0(\eta)) + L\eta^{-1} h^\theta) \\
&+ C \int_0^t \int_{\Pi^{4d}} K_h(x-y) \chi(\delta\rho_k) W_0(z, w) \bar{K}_h(x-z) \bar{K}_h(y-w).
\end{aligned}$$

Just optimizing in L and η leads to the desired $\varepsilon(h)$ and concludes the proof of Lemma 8.5.

Proof of Lemma 8.4: The term A_i with $i = 1, 2$. It now remains to analyze more precisely the terms $(P_k(\rho_k(x)) - P_k(\rho_k(y))) G_i$ for $i=1,2$ concerning the case without diffusion but with non monotone pressure. We will split the study in three cases.

Case 1). The case $(P_k(\rho_k(x)) - P_k(\rho_k(y)))\delta\rho_k \geq 0$. Since G_2 obviously have the same sign as $\delta\rho_k$, one simply has

$$(P_k(\rho_k(x)) - P_k(\rho_k(y))) G_i \geq 0,$$

for $i = 2$. In the same case, for G_1

$$\begin{aligned} & (P_k(\rho_k(x)) - P_k(\rho_k(y))) G_1 \\ &= (P_k(\rho_k(x)) - P_k(\rho_k(y))) \left(\frac{1}{2}\chi'(\delta\rho_k) \bar{\rho}_k + \chi(\delta\rho_k) - \frac{1}{2}\chi'(\delta\rho_k) \delta\rho_k \right) w_1(x) \\ &\geq |P_k(\rho_k(x)) - P_k(\rho_k(y))| \left(\frac{1}{2}|\chi'(\delta\rho_k)| \bar{\rho}_k - \left| \chi(\delta\rho_k) - \frac{1}{2}\chi'(\delta\rho_k) \delta\rho_k \right| \right) w_1(x) \\ &\geq 0, \end{aligned}$$

by (8.1) as $\bar{\rho}_k \geq |\delta\rho_k|$. Therefore in that case the terms have the right sign and can be dropped.

Case 2). The case $(P_k(\rho_k(x)) - P_k(\rho_k(y)))\delta\rho_k < 0$ and $\rho_k(y) \leq \rho_k(x)/2$ or $\rho_k(y) \geq 2\rho_k(x)$ for some constant C .

– For $i = 1$, first assume that $P_k(\rho_k(x)) \geq P_k(\rho_k(y))$ while $\rho_k(y) \geq 2\rho_k(x)$.

$$(P_k(\rho_k(x)) - P_k(\rho_k(y))) G_1 \geq -P_k(\rho_k(x)) (|\chi'(\delta\rho_k)| \bar{\rho}_k + \chi(\delta\rho_k)) w_1(x).$$

Now observe that since $\rho_k(y) \geq 2\rho_k(x)$ then

$$|\chi'(\delta\rho_k)| \bar{\rho}_k \leq \frac{3}{2} |\chi'(\delta\rho_k)| \rho_k(y) \leq 3 |\chi'(\delta\rho_k)| |\delta\rho_k| \leq C \chi(\delta\rho_k),$$

by (8.1). Therefore in that case

$$(P_k(\rho_k(x)) - P_k(\rho_k(y))) G_1 \geq -C \rho_k(x)^{\tilde{\gamma}} \chi(\delta\rho_k) w_1(x),$$

since by (5.9), $P_k(\xi) \leq P_k(0) + C \xi^{\tilde{\gamma}-1} \xi \leq C \xi^{\tilde{\gamma}}$.

Note that the result is not symmetric in x and y and we have to check also $P_k(\rho_k(x)) \leq P_k(\rho_k(y))$ and $\rho_k(y) \leq \rho_k(x)/2$. Then simply bound

$$\begin{aligned} (P_k(\rho_k(x)) - P_k(\rho_k(y))) G_1 &\geq -C (\rho_k(y))^{\tilde{\gamma}} \chi(\delta\rho_k) w_1(x) \\ &\geq -C (\rho_k(x))^{\tilde{\gamma}} \chi(\delta\rho_k) w_1(x), \end{aligned}$$

since now $\rho_k(y) \leq \rho_k(x)$.

– For $i = 2$. The calculations are similar (simpler in fact) for G_2 and this lets us deduce that if $P_k(\rho_k(x)) - P_k(\rho_k(y))$ and $\rho_k(x) - \rho_k(y)$ have different signs but $\rho_k(y) \leq \rho_k(x)/2$ or $\rho_k(y) \geq 2\rho_k(x)$ then

$$(P_k(\rho_k(x)) - P_k(\rho_k(y))) G_2 \geq -C (\rho_k(x))^{\tilde{\gamma}} \chi(\delta\rho_k) w_1(x) w_1(y).$$

Case 3). For $i = 1, 2$, the situation where $P_k(\rho_k(x)) - P_k(\rho_k(y))$ and $\rho_k(x) - \rho_k(y)$ have different signs but $\rho_k(x)/2 \leq \rho_k(y) \leq 2\rho_k(x)$. Then one bluntly estimates using the Lipschitz bound on P_k given by (5.9)

$$|(P_k(\rho_k(x)) - P_k(\rho_k(y))) G_i| \leq C (1 + \rho_k^{\tilde{\gamma}-1}(x) + \rho_k^{\tilde{\gamma}-1}(y)) |\delta\rho_k| G_i.$$

Bounding now the G_i by (8.1),

$$(P_k(\rho_k(x)) - P_k(\rho_k(y))) G_2 \leq C (1 + \rho_k^{\tilde{\gamma}}(x) + \rho_k^{\tilde{\gamma}}(y)) \chi(\delta\rho_k) w_1(x) w_1(y),$$

and

$$\begin{aligned} (P_k(\rho_k(x)) - P_k(\rho_k(y))) G_1 &\leq C (1 + \rho_k^{\tilde{\gamma}}(x) + \rho_k^{\tilde{\gamma}}(y)) \chi(\delta\rho_k) w_1(x) \\ &\leq C (1 + \rho_k^{\tilde{\gamma}}(x)) \chi(\delta\rho_k) w_1(x), \end{aligned}$$

as $\rho_k(x)$ and $\rho_k(y)$ are of the same order.

From the analysis of these three cases, one has that

$$A_1 \leq C \int K_h(x-y) (1 + \rho_k^{\tilde{\gamma}}(x)) \chi(\delta\rho_k) w_1(x) dx dy, \quad (8.12)$$

and

$$A_2 \leq C \int K_h(x-y) (1 + \rho_k^{\tilde{\gamma}}(x) + \rho_k^{\tilde{\gamma}}(y)) \chi(\delta\rho_k) w_1(x) w_1(y) dx dy. \quad (8.13)$$

Conclusion of the proof of Lemma 8.4. Summing up every term, namely (8.7)-(8.8) and (8.12)-(8.13), we eventually find that

$$\begin{aligned} & - \int_0^t \int_{\Pi^{2d}} K_h(x-y) (\operatorname{div} u_k(x) - \operatorname{div} u_k(y)) G_1 \\ & \leq C \|K_h\|_{L^1} (L^{-\theta} + L(\varepsilon_0(h) + \varepsilon_0(\eta)) + L\eta^{-1} h^\theta) \\ & \quad + C \int_0^t \int_{\Pi^{2d}} K_h(x-y) (1 + \rho_k(x)^{\tilde{\gamma}}) \chi(\delta\rho_k) w_1(x), \end{aligned}$$

while

$$\begin{aligned} & - \int_0^t \int_{\Pi^{2d}} K_h(x-y) (\operatorname{div} u_k(x) - \operatorname{div} u_k(y)) G_2 \\ & \leq C \|K_h\|_{L^1} (L^{-\theta} + L(\varepsilon_0(h) + \varepsilon_0(\eta)) + L\eta^{-1} h^\theta) \\ & \quad + C \int_0^t \int_{\Pi^{2d}} K_h(x-y) (1 + \rho_k(x)^{\tilde{\gamma}} + \rho_k(y)^{\tilde{\gamma}}) \chi(\delta\rho_k) w_1(x) w_1(y). \end{aligned}$$

To conclude the proof of Lemma 8.4, one optimizes in η and L . Just remark that since the inequalities depend polynomially in L and η then the result depends on ε_0^θ for some θ . \square

8.4 Conclusion of the proofs of Theorems 5.1 and 5.2.

We combine Lemma 8.1 with Lemma 8.5 or 8.4 and we finally use Prop. 7.2. Let us summarize the required assumptions. In all cases one assumes that ρ_k solves (5.1) and that $\operatorname{div} u_k$ is coupled with ρ_k through (5.2); bounds are assumed on the viscosity as per (5.3), on the time derivative of $\rho_k u_k$ per (5.6), on u_k per (5.5). Finally the viscosity μ_k and the force term F_k are assumed to be compact in L^1 . Moreover

- In the case with diffusion, $\alpha_k > 0$, one assumes that the pressure term P_k satisfies (5.8) and the bounds (5.4) on ρ_k with $\gamma > d/2$ and $p > 2$.
- In the case without diffusion, $\alpha_k = 0$, one needs only (5.9) on the pressure P_k and the bounds (5.4) on ρ_k with $\gamma > d/2$ and $p > 2$. Moreover for Prop. 7.2, it is necessary that $p \geq \tilde{\gamma}$ (in general $\tilde{\gamma} = \gamma < p$ so this is not a big issue).

Then one obtains by taking λ large enough and using a simple Gronwall lemma

$$\begin{aligned} & \int_{h_0}^1 \int_{\Pi^{4d}} \bar{K}_h(x-z) \bar{K}_h(y-w) (w_0(t,z) + w_0(t,w)) K_h(x-y) \chi(\delta\rho_k) dx dy dz dw \frac{dh}{h} \\ &= \int_{\Pi^{4d}} \bar{K}_h(x-z) \bar{K}_h(y-w) (w_0(t,z) + w_0(t,w)) \mathcal{K}_{h_0}(x-y) \chi(\delta\rho_k) dx dy dz dw \\ &\leq C \left(|\log h_0|^{1/2} + \varepsilon_{h_0}(k) + \int_{h_0}^1 \varepsilon(h) \frac{dh}{h} \right), \end{aligned} \tag{8.14}$$

and

$$\begin{aligned} & \int_{h_0}^1 \int_{\Pi^{2d}} (w_1(t,x) + w_1(t,y)) \bar{K}_h(x-y) \chi(\delta\rho_k) dx dy \frac{dh}{h} \\ &= \int_{\Pi^{2d}} (w_1(t,x) + w_1(t,y)) \mathcal{K}_{h_0}(x-y) \chi(\delta\rho_k) dx dy \\ &\leq C \left(|\log h_0|^{1/2} + \int_{h_0}^1 \varepsilon(h) \frac{dh}{h} \right), \end{aligned} \tag{8.15}$$

with finally

$$\begin{aligned} & \int_{\Pi^{2d}} w_1(t,x) w_1(t,y) K_h(x-y) \chi(\delta\rho_k) dx dy \\ &\leq C \|K_h\|_{L^1} (h^\theta + \varepsilon(h)), \end{aligned} \tag{8.16}$$

where ε depends only on the smoothness of μ_k and F_k and of $p > 2$.

The key point in all three cases is to be able to remove the weights from those estimates. For that, one uses point ii) of Prop. 7.2.

The case with $w_0(x) + w_0(y)$. Denote $\omega_\eta = \{\bar{K}_h \star w_0(t, x) \leq \eta\} \subset [0, T] \times \Pi^d$ and remark that

$$\begin{aligned} \int_{\Pi^{2d}} \mathcal{K}_{h_0}(x-y) \chi(\delta\rho_k) &= \int_{h_0}^1 \int_{\Pi^{2d}} \bar{K}_h(x-y) \chi(\delta\rho_k) \frac{dh}{h} \\ &= \int_{h_0}^1 \int_{x \in \omega_\eta^c \text{ or } y \in \omega_\eta^c} \bar{K}_h(x-y) \chi(\delta\rho_k) \frac{dh}{h} \\ &\quad + \int_{h_0}^1 \int_{x \in \omega_\eta \text{ and } y \in \omega_\eta} \bar{K}_h(x-y) \chi(\delta\rho_k) \frac{dh}{h}. \end{aligned}$$

Now

$$\begin{aligned} \int_{h_0}^1 \int_{x \in \omega_\eta^c \text{ or } y \in \omega_\eta^c} \bar{K}_h(x-y) \chi(\delta\rho_k) \frac{dh}{h} \\ \leq \frac{1}{\eta} \int_{h_0}^1 \int_{\Pi^{2d}} \bar{K}_h(x-y) (\bar{K}_h \star w_0(x) + \bar{K}_h \star w_0(y)) \chi(\delta\rho_k) \frac{dh}{h}, \end{aligned}$$

while by point iii) in Prop. 7.2, using that $\rho \in L^p((0, T) \times \Pi^d)$ with $p > 2$ and recalling that $\chi(\xi) \leq C|\xi|$

$$\begin{aligned} \int_{h_0}^1 \int_{x \in \omega_\eta \text{ and } y \in \omega_\eta} \bar{K}_h(x-y) \chi(\delta\rho_k) \frac{dh}{h} &\leq 2 \int_{h_0}^1 \int_{\Pi^{2d}} \bar{K}_h(x-y) \rho_k \mathbb{I}_{\bar{K}_h \star w_0 \leq \eta} \frac{dh}{h} \\ &\leq \frac{C|\log h_0|}{|\log \eta|^{1/2}}. \end{aligned}$$

Therefore combining this with (8.14), one obtains

$$\int_{\Pi^{2d}} \mathcal{K}_{h_0}(x-y) \chi(\delta\rho_k) dx dy \leq C \left(\frac{\varepsilon_{h_0}(k) + |\log h_0|^{1/2} + \int_{h_0}^1 \varepsilon(h) \frac{dh}{h}}{\eta} + \frac{\|\mathcal{K}_{h_0}\|_{L^1}}{|\log \eta|^{1/2}} \right)$$

recalling that

$$\varepsilon_{h_0}(k) = \alpha_k \int_{h_0}^1 \frac{dh}{h} h_0^{-2},$$

and denoting

$$\bar{\varepsilon}(h_0) = \frac{1}{|\log h_0|} \int_{h_0}^1 \varepsilon(h) \frac{dh}{h}$$

Remark that $\bar{\varepsilon}(h_0) \rightarrow 0$ since $\varepsilon(h) \rightarrow 0$: For instance if $\varepsilon(h) = h^\theta$ then $\bar{\varepsilon}(h_0) \sim |\log h_0|^{-1}$. The estimate then reads

$$\begin{aligned} \int_{\Pi^{2d}} \mathcal{K}_{h_0}(x-y) \chi(\delta\rho_k) dx dy &\leq C \left(|\log h_0| \frac{\alpha_k h_0^{-2} + |\log h_0|^{-1/2} + \bar{\varepsilon}(h_0)}{\eta} \right. \\ &\quad \left. + \frac{\|\mathcal{K}_{h_0}\|_{L^1}}{|\log \eta|^{1/2}} \right). \end{aligned}$$

As $\|\mathcal{K}_{h_0}\|_{L^1} \sim |\log h_0|$, by optimizing in η , the following estimate is obtained

$$\int_{\Pi^{2d}} \mathcal{K}_{h_0}(x-y) \chi(\delta\rho_k) dx dy \leq \frac{C\|\mathcal{K}_{h_0}\|_{L^1}}{|\log(\alpha_k h_0^{-2} + |\log h_0|^{-1/2} + \bar{\varepsilon}(h_0))|^{1/2}}.$$

Per Prop. 4.1, this gives the compactness of $\delta\rho_k$ as the r.h.s. is negligible against $\|\mathcal{K}_{h_0}\|_{L^1}$ as $h_0 \rightarrow 0$. And it proves the case i) of Theorem 5.1.

The case with $w_1(x) + w_1(y)$. Similarly, from (8.15), one then proves that in the corresponding case

$$\begin{aligned} \int_{\Pi^{2d}} \mathcal{K}_{h_0}(x-y) \chi(\delta\rho_k) dx dy &\leq C |\log h_0| \left(\frac{|\log h_0|^{-1/2} + \bar{\varepsilon}(h_0)}{\eta} + \frac{1}{|\log \eta|^\theta} \right) \\ &\leq \frac{C\|\mathcal{K}_{h_0}\|_{L^1}}{|\log(|\log h_0|^{-1/2} + \bar{\varepsilon}(h_0))|^\theta}, \end{aligned}$$

again using part ii) of Prop. 7.2.

In both cases, using Prop. 4.1 together with Lemma 6.6 in the second case, one concludes that ρ_k is locally compact in x and then in t, x . Thus we've shown the case ii) and concluded the proof of Theorem 5.1.

The case with $w_1(y)w_1(x)$. The situation is more complicated for (8.16) and the product $w_1(y)w_1(x)$. Indeed $w_0(x) + w_0(y)$ or $w_1(x) + w_1(y)$ are small only if both $w_0(x)$ and $w_0(y)$ are small (or the corresponding terms for w_1). But $w_1(x)w_1(y)$ can be small if either $w_1(x)$ or $w_1(y)$ is small. This was previously an advantage with then simpler computations but not here and (8.16) does not provide compactness.

This is due to the fact that one does not control the size of $\{w_w \leq \eta\}$ but only the mass of ρ_k over that set. The difference between the two is the famous vacuum problem for compressible fluid dynamics which is still unsolved.

The best that can be done by part ii) of Prop. 7.2 is for any η, η'

$$\begin{aligned} \int_{\Pi^{2d}} \mathbb{I}_{\rho_k(x) \geq \eta} \mathbb{I}_{\rho_k(y) \geq \eta} K_h(x-y) \chi(\delta\rho_k) &\leq \frac{1}{\eta'^2} \int w_1(x) w_1(y) K_h(x-y) \chi(\delta\rho_k) \\ &+ C \frac{\|K_h\|_{L^1}}{\eta^{1/2} |\log \eta'|^{\theta/2}}, \end{aligned}$$

using that $\rho_k \in L^2$ uniformly. Using (8.16) and optimizing in η' , one finds for some $\theta > 0$

$$\int_{\Pi^{2d}} \mathbb{I}_{\rho_k(x) \geq \eta} \mathbb{I}_{\rho_k(y) \geq \eta} K_h(x-y) \chi(\delta\rho_k) \leq C \frac{\|K_h\|_{L^1}}{\eta^{1/2} |\log(\varepsilon(h) + h^\theta)|^{\theta/2}}.$$

If μ_k and F_k are uniformly in $W^{s,1}$ for $s > 0$, then

$$\int_{\Pi^{2d}} \mathbb{I}_{\rho_k(x) \geq \eta} \mathbb{I}_{\rho_k(y) \geq \eta} K_h(x-y) \chi(\delta\rho_k) \leq C \frac{\|K_h\|_{L^1}}{\eta^{1/2} |\log h|^{\theta/2}},$$

which concludes the proof of Th. 5.2. Note however that in many senses (8.16) is more precise than the final result.

8.5 The coupling with the pressure in the anisotropic case

In that case we need the weight w_a and its regularization $w_{a,h}$, defined by (7.2) with (7.5) in order to compensate some terms coming from the anisotropic non-local part of the stress tensor.

Lemma 8.6 *There exists $C_* > 0$ s.t. assuming that ρ_k solves (5.1) with $\alpha_k = 0$, that (5.6), (5.5), (5.4) with $\gamma > d/2$ and $p > \gamma + 1 + \ell = \gamma^2/(\gamma - 1)$ hold where $\ell = 1/(\gamma - 1)$; assuming moreover that P_k satisfying (5.8), that u solves (5.10) with*

$$a_\mu \leq C_*. \quad (8.17)$$

Then there exists $0 < \theta < 1$ s.t. for χ_a verifying (8.2) for this choice of ℓ

$$\begin{aligned} & \int_{h_0}^1 \int_{\Pi^{2d}} \frac{\overline{K}_h(x-y)}{h} (w_{a,h}(x) + w_{a,h}(y)) \chi_a(\delta\rho_k)(t) \\ & \leq \int_{h_0}^1 \int_{\Pi^{2d}} \frac{\overline{K}_h(x-y)}{h} (w_{a,h}(x) + w_{a,h}(y)) \chi_a(\delta\rho_k)|_{t=0} + C(1+\ell) |\log h_0|^\theta. \end{aligned}$$

Proof To simplify the estimate, we assume in this proof that $P_k(\rho_k) = \rho_k^\gamma$, the extension when P_k satisfies instead 5.8 being straightforward. We also recall that χ_a satisfies (8.2), meaning that for all practical purposes $\chi_a(\xi) \sim |\xi|^{1+\ell}$.

We use the point iii) in Lemma 8.2

$$\begin{aligned} & \int_{h_0}^1 \int_{\Pi^{2d}} \frac{\overline{K}_h(x-y)}{h} (w_{a,h}(x) + w_{a,h}(y)) \chi_a(\delta\rho_k)(t) \\ & \quad - \int_{h_0}^1 \int_{\Pi^{2d}} \frac{\overline{K}_h(x-y)}{h} (w_{a,h}(x) + w_{a,h}(y)) \chi_a(\delta\rho_k)|_{t=0} \\ & \leq C |\log h_0|^\theta + I + II - \Pi_a \end{aligned}$$

where $0 < \theta < 1$ and with the dissipation term by symmetry

$$\Pi_a = \lambda \int_0^t \int_{h_0}^1 \int_{\Pi^{2d}} w_{a,h}(x) \chi_a(\delta\rho_k) \overline{K}_h \star (|\operatorname{div} u_k| + |A_\mu P_k(\rho_k)|)(x) \overline{K}_h \frac{dh}{h}$$

while still by symmetry

$$I = -\frac{1}{2} \int_{h_0}^1 \int_0^t \int_{\Pi^{2d}} \frac{\overline{K}_h(x-y)}{h} (\operatorname{div} u_k(x) - \operatorname{div} u_k(y)) \chi'_a(\delta\rho_k) \bar{\rho}_k w_{a,h}(x),$$

and

$$\begin{aligned} II = & -\frac{1}{2} \int_{h_0}^1 \int_0^t \int_{\Pi^{2d}} \frac{\overline{K}_h(x-y)}{h} (\operatorname{div} u_k(x) + \operatorname{div} u_k(y)) \\ & (\chi'_a(\delta\rho_k) \delta\rho_k - 2\chi_a(\delta\rho_k)) w_{a,h}(x). \end{aligned}$$

I) The quantity I . We recall that in this case one has the formula (5.10) on $\operatorname{div} u_k$

$$\begin{aligned} \operatorname{div} u_k &= \nu_k P_k(\rho_k) + \nu_k a_\mu A_\mu P_k(\rho_k) \\ &\quad + \nu_k (\Delta_\mu - a_\mu E_k)^{-1} \operatorname{div} (\partial_t (\rho_k u_k) + \operatorname{div} (\rho_k u_k \otimes u_k)), \end{aligned} \quad (8.18)$$

leading to the notation

$$\tilde{D}\rho u_k = \nu_k (\Delta_\mu - a_\mu E_k)^{-1} \operatorname{div} (\partial_t (\rho_k u_k) + \operatorname{div} (\rho_k u_k \otimes u_k)).$$

Therefore, one may decompose

$$I = I_0 + I^D + I^R,$$

with

$$\begin{aligned} I_0 &= -\frac{1}{2} \int_{h_0}^1 \int_0^t \int_{\Pi^{2d}} \frac{\bar{K}_h(x-y)}{h} (\tilde{D}\rho u_k(x) - \tilde{D}\rho u_k(y)) \\ &\quad \chi'_a(\delta\rho_k) \bar{\rho}_k (w_{a,h}(x) + w_{a,h}(y)), \end{aligned}$$

while

$$\begin{aligned} I^D &= -\frac{\nu_k}{2} \int_{h_0}^1 \int_0^t \int_{\Pi^{2d}} \frac{\bar{K}_h(x-y)}{h} (P_k(\rho_k(x)) - P_k(\rho_k(y))) \\ &\quad \chi'_a(\delta\rho_k) \bar{\rho}_k w_{a,h}(x), \end{aligned}$$

and

$$\begin{aligned} I^R &= -\frac{a_\mu \nu_k}{2} \int_{h_0}^1 \int_0^t \int_{\Pi^{2d}} \frac{\bar{K}_h(x-y)}{h} (A_\mu P_k(\rho_k(x)) - A_\mu P_k(\rho_k(y))) \\ &\quad \chi'_a(\delta\rho_k) \bar{\rho}_k w_{a,h}(x). \end{aligned}$$

I-1) The term I_0 . This term is handled just as in the proof of Lemmas 8.4-8.5 by using Lemma 8.3 and for this reason we do not fully detail all the steps here. First note that Lemma 8.3 applies to $\tilde{D}\rho_k u_k$ as well as for $D\rho_k u_k$ as

$$\tilde{D}\rho_k u_k = (\nu_k (\Delta_\mu - a_\mu E_k)^{-1} \Delta) D\rho_k u_k.$$

Then as before, we first truncate by using some smooth function $I_k^L(t, x, y) = \phi(\rho_k(t, x)/L)\phi(\rho_k(t, y)/L)$ with some smooth and compactly supported function ϕ leading to $I_0 = I_0^L + I_0^{RL}$ with

$$\begin{aligned} I_0^L &= -\frac{1}{2} \int_{h_0}^1 \int_0^t \int_{\Pi^{2d}} \frac{\bar{K}_h(x-y)}{h} (\tilde{D}\rho u_k(x) - \tilde{D}\rho u_k(y)) \\ &\quad \chi'_a(\delta\rho_k) I_k^L \bar{\rho}_k (w_{a,h}(x) + w_{a,h}(y)), \end{aligned}$$

and

$$I_0^{RL} = -\frac{1}{2} \int_{h_0}^1 \int_0^t \int_{\Pi^{2d}} \frac{\bar{K}_h(x-y)}{h} (\tilde{D}\rho u_k(x) - \tilde{D}\rho u_k(y)) \chi'_a(\delta\rho_k) (1 - I_k^L) \bar{\rho}_k(w_{a,h}(x) + w_{a,h}(y)).$$

Remark that $\operatorname{div} u_k \in L_{t,x}^2$, $P_k(\rho_k) \in L_{t,x}^{p/\gamma}$ and since A_μ is an operator of 0 order, $A_\mu P_k(\rho_k) \in L_{t,x}^{p/\gamma}$. Therefore by Eq. (8.18)

$$\sup_k \|\tilde{D}\rho_k u_k\|_{L_{t,x}^{\min(2,p/\gamma)}} < \infty.$$

On the other hand $|\chi'_a(\delta\rho_k)| \leq C(1+\ell)(|\rho_k(x)|^\ell + |\rho_k(y)|^\ell)$ and this lets us bound very simply I_0^{RL} by Hölder estimates

$$\begin{aligned} I_0^{RL} &\leq C(1+\ell) |\log h_0| \|\tilde{D}\rho_k u_k\|_{L_{t,x}^{\min(2,p/\gamma)}} \|(1 - I_k^L) \rho_k^{1+\ell}\|_{L_{t,x}^{\max(2,q)}} \\ &\leq C(1+\ell) |\log h_0| \|(1 - I_k^L) \rho_k^{1+\ell}\|_{L_{t,x}^{\max(2,q)}}, \end{aligned}$$

with $1/q + \gamma/p = 1$. But $q(1+\ell) < p$ by the assumption $p > \gamma + 1 + \ell$ and similarly $2(1+\ell) < p$. As a consequence for some exponent $\theta_1 > 0$

$$I_0^{RL} \leq C(1+\ell) |\log h_0| L^{-\theta_1}. \quad (8.19)$$

We now use Lemma 8.3 for $\tilde{D}\rho_k u_k$ and $\Phi = \chi'_a(\delta\rho_k) I_k^L \bar{\rho}_k w_{a,h}(x)$. We note that $\|\Phi\|_{L^\infty} \leq C(1+\ell) L^{1+\ell}$. Moreover just as in the proof of Lemmas 8.4-8.5, we can show that Φ satisfy a transport equation giving that

$$\begin{aligned} C_\Phi &= \left\| \int_{\Pi^d} \bar{K}_h(x-y) \Phi(t,x,y) dy \right\|_{W_t^{1,1} W_x^{-1,1}} \\ &\quad + \left\| \int_{\Pi^d} \bar{K}_h(x-y) \Phi(t,x,y) dx \right\|_{W_t^{1,1} W_x^{-1,1}} \leq C(1+\ell) L^{1+\ell}. \end{aligned}$$

By Lemma 8.3, we obtain that for some $\theta_2 > 0$

$$I_0^L \leq C(1+\ell) L^{1+\ell} \int_{h_0}^1 h^{\theta_2} \frac{dh}{h} \leq C(1+\ell) L^{1+\ell}. \quad (8.20)$$

By optimizing in L , this lets us conclude that again for some $0 < \theta < 1$ and provided that $p > \gamma + 1 + \ell$

$$I_0 \leq C(1+\ell) |\log h_0|^\theta. \quad (8.21)$$

1-2) The term I^D . This term has the right sign as

$$\begin{aligned} &\int_{h_0}^1 \int_0^t \int_{\Pi^{2d}} \frac{\bar{K}_h(x-y)}{h} (\rho_k^\gamma(x) - \rho_k^\gamma(y)) \chi'_a(\delta\rho_k) \bar{\rho}_k(w_{a,h}(x) + w_{a,h}(y)) \\ &\geq C \int_{h_0}^1 \int_0^t \int_{\Pi^{2d}} \frac{\bar{K}_h(x-y)}{h} \chi'_a(\delta\rho_k) \delta\rho_k \bar{\rho}_k^\gamma(w_{a,h}(x) + w_{a,h}(y)). \end{aligned}$$

We will actually give a more precise control on $I^D + II^D$ later on when the corresponding decomposition of $II = II_0 + II^D + II^R$ will be introduced.

I-3) The term I^R . The difficulty is thus in this quantity. From its definition, A_μ is a convolution operator. With a slight abuse of notation, we denote by A_μ as well its kernel or

$$A_\mu f = \int_{\Pi^d} A_\mu(x-y) f(y) dy,$$

and note that A_μ corresponds to an operator of 0 order, *i.e.* it for instance satisfies the property $\int_A A_\mu = 0$ for any annulus A centered at the origin, $|A_\mu(x)| \leq C|x|^{-d}$. Decompose

$$A_\mu = L_h + R_h, \quad \text{supp } L_h \subset \{|x| \leq \delta_h\},$$

such that both L_h and R_h remain bounded on any L^p space, $1 < p < \infty$, and moreover R_h is a regularization of A_μ that is $R_h = A_\mu \star N_{\delta_h}$ for some smooth kernel N_{δ_h} . The scale δ_h has to satisfy that

$$\delta_h \ll h, \quad \log \frac{h}{\delta_h} \ll |\log h|.$$

For simplicity we choose here $\delta_h = h/|\log h|$.

Contribution of the R_h part.

The first step is to decompose R_h into dyadic blocks in Fourier. Introduce a decomposition of identity Ψ_l as in sections 6 and 13 s.t. $1 = \sum_l \Psi_l$ and write

$$R_h = \sum_{l=|\log_2 h|}^{|\log_2 \delta_h|} \Psi_l \star R_h + \tilde{R}_h, \quad \tilde{R}_h = \sum_{l < |\log_2 h|} \Psi_l \star R_h = \tilde{N}_h \star N_{\delta_h} \star A_\mu. \quad (8.22)$$

Note that we of course require of the Ψ_l to satisfy all the assumptions specified in section 13 for the definition of Besov spaces. Define now $\bar{N}_h = \tilde{N}_h \star N_{\delta_h}$, this kernel \bar{N}_h therefore satisfies that for any $s > 0$

$$\|\bar{N}_h\|_{W^{s,1}} \leq C h^{-s}, \quad (8.23)$$

and moreover by the localization property of the Ψ_k , one has that for $s > 0$ and any $|\omega| \leq 1$

$$\int_{\Pi^d} |z|^s |\bar{N}_h(z) + \bar{N}_h(z + \omega r)| dr \leq C h^s. \quad (8.24)$$

Fix t for the moment and decompose accordingly

$$\begin{aligned} & \int_{h_0}^1 \int_{\Pi^d} \frac{\bar{K}_h(z)}{h} \|R_h \star \rho_k^\gamma(t, \cdot) - R_h \star \rho_k^\gamma(t, \cdot + z)\|_{L_x^q} \\ & \leq \int_{h_0}^1 \int_{\Pi^d} \frac{\bar{K}_h(z)}{h} \|\tilde{R}_h \star \rho_k^\gamma(t, \cdot) - \tilde{R}_h \star \rho_k^\gamma(t, \cdot + z)\|_{L_x^q} \\ & + \int_{h_0}^1 \sum_{l=|\log_2 h|}^{|\log_2 \delta_h|} \int_{\Pi^d} \frac{\bar{K}_h(z)}{h} \|\Psi_l \star R_h \star \rho_k^\gamma(t, \cdot) - \Psi_l \star R_h \star \rho_k^\gamma(t, \cdot + z)\|_{L_x^q}. \end{aligned}$$

By (8.23) and (8.24), the kernel \bar{N}_h satisfies the assumptions of Lemma 6.4 and thus with $U_k = A_\mu \star \rho_k^\gamma$, applying Lemma 6.4, for any $q > 1$

$$\begin{aligned} & \int_{h_0}^1 \int_{\Pi^d} \frac{\bar{K}_h(z)}{h} \|\tilde{R}_h \star \rho_k^\gamma(t, \cdot) - \tilde{R}_h \star \rho_k^\gamma(t, \cdot + z)\|_{L_x^q} \\ &= \int_{h_0}^1 \int_{\Pi^d} \frac{\bar{K}_h(z)}{h} \|\bar{N}_h \star U_k(t, \cdot) - \bar{N}_h \star U_k(\cdot + z)\|_{L_x^q} \\ &\leq C |\log h_0|^{1/2} \|U_k(t, \cdot)\|_{L_x^q}. \end{aligned}$$

Recalling that A_μ is continuous on every L^p space, one has that $\|U_k\|_{L_x^p} \leq C \|\rho_k^\gamma\|_{L_x^q}$ hence

$$\int_{h_0}^1 \int_{\Pi^d} \frac{\bar{K}_h(z)}{h} \|\tilde{R}_h \star \rho_k^\gamma(t, \cdot) - \tilde{R}_h \star \rho_k^\gamma(t, \cdot + z)\|_{L_x^q} \leq C |\log h_0|^{1/2} \|\rho_k(t, \cdot)\|_{L_x^{q\gamma}}^\gamma.$$

On the other hand simply by bounding

$$|\Psi_l \star R_h \star \rho_k^\gamma(x) - \Psi_l \star R_h \star \rho_k^\gamma(y)|^q \leq C |\Psi_l \star R_h \star \rho_k^\gamma(x)|^q + |\Psi_l \star R_h \star \rho_k^\gamma(y)|^q$$

we write

$$\begin{aligned} & \int_{h_0}^1 \sum_{l=|\log_2 h|}^{|\log_2 \delta_h|} \int_{\Pi^d} \frac{\bar{K}_h(z)}{h} \|\Psi_l \star R_h \star \rho_k^\gamma(t, \cdot) - \Psi_l \star R_h \star \rho_k^\gamma(t, \cdot + z)\|_{L_x^q} \\ &\leq C \sum_{l \leq |\log_2 h_0| + |\log_2 |\log_2 h_0||} \|\Psi_l \star R_h \star \rho_k^\gamma(t, \cdot)\|_{L_x^q} \int_{2^{-l}}^{l 2^{-l}} \frac{dh}{h}. \end{aligned}$$

recalling that $\delta_h = h/|\log_2 h|$. This leads to

$$\begin{aligned} & \int_{h_0}^1 \sum_{l=|\log_2 h|}^{|\log_2 \delta_h|} \int_{\Pi^d} \frac{\bar{K}_h(z)}{h} \|\Psi_l \star R_h \star \rho_k^\gamma(t, \cdot) - \Psi_l \star R_h \star \rho_k^\gamma(t, \cdot + z)\|_{L_x^q} \\ &\leq C \sum_{l \leq 2|\log_2 h_0|} \log l \|\Psi_l \star R_h \star \rho_k^\gamma(t, \cdot)\|_{L_x^q}, \end{aligned}$$

and can in turn be directly bounded by

$$\begin{aligned} & \leq C \log |\log h_0| \sum_{l \leq 2|\log_2 h_0|} \|\Psi_l \star R_h \star \rho_k^\gamma(t, \cdot)\|_{L_x^q} \\ & \leq C \log |\log h_0| |\log h_0|^{1/2} \|R_h \star \rho_k^\gamma(t, \cdot)\|_{L_x^q} \leq C |\log h_0|^\theta \|\rho_k(t, \cdot)\|_{L_x^{q\gamma}}^\gamma, \end{aligned}$$

with $0 < \theta < 1$ by Lemma 13.3. Combining with the previous estimate, we deduce that

$$\begin{aligned} & \int_{h_0}^1 \int_{\Pi^d} \frac{\bar{K}_h(z)}{h} \|R_h \star \rho_k^\gamma(t, \cdot) - R_h \star \rho_k^\gamma(t, \cdot + z)\|_{L_x^q} \\ & \leq C |\log h_0|^\theta \|\rho_k(t, \cdot)\|_{L_x^{q\gamma}}^\gamma. \end{aligned} \tag{8.25}$$

with $0 < \theta < 1$. And therefore since $\chi'_a(\xi) \leq (1 + \ell) |\xi|^\ell$, by Hölder's inequality with the relation $1/q + (1 + \ell)/(1 + \gamma + \ell) = 1$, that is $q = (1 + \ell + \gamma)/\gamma$

$$\begin{aligned} & \int_{h_0}^1 \int_0^t \int_{\Pi^{2d}} \frac{\overline{K}_h(x-y)}{h} (R_h \star \rho_k^\gamma(x) - R_h \star \rho_k^\gamma(y)) \chi'_a(\delta \rho_k) \bar{\rho}_k (w_{a,h}(x) + w_{a,h}(y)) \\ & \geq -C(1+\ell) \int_0^t \|\rho_k(t, \cdot)\|_{L_x^{1+\ell+\gamma}}^{1+\ell} \int_{h_0}^1 \int_{\Pi^d} \frac{\overline{K}_h(z)}{h} \|R_h \star \rho_k^\gamma(t, \cdot) - R_h \star \rho_k^\gamma(t, \cdot + z)\|_{L_x^q}. \end{aligned}$$

Finally by (8.25) there exists $0 < \theta < 1$ s.t.

$$\begin{aligned} & \int_{h_0}^1 \int_0^t \int_{\Pi^{2d}} \frac{\overline{K}_h(x-y)}{h} (R_h \star \rho_k^\gamma(x) - R_h \star \rho_k^\gamma(y)) \\ & \qquad \qquad \qquad \chi'_a(\delta \rho_k) \bar{\rho}_k (w_{a,h}(x) + w_{a,h}(y)) \\ & \geq -C(1+\ell) |\log h_0|^\theta \|\rho_k\|_{L^{\gamma+\ell+1}}^{\gamma+\ell+1}. \end{aligned} \tag{8.26}$$

Contribution of the L_h part. It remains the term with L_h where we symmetrize the position of the weight with respect to the convolution with L_h by

$$\begin{aligned} & \int_{h_0}^1 \int_0^t \int_{\Pi^{2d}} \frac{\overline{K}_h(x-y)}{h} (L_h \star \rho_k^\gamma(x) - L_h \star \rho_k^\gamma(y)) \chi'_a(\delta \rho_k) \bar{\rho}_k w_{a,h}(x) \\ & = I_h^L - \text{Diff}, \end{aligned}$$

with

$$\begin{aligned} I_h^L & = \int_{h_0}^1 \int_0^t \int_{\Pi^{3d}} \frac{\overline{K}_h(x-y)}{h} L_h(z) (\rho_k^\gamma(x-z) - \rho_k^\gamma(y-z)) \\ & \qquad \qquad \qquad \chi'_a(\delta \rho_k) \bar{\rho}_k w_h^{1-n}(x) w_h^n(x-z), \end{aligned}$$

for $n = 1 - 1/\gamma$. Recall that since $w_{a,h} = \overline{K}_h \star w_a$, $|w_{a,h}(x) - w_{a,h}(x-z)| \leq h^{-1} |z|$ while $|z| \sim \delta_h$ on the support of L_h . Thus using that $|\chi'_a(\xi)| \leq C |\xi|^\ell$ from (8.2) and that $|L_h(z)| \leq C |z|^{-d}$, one has

$$\begin{aligned} \text{Diff} & = \int_{h_0}^1 \int_0^t \int_{\Pi^{3d}} \frac{\overline{K}_h(x-y)}{h} L_h(z) (\rho_k^\gamma(x-z) - \rho_k^\gamma(y-z)) \\ & \qquad \qquad \qquad \chi'_a(\delta \rho_k) \bar{\rho}_k w_{a,h}^{1-n}(x) (w_{a,h}^n(x) - w_{a,h}^n(x-z)) \\ & \leq C(1+\ell) \int_{h_0}^1 \int_0^t \int_{\Pi^{3d}} \mathbb{I}_{|z| \leq \delta_h} \frac{\overline{K}_h(w)}{h |z|^d} |\rho_k^\gamma(x-z) + \rho_k^\gamma(x-z+w)| \bar{\rho}^{\ell+1} h^{-n} |z|^n \\ & \leq C(1+\ell) \int_{h_0}^1 \int_0^t \int_{\Pi^{3d}} \mathbb{I}_{|z| \leq \delta_h} \frac{\overline{K}_h(w)}{h^{1+n} |z|^{d-n}} (\rho_k^{\gamma+\ell+1}(x-z) + \rho_k^{\gamma+\ell+1}(x-z+w) \\ & \qquad \qquad \qquad + \rho_k^{\gamma+\ell+1}(x) + \rho_k^{\gamma+\ell+1}(x-w)). \end{aligned}$$

Using $|z| \leq \delta_h = \frac{h}{|\log_2 h|}$, we obtain on the other hand that

$$\begin{aligned} & \int_{h_0}^1 \frac{dh}{h^{1+n}} \int_{|z| \leq \delta_h} \frac{dz}{|z|^{d-n}} \leq C \int_{h_0}^1 \frac{\delta_h^n dh}{h^{1+n}} = C \int_{h_0}^1 \frac{dh}{h |\log h|^n} \\ & \leq C |\log h_0|^{1-n}. \end{aligned}$$

As $n = 1 - 1/\gamma$, this leads to

$$\text{Diff} \leq C(1 + \ell) \|\rho_k\|_{L^{\gamma+\ell+1}}^{\gamma+\ell+1} |\log h_0|^{1/\gamma}. \quad (8.27)$$

As for the first term by Hölder inequality, using again that $|\chi'_a(\xi)| \leq C\ell|\xi|^\ell$

$$I_h^L \leq C(1 + \ell) \int_{h_0}^1 \int_0^t \int_{\Pi^d} \left(\int_{\Pi^d} |\delta\rho_k|^{\ell+1} \bar{\rho}_k^{(\ell+1)/\ell} w_{a,h}^{(1-\theta)(\ell+1)/\ell}(x) dx \right)^{\ell/(\ell+1)} \\ \left(\int_{\Pi^d} (L_h \star ((\rho_k^\gamma(\cdot) - \rho_k^\gamma(\cdot + w)) w_{a,h}^\theta(\cdot)))_{|x}^{(\gamma+\ell+1)/\gamma} dx \right)^{\gamma/(\gamma+\ell+1)} \frac{\bar{K}_h(w)}{h} dw,$$

provided that l is chosen s.t.

$$\frac{\gamma}{\gamma + \ell + 1} + \frac{\ell}{\ell + 1} = 1, \quad \text{or} \quad \ell + 1 = \frac{\gamma}{\gamma - 1},$$

implying for instance that

$$\frac{\ell + 1}{\ell} = \gamma, \quad \frac{\gamma + \ell + 1}{\gamma} = \ell + 1 \dots$$

Given those algebraic relations and recalling that $L_h \star$ is continuous on every L^p for any $1 < p < \infty$

$$I_h^L \leq C(1 + \ell) \int_{h_0}^1 \int_0^t \int_{\Pi^d} \frac{\bar{K}_h(w)}{h} \left(\int_{\Pi^d} |\rho_k^\gamma(x) - \rho_k^\gamma(x + w)|^{\frac{\gamma+\ell+1}{\gamma}} w_{a,h} dx \right)^{\frac{\gamma}{\gamma+\ell+1}} \\ \left(\int_{\Pi^d} |\chi'_0(\delta_k)|^\gamma \bar{\rho}_k^\gamma w_{a,h} dx \right)^{\ell/(\ell+1)} dw.$$

Since, using the definition of ℓ ,

$$|\rho_k^\gamma(x) - \rho_k^\gamma(x + w)|^{(\gamma+\ell+1)/\gamma} \leq \gamma \bar{\rho}^\gamma |\delta\rho_k|^{(\gamma+\ell+1)/\gamma} = \gamma \bar{\rho}^\gamma |\delta\rho_k|^{\ell+1},$$

one has

$$I_h^L \leq C(1 + \ell) \gamma \int_{h_0}^1 \int_{\Pi^{2d}} \frac{\bar{K}_h(w)}{h} |\delta\rho_k|^{\ell+1} \bar{\rho}_k^\gamma w_{a,h}(x) dx dw, \quad (8.28)$$

which multiplied by $-a_\mu \nu_k/2$ will be bounded by $I^D + II^D$ provided $|a_\mu|$ is small enough.

II) The quantity II. Let us turn to II and decompose it as for I

$$II = II_0 + II^D + II^R,$$

where

$$II_0 = -\frac{1}{2} \int_{h_0}^1 \int_0^t \int_{\Pi^{2d}} \frac{\bar{K}_h(x-y)}{h} (\tilde{D}\rho u_k(x) + \tilde{D}\rho u_k(y)) \\ (\chi'_a(\delta\rho_k)\delta\rho_k - 2\chi_a(\delta\rho_k))(w_{a,h}(x) + w_{a,h}(y)),$$

while

$$II^D = -\frac{\nu_k}{2} \int_{h_0}^1 \int_0^t \int_{\Pi^{2d}} \frac{\bar{K}_h(x-y)}{h} (P_k(\rho_k(x)) + P_k(\rho_k(y))) \\ (\chi'_a(\delta\rho_k)\delta\rho_k - 2\chi_a(\delta\rho_k)) (w_{a,h}(x) + w_{a,h}(y)),$$

and

$$II^R = -\frac{a_\mu \nu_k}{2} \int_{h_0}^1 \int_0^t \int_{\Pi^{2d}} \frac{\bar{K}_h(x-y)}{h} (A_\mu P_k(\rho_k(x)) + A_\mu P_k(\rho_k(y))) \\ (\chi'_a(\delta\rho_k)\delta\rho_k - 2\chi_a(\delta\rho_k)) (w_{a,h}(x) + w_{a,h}(y)).$$

II-1) Term II_0 . For the term II_0 , using Lemma 8.3 in a manner identical to I_0

$$II_0 \leq - \int_{h_0}^1 \int_0^t \int_{\Pi^{2d}} \frac{\bar{K}_h(x-y)}{h} \bar{K}_h \star \tilde{D}\rho u_k(x) (\chi'_a(\delta\rho_k)\delta\rho_k - 2\chi_a(\delta\rho_k)) w_{a,h}(x) \\ + C(1+\ell) \|\rho_k\|_{L^{\gamma+\ell+1}}^{1+\ell} |\log h_0|^\theta,$$

for some $0 < \theta < 1$. Using formula (5.10) or (8.18), one has that

$$\operatorname{div} u_k - a_\mu A_\mu P_k(\rho_k) \geq \tilde{D}\rho u_k,$$

and hence since $-\chi'_a \xi + 2\chi_a \geq -C(1+\ell)\chi_a$

$$II_0 \leq C\ell \|\rho_k\|_{L^{\gamma+\ell+1}}^{1+\ell} |\log h_0|^\theta \\ + C(1+\ell) \int_{h_0}^1 \int_0^t \int_{\Pi^{2d}} \frac{\bar{K}_h(x-y)}{h} (\bar{K}_h \star (|\operatorname{div} u_k| + a_\mu |A_\mu P_k(\rho_k)|)) \chi_a(\delta\rho_k) w_{a,h}(x), \quad (8.29)$$

and the first integral will be bounded by $\Pi_a/2$ for λ large enough.

II-2) Term II^D . The term II^D is controlled by I^D : For $a \geq b$, by (8.2)

$$(a^\gamma + b^\gamma) (-\chi'_a(a-b)(a-b) + 2\chi_a(a-b)) \geq -(a^\gamma - b^\gamma) \frac{\ell-1}{\ell} \chi'_a(a-b)(a+b).$$

Therefore

$$I^D + II^D \leq -C\gamma \frac{\nu_k}{2} \int_0^t \int_{\Pi^{2d}} \frac{\bar{K}_h(x-y)}{h} |\delta\rho_k|^{\ell+1} \bar{\rho}_k^{-\gamma} (w_h(x) + w_h(y)), \quad (8.30)$$

for some C independent of ℓ and γ .

II-3) Term II^R . The control on the last term, II^R , requires the use of the penalization Π_a

$$II^R + \frac{1}{2}\Pi_a \leq -a_\mu \nu_k \int_{h_0}^1 \int_0^t \int_{\Pi^{2d}} \frac{\bar{K}_h(x-y)}{h} (A_\mu \rho_k^\gamma(x) - A_\mu \bar{K}_h \star \rho_k^\gamma(x)) \\ (\chi'_a(\delta\rho_k)\delta\rho_k - 2\chi_a(\delta\rho_k)) w_{a,h}(x).$$

We use the same decomposition of $A_\mu = L_h + R_h$ as for I^R .

Contribution of the R_h part.

Note that, as $\chi_a(\xi) \leq C |\xi|^{1+\ell}$ and $|\chi'_a| \leq C(1+\ell) |\xi|^\ell$, for $q = (1+\ell+\gamma)/\gamma$ or $1/q + (1+\ell)/(1+\ell+\gamma) = 1$

$$\begin{aligned} & -a_\mu \nu_k \int_{h_0}^1 \int_0^t \int_{\Pi^{2d}} \frac{\overline{K}_h(x-y)}{h} (R_h \star \rho_k^\gamma(x) - \overline{K}_h \star R_h \star \rho_k^\gamma(x)) \\ & \quad (\chi'_a(\delta\rho_k)\delta\rho_k - 2\chi_a(\delta\rho_k)) w_{a,h}(x) \\ & \leq C(1+\ell) \int_0^t \|\rho_k(t, \cdot)\|_{L_x^{1+\ell+\gamma}}^{1+\ell} \int_{h_0}^1 \int_{\Pi^d} \frac{\overline{K}_h(z)}{h} \|R_h \star \rho_k^\gamma(t, \cdot) - R_h \star \rho_k^\gamma(t, \cdot + z)\|_{L_x^q}. \end{aligned}$$

Now by estimate (8.25), we have that

$$\begin{aligned} & -a_\mu \nu_k \int_{h_0}^1 \int_0^t \int_{\Pi^{2d}} \frac{\overline{K}_h(x-y)}{h} (R_h \star \rho_k^\gamma(x) - \overline{K}_h \star R_h \star \rho_k^\gamma(x)) \\ & \quad (\chi'_a(\delta\rho_k)\delta\rho_k - 2\chi_a(\delta\rho_k)) w_{a,h}(x) \\ & \leq C(1+\ell) |\log h_0|^{3/4} \int_0^t \|\rho_k(t, \cdot)\|_{L_x^{1+\ell+\gamma}}^{1+\ell} \|\rho_k(t, \cdot)\|_{L_x^\gamma}^\gamma \\ & \leq C(1+\ell) |\log h_0|^{3/4} \|\rho_k\|_{L_{t,x}^{1+\ell+\gamma}}^{1+\ell+\gamma}. \end{aligned} \tag{8.31}$$

Contribution of the L_h part. Similarly as for I^R , we symmetrize the weights leading to the following decomposition

$$\begin{aligned} & -a_\mu \nu_k \int_{h_0}^1 \int_0^t \int_{\Pi^{2d}} \frac{\overline{K}_h(x-y)}{h} (L_h \star \rho_k^\gamma(x) - \overline{K}_h \star L_h \star \rho_k^\gamma(x)) \\ & \quad (\chi'_a(\delta\rho_k)\delta\rho_k - 2\chi_a(\delta\rho_k)) w_{a,h}(x) \\ & = II_h^L + \text{Diff}_2, \end{aligned}$$

where

$$\begin{aligned} II_h^L & = a_\mu \nu_k \int_{h_0}^1 \int_0^t \int_{\Pi^{2d}} \frac{\overline{K}_h(x-y)}{h} L_h \star (w_{a,h}^n(x) (\rho_k^\gamma(x) - \overline{K}_h \star \rho_k^\gamma(x))) \\ & \quad (-\chi'_a(\delta\rho_k)\delta\rho_k + 2\chi_a(\delta\rho_k)) w_{a,h}^{1-n}(x), \end{aligned}$$

with still $n = 1 - 1/\gamma$. The term Diff_2 is controlled as the term Diff in I^R using the regularity of $w_{a,h}$ and yielding

$$\text{Diff}_2 \leq C(1+\ell) |\log h_0|^\gamma \|\rho_k\|_{L_{\gamma+1+\ell}^{\gamma+1+\ell}}. \tag{8.32}$$

We handle II_h^L with Hölder estimates quite similar to the ones used for the

term I_h^L , recalling that $L_h \star$ is bounded on any L^q space for $1 < q < \infty$

$$\begin{aligned}
II_h^L &\leq C a_\mu \nu_k (1 + \ell) \int_{h_0}^1 \int_0^t \int_{\Pi^d} \frac{\overline{K}_h(w)}{h} \|w_{a,h}^\theta (\rho_k^\gamma(\cdot) - \overline{K}_h \star \rho_k^\gamma(\cdot))\|_{L^{\ell+1}} \\
&\quad \| |\delta \rho_k|^{\ell+1} w_{a,h}^{1-\theta}(x) \|_{L^{(\ell+1)/\ell}} dw \\
&\leq C a_\mu \nu_k (1 + \ell) \int_{h_0}^1 \int_0^t \int_{\Pi^d} \frac{\overline{K}_h(w)}{h} \int_{\Pi^d} w_{a,h}(x) |\rho_k^\gamma(x) - \overline{K}_h \star \rho_k^\gamma(x)|^{\ell+1} dx dw \\
&+ C a_\mu \nu_k (1 + \ell) \int_{h_0}^1 \int_0^t \int_{\Pi^d} \frac{\overline{K}_h(w)}{h} \int_{\Pi^d} w_{a,h}(x) |\delta \rho_k|^{(\ell+1)^2/\ell} dx dw.
\end{aligned}$$

One has immediately that

$$\begin{aligned}
&\int_{h_0}^1 \int_0^t \int_{\Pi^{2d}} \frac{\overline{K}_h(w)}{h} w_{a,h}(x) |\delta \rho_k|^{(\ell+1)^2/\ell} dx dw \\
&\leq \int_{h_0}^1 \int_0^t \int_{\Pi^{2d}} \frac{\overline{K}_h(w)}{h} w_{a,h}(x) |\delta \rho_k|^{\ell+1} \bar{\rho}_k^\gamma dx dw
\end{aligned}$$

as $|\delta \rho_k| \leq \bar{\rho}_k$ and again $(\ell + 1)/\ell = \gamma$.

Finally as $(\ell + 1)(\gamma - 1) = \gamma$

$$\begin{aligned}
&\int_{h_0}^1 \int_0^t \int_{\Pi^{2d}} \frac{\overline{K}_h(w)}{h} w_{a,h}(x) |\rho_k^\gamma(x) - \overline{K}_h \star \rho_k^\gamma(x)|^{\ell+1} dx dw \\
&\leq \gamma \int_{h_0}^1 \int_0^t \int_{\Pi^{3d}} \frac{\overline{K}_h(w)}{h} \overline{K}_h(z) w_{a,h}(x) |\rho_k(x) - \rho_k(x+z)|^{\ell+1} (\rho_k(x) + \rho_k(x+z))^\gamma \\
&\leq \gamma \int_{h_0}^1 \int_0^t \int_{\Pi^{2d}} \frac{\overline{K}_h(z)}{h} w_{a,h}(x) |\rho_k(x) - \rho_k(x+z)|^{1+\ell} (\rho_k(x) + \rho_k(x+z))^\gamma,
\end{aligned}$$

as $\overline{K}_h(w)$ is the only term depending on w and is of integral 1. Therefore

$$II_h^L \leq C a_\mu \nu_k \gamma (1 + \ell) \int_{h_0}^1 \int_0^t \int_{\Pi^{2d}} \frac{\overline{K}_h(z)}{h} w_{a,h}(x) |\delta \rho_k|^{1+\ell} \bar{\rho}_k^\gamma. \quad (8.33)$$

To conclude, we sum all the contributions, more precisely (8.21), (8.26), (8.27), (8.28), (8.29), (8.30), (8.31), (8.32), and (8.33), to find for some $0 < \theta < 1$ and provided $p > \gamma + 1 + \ell$

$$\begin{aligned}
&\int_{h_0}^1 \int_{\Pi^{2d}} \frac{\overline{K}_h(x-y)}{h} (w_{a,h}(x) + w_{a,h}(y)) \chi_a(\delta \rho_k)(t) \\
&\leq \int_{h_0}^1 \int_{\Pi^{2d}} \frac{\overline{K}_h(x-y)}{h} (w_{a,h}(x) + w_{a,h}(y)) \chi_a(\delta \rho_k)|_{t=0} + C (1 + \ell) |\log h_0|^\theta \\
&\quad + C \left(a_\mu \nu_k (1 + \ell) - C \frac{\nu_k}{2} \right) \gamma \int_{h_0}^1 \int_0^t \int_{\Pi^{2d}} \frac{\overline{K}_h(z)}{h} w_{a,h}(x) |\delta \rho_k|^{1+\ell} \bar{\rho}_k^\gamma.
\end{aligned}$$

This finishes the proof of the lemma: As $1 + \ell = \gamma/(\gamma - 1)$ is bounded (we recall that $\gamma > d/2$), if $a_\mu \leq C_*$ for $C_* > 0$ well chosen, the last term in the r.h.s. is non positive. \square

8.6 Conclusion of the proof of Theorem 5.4

We combine Lemmas 8.2 and 8.6 to get the following estimate

$$\begin{aligned} & \int_{h_0}^1 \int_{\Pi^{2d}} \bar{K}_h(x-y)(w_{a,h}(x) + w_{a,h}(y))\chi_a(\delta\rho_k)(t) dx dy \frac{dh}{h} \\ & \leq C |\log h_0|^\theta + \text{initial value}, \end{aligned}$$

with $0 < \theta < 1$. We now follow the same steps as in the proof of Theorem 5.1 with the weight $w_0(t, x) + w_0(t, y)$. We define $\omega_\eta = \{w_{a,h}(t, x) \leq \eta\}$ and note that

$$\begin{aligned} \int_{\Pi^{2d}} \mathcal{K}_{h_0}(x-y) \chi_a(\delta\rho_k) &= \int_{h_0}^1 \int_{\Pi^{2d}} \bar{K}_h(x-y) \chi_a(\delta\rho_k) \frac{dh}{h} \\ &= \int_{h_0}^1 \int_{x \in \omega_\eta^c \text{ or } y \in \omega_\eta^c} \bar{K}_h(x-y) \chi_a(\delta\rho_k) \frac{dh}{h} \\ &+ \int_{h_0}^1 \int_{x \in \omega_\eta \text{ and } y \in \omega_\eta} \bar{K}_h(x-y) \chi_a(\delta\rho_k) \frac{dh}{h}. \end{aligned}$$

Now

$$\begin{aligned} & \int_{h_0}^1 \int_{x \in \omega_\eta^c \text{ or } y \in \omega_\eta^c} \bar{K}_h(x-y) \chi_a(\delta\rho_k) \frac{dh}{h} \\ & \leq \frac{1}{\eta} \int_{h_0}^1 \int_{\Pi^{2d}} \bar{K}_h(x-y)(w_{a,h}(x) + w_{a,h}(y))\chi_a(\delta\rho_k) \frac{dh}{h} \leq C |\log h_0|^\theta, \end{aligned}$$

while by point iii) in Prop. 7.2 and using the L^p bound on ρ , for some $\theta > 0$

$$\begin{aligned} \int_{h_0}^1 \int_{x \in \omega_\eta \text{ and } y \in \omega_\eta} \bar{K}_h(x-y) \chi_a(\delta\rho_k) \frac{dh}{h} &\leq 2 \int_{h_0}^1 \int_{\Pi^{2d}} \bar{K}_h(x-y) \rho_k^{1+\ell} \mathbb{I}_{\bar{K}_h \star w_a \leq \eta} \frac{dh}{h} \\ &\leq \frac{C |\log h_0|}{|\log \eta|^\theta}. \end{aligned}$$

Hence we have

$$\begin{aligned} \int_{\Pi^{2d}} \mathcal{K}_{h_0}(x-y) \chi_a(\delta\rho_k)(t) &\leq C |\log h_0| \left(\frac{|\log h_0|^{\theta-1}}{\eta} + \frac{1}{|\log \eta|^\theta} \right) \\ &\leq \frac{C \|\mathcal{K}_{h_0}\|_{L^1}}{|\log |\log h_0||^\theta}, \end{aligned}$$

by optimizing in η and recalling that $\|\mathcal{K}_{h_0}\|_{L^1} = |\log h_0|$.

Using Prop. 4.1 together with Lemma 6.6, one concludes that ρ_k is locally compact in t, x . Thus we conclude the proof of Theorem 5.4.

9 Proof of Theorems 3.1 and 3.2: Approximate sequences

In this section, we construct approximate systems that allow to use Theorems 5.1 and 5.4 to prove Theorems 3.1 and 3.2.

9.1 From regularized systems with added viscosity to no viscosity

Our starting point for global existence is the following regularized system

$$\begin{cases} \partial_t \rho_k + \operatorname{div}(\rho_k u_k) = \alpha_k \Delta \rho_k, \\ \partial_t(\rho_k u_k) + \operatorname{div}(\rho_k u_k \otimes u_k) - \mu \Delta u_k - (\lambda + \mu) \nabla \operatorname{div} u_k - \mathcal{A}_\varepsilon \star u_k \\ \quad + \nabla P_\varepsilon(\rho_k) + \alpha_k \nabla \rho_k \cdot \nabla u_k = \rho_k f, \end{cases} \quad (9.1)$$

with the fixed initial data

$$\rho_k|_{t=0} = \rho^0, \quad \rho_k u_k|_{t=0} = \rho^0 u^0. \quad (9.2)$$

The pressure P_ε satisfies the bound (3.2) with $\gamma > 3d/(d+2)$ uniformly in ε , that is

$$C^{-1} \rho^\gamma - C \leq P_\varepsilon(\rho) \leq C \rho^\gamma + C,$$

implying that $e(\rho) \geq C^{-1} \rho^{\gamma-1} - C$. In addition we ask that P_ε satisfies the quasi-monotone property (5.8) but possibly depending on ε , *i.e.* there exists $\rho_{0,\varepsilon}$ s.t.

$$(P_\varepsilon(s)/s)' \geq 0 \text{ for all } s \geq \rho_{0,\varepsilon}.$$

Similarly \mathcal{A}_ε is assumed to be a given smooth function, possibly depending on ε but such that the operator defined by

$$\mathcal{D}_\varepsilon f = -\mu \Delta f - (\lambda + \mu) \nabla \operatorname{div} f - \mathcal{A}_\varepsilon \star f$$

satisfies (2.2) and (2.3) uniformly in ε .

As usual the equation of continuity is regularized by means of an artificial viscosity term and the momentum balance is replaced by a Faedo-Galerkin approximation to eventually reduce the problem on X_n , a finite-dimensional vector space of functions.

This approximate system can then be solved by a standard procedure: The velocity u of the approximate momentum equation is looked as a fixed point of a suitable integral operator. Then given u , the approximate continuity equation is solved directly by means of the standard theory of linear parabolic equations. This methodology concerning the compressible Navier–Stokes equations is well explained and described in the reference books [34], [35], [53]. We omit the rest of this classical (but tedious) procedure and we assume that we have well posed and smooth solutions to (9.1).

We now use the classical energy estimates detailed in subsection 2.1. Note that they remain the same in spite of the added viscosity in the continuity equation because in particular of the added term $\alpha_k \nabla \rho_k \cdot \nabla u_k$ in the momentum equation. Let us summarize the *a priori* estimates that are obtained

$$\sup_{k,\varepsilon} \sup_t \int_{\Pi^d} (\rho_k |u_k|^2 + \rho_k^\gamma) dx < \infty, \quad \sup_{k,\varepsilon} \int_0^T \int_{\Pi^d} |\nabla u_k|^2 dx dt < \infty,$$

from (2.6); while (2.9) provides

$$\sup_{k,\varepsilon} \int_0^T \int_{\Pi^d} \rho_k^p(t,x) dx dt < \infty,$$

with $p = \gamma + 2\gamma/d - 1$ which means $p > 2$ as $\gamma > 3d/(d+2)$. Therefore we obtain (5.4) and (5.5) uniformly in ε .

From those bounds it is straightforward to deduce that $\rho_k u_k$ and $\rho_k |u_k|^2$ belong to $L^q_{t,x}$ for some $q > 1$, uniformly in k and ε . Therefore using the continuity equation in (9.1), we deduce (5.7). Using the momentum equation, we obtain (5.6) but this bound (and only this bound) is not independent of ε because of \mathcal{A}_ε .

Finally taking the divergence of the momentum equation and inverting Δ

$$\begin{aligned} (\lambda + 2\mu) \operatorname{div} u_k &= P_\varepsilon(\rho_k) + \Delta^{-1} \operatorname{div} (\partial_t(\rho_k u_k) + \operatorname{div}(\rho_k u_k \otimes u_k)) \\ &\quad - \Delta^{-1} \operatorname{div}(\rho_k f + \mathcal{A}_\varepsilon \star u_k) + \alpha_k \Delta^{-1} \operatorname{div}(\nabla \rho_k \cdot \nabla u_k), \end{aligned}$$

which is exactly (5.2) with $\mu_k = \lambda + 2\mu$ satisfying (5.3) and compact, while

$$F_k = -\Delta^{-1} \operatorname{div}(\rho_k f + \mathcal{A}_\varepsilon \star u_k) + \alpha_k \Delta^{-1} \operatorname{div}(\nabla \rho_k \cdot \nabla u_k).$$

The first term in F_k is also compact in L^1 since \mathcal{A}_ε is smooth for a fixed ε . On the other hand

$$\alpha_k \Delta^{-1} \operatorname{div}(\nabla \rho_k \cdot \nabla u_k)$$

converges to 0 in L^1 since $\sqrt{\alpha_k} \nabla \rho_k$ is uniformly bounded in L^2 and ∇u_k is as well in L^2 . Therefore F_k is compact in L^1 . We may hence apply point i) of Theorem 5.1 to obtain the compactness of ρ_k in L^1 . Then extracting converging subsequences, we can pass to the limit in every term (see subsection 2.2 for instance) and obtain the existence of weak solutions to

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mu \Delta u - (\lambda + \mu) \nabla \operatorname{div} u - \mathcal{A}_\varepsilon \star u + \nabla P_\varepsilon(\rho) = \rho f. \end{cases} \quad (9.3)$$

9.2 General pressure laws: End of proof (Theorem 3.1)

Consider now a nonmonotone pressure P satisfying (3.2) and (3.3). Let us fix $c_{0,\varepsilon} = 1/\varepsilon$ and define

$$P_\varepsilon(\rho) = P(\rho) \quad \text{if } \rho \leq c_{0,\varepsilon}, \quad P_\varepsilon(\rho) = P(c_{0,\varepsilon}) + C(\rho - c_{0,\varepsilon})^\gamma, \quad \text{if } \rho \geq c_{0,\varepsilon}.$$

Note that P_ε is Lipschitz, converges uniformly to P on any compact interval. Due to (3.2) there exists $\rho_{0,\varepsilon}$ with $\rho_{0,\varepsilon} \rightarrow +\infty$ as $\varepsilon \rightarrow +\infty$ such that for $\rho \geq \rho_{0,\varepsilon}$,

$$(P_\varepsilon(s)/s)' = (P'_\varepsilon(s)s - P(s))/s^2 \geq (C(\gamma - 1)(\rho - c_{0,\varepsilon})^\gamma - P(c_{0,\varepsilon}))/s^2 \geq 0.$$

The approximate pressure P_ε still satisfies (3.2) with γ and due to the previous inequality it satisfies (5.8) for $\rho \geq \rho_{0,\varepsilon}$ and (5.9) in the following sense: For all $s \geq 0$

$$|P'_\varepsilon(s)| \leq \bar{P}s^{\tilde{\gamma}-1}\mathbb{1}_{s \leq c_{0,\varepsilon}} + C(\gamma - 1)s^{\gamma-1}\mathbb{1}_{s \geq c_{0,\varepsilon}}.$$

As a consequence, we have existence of weak solutions (ρ, u) to (9.1) for this P_ε (assuming $\mathcal{A}_\varepsilon = 0$) for any $\varepsilon > 0$. Consider a sequence $\varepsilon_k \rightarrow 0$ and the corresponding sequence (ρ_k, u_k) of weak solutions to (9.1).

Because the previous *a priori* estimates were uniform in ε , (including (5.6) since $\mathcal{A}_\varepsilon = 0$), then the sequence (ρ_k, u_k) satisfies all the bounds (5.4), (5.5), (5.6), (5.7) and (5.9).

Moreover the representation (5.2) still holds with $\mu_k = 2\mu + \lambda$, compact in L^1 and satisfying (5.3). Finally the exponent p in (5.4) can be chosen up to $\gamma + 2\gamma/d - 1$. Since $\gamma > 3d/(d+2)$ then $p > 2$ and since $\gamma > (\tilde{\gamma} + 1)d/(d+2)$ then one has $p > \tilde{\gamma}$ as well.

Therefore all the assumptions of point ii) of Theorem 5.1 are satisfied and one has the compactness of ρ_k . Extracting converging subsequences of ρ_k and u_k , one passes to the limit in every term. Note in particular that $P_{\varepsilon_k}(\rho_k)$ converges in L^1 to $P(\rho)$ by the compactness of ρ_k , the uniform convergence of P_{ε_k} to P on compact intervals and by truncating $P_{\varepsilon_k}(\rho_k)$ for large values of ρ_k since the exponent p in (5.4) is strictly larger than γ .

This proves the global existence in Theorem 3.1. The regularity of ρ follows from Theorem 5.2, which concludes the proof of Theorem 3.1.

9.3 Anisotropic viscosities: End of proof (Theorem 3.2)

For simplicity, we take $f = 0$. Consider now a “quasi-monotone” pressure P satisfying (3.7). Observe that P then automatically satisfies (3.2) since $P(0) = 0$. To satisfy (5.8), we have to modify P on an interval $(c_{0,\varepsilon}, +\infty)$ with $c_{0,\varepsilon} \rightarrow +\infty$ when $\varepsilon \rightarrow +\infty$. More precisely we consider P_ε as defined in the previous subsection

$$P_\varepsilon(\rho) = P(\rho) \quad \text{if } \rho \leq c_{0,\varepsilon}, \quad P_\varepsilon(\rho) = P(c_{0,\varepsilon}) + C(\rho - c_{0,\varepsilon})^\gamma, \quad \text{if } \rho \geq c_{0,\varepsilon}.$$

Now given any smooth kernel, for instance \bar{K} , we define

$$\mathcal{A}_\varepsilon \star u = \text{div}(\delta A(t) \nabla \bar{K}_\varepsilon \star u).$$

Because of the smallness assumption on $\delta A(t)$, the operator \mathcal{D}_ε satisfies (2.2) and (2.3) uniformly in ε . Therefore we have existence of global weak solutions to (9.3) with this choice of P_ε and \mathcal{A}_ε . We again consider a sequence of such solutions (ρ_k, u_k) corresponding to some sequence $\varepsilon_k \rightarrow 0$. Because the estimates are uniform

in ε for (9.3), we have again that this sequence satisfies the bounds (5.4), (5.5), (5.7). We now assume that

$$\gamma > \frac{d}{2} \left[\left(1 + \frac{1}{d} \right) + \sqrt{1 + \frac{1}{d^2}} \right],$$

implying that p in (5.4) is strictly larger than $\gamma^2/(\gamma - 1)$. Moreover observe that

$$\|\mathcal{A}_{\varepsilon_k} u_k\|_{L^2_t H_x^{-1}} \leq C \|\nabla u_k\|_{L^2_{t,x}},$$

such that (5.6) is also satisfied.

For simplicity, we assume that δA has a vanishing trace (otherwise just add the corresponding trace to μ). Denote $a_\mu = 2 \|\delta A\|_{L^\infty} / (2\mu + \lambda)$ and the operator E_k

$$E_k u = -\operatorname{div} \left(\frac{\delta A(t)}{2 \|\delta A\|_{L^\infty}} \nabla \bar{K}_\varepsilon \star u \right) = - \sum_{ij} \frac{\delta A_{ij}(t)}{2 \|\delta A\|_{L^\infty}} \partial_{ij} \bar{K}_{\varepsilon_k} \star u.$$

For a_μ small enough, $\Delta - a_\mu E_k$ is a uniform elliptic operator so that $(\Delta - a_\mu E_k)^{-1} \Delta$ is bounded on every L^p space, uniformly in k . For the same reason, $A_\mu = (\Delta - a_\mu E_k)^{-1} E_k$ is bounded on every L^p space with norm less than 1 and can be represented by a convolution with a singular integral.

Taking the divergence of the momentum equation in (9.1), one has

$$(2\mu + \lambda) (\Delta \operatorname{div} u_k - a_\mu E_k \operatorname{div} u) = \Delta P(\rho_k) + \operatorname{div} (\partial_t(\rho_k u_k) + \operatorname{div}(\rho_k u_k \otimes u_k)).$$

Just write $\Delta P_\varepsilon(\rho_k) = (\Delta - a_\mu E_k) P_\varepsilon(\rho_k) + a_\mu E_k P_\varepsilon$, take the inverse of $\Delta - a_\mu E_k$ to obtain

$$\begin{aligned} (2\mu + \lambda) \operatorname{div} u &= P(\rho_k) + a_\mu (\Delta - a_\mu E_k)^{-1} E_k P(\rho_k) \\ &\quad + (\Delta - a_\mu E_k)^{-1} \operatorname{div} (\partial_t(\rho_k u_k) + \operatorname{div}(\rho_k u_k \otimes u_k)), \end{aligned}$$

which is exactly (5.10) with $\nu_k = (2\mu + \lambda)^{-1}$. As a consequence, if $a_\mu \leq C_*$, which is implied by $\|\delta A\|_{L^\infty}$ small enough, then Theorem 5.4 applies and ρ_k is compact. Passing to the limit again in every term proves Theorem 3.2. Note that $P_{\varepsilon_k}(\rho_k)$ converges in L^1 to $P(\rho)$ for the same reason than in the previous subsection.

The case with $D(u)$ instead of ∇u . Let us finish this proof by remarking on the different structure in the case with symmetric stress tensor $\operatorname{div}(A D u)$. In that case, one cannot find $\operatorname{div} u_k$ by taking the divergence of the momentum equation but instead we have to consider the whole momentum equation. Let us write it as

$$\mathcal{E} u_k = \nabla P(\rho_k) + \partial_t(\rho_k u_k) + \operatorname{div}(\rho_k u_k \otimes u_k),$$

with \mathcal{E} the elliptic vector-valued operator

$$\mathcal{E} u = \mu \Delta u + (\mu + \lambda) \nabla \operatorname{div} u + \operatorname{div}(\delta A D u).$$

The operator \mathcal{E} is invertible for δA small enough as one can readily check in Fourier for instance where $-\hat{\mathcal{E}}$ becomes a perturbation of $\mu |\xi|^2 I + (\mu + \lambda) \xi \otimes \xi$. Its inverse has

most of the usual properties of inverses of scalar elliptic operator (with the exception of the maximum principle for instance). Therefore, one may still write

$$\operatorname{div} u_k = \operatorname{div} \mathcal{E}^{-1} \nabla P(\rho_k) + \operatorname{div} \mathcal{E}^{-1} (\partial_t(\rho_k u_k) + \operatorname{div}(\rho_k u_k \otimes u_k)),$$

leading to the variant (5.11) of the simpler formula (5.10).

10 Extension to the Navier–Stokes–Fourier system

We present here a direct application of our compactness results to compressible Navier–Stokes systems with temperature. Those systems are considered to be more physically realistic. They also offer many examples of non monotone pressure laws thus being especially relevant to our approach without any thermodynamic stability assumption.

Our goal is not to give the most optimal result as it is likely that this would require estimates specifically tailored to the case under consideration, and for the same reason would require to specifically tailor the compactness Theorem 5.1. But we believe that the result shown here can provide a good general basis for future works.

10.1 Model and estimates

The heat-conducting compressible Navier–Stokes equations read

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div} \mathcal{S} + \nabla P(\rho, \vartheta) = 0, \\ \partial_t(\rho E) + \operatorname{div}(\rho u E) + \operatorname{div}(P(\rho, \vartheta)u) = \operatorname{div}(\mathcal{S}u) + \operatorname{div}(\kappa(\vartheta)\nabla\vartheta), \end{cases} \quad (10.1)$$

where $E = |u|^2/2 + e$ is the total energy with $P = P(\rho, \vartheta)$ and $e = e(\rho, \vartheta)$ respectively stand for the pressure and the (specific) internal energy. As usual the system is supplemented with initial conditions.

We denote the stress tensor $\mathcal{D} = \operatorname{div} \mathcal{S}$ and as in the barotropic case, we always assume that for some function $\bar{\mu}$

$$\int \nabla u : \mathcal{S} \, dx \sim \int \bar{\mu} |\nabla u|^2 \, dx.$$

Instead of the temperature ϑ , one could also choose as a third unknown the internal energy (or the entropy as defined below). Some formulas are easier when using the set of variables (ρ, ϑ) and some with the set (ρ, e) . For this reason and simplicity we here follow the classical notations of Thermodynamics for partial derivatives, denoting for instance $\partial_\vartheta f|_\rho$ if f is the function $f(\rho, \vartheta)$ and $\partial_e f|_\rho$ if instead one considers the composition $f(\rho, \vartheta(\rho, e))$.

The total mass as well as the total energy of the system are constants of motion namely

$$\int \rho(t, \cdot) \, dx = \int \rho_0 \, dx$$

and

$$\int \left(\frac{1}{2} \rho |u|^2 + \rho e(\rho, \vartheta) \right)(t, \cdot) dx = \int \left(\frac{1}{2} \rho_0 |u_0|^2 + \rho_0 e(\rho_0, \vartheta_0) \right) dx.$$

From the mathematical viewpoint, the heat-conducting compressible Navier–Stokes system suffers the deficiency of strong *a priori* bounds. For instance in comparison with the previous barotropic Navier–Stokes, the energy bound does not yield any $L_t^2 H_x^1$ bound on the velocity.

In order to be consistent with the second principle of Thermodynamics which implies the existence of the entropy as a closed differential form in the energy balance, the following compatibility condition, called "Maxwell equation" between P and e has to be satisfied

$$P = \rho^2 \frac{\partial e}{\partial \rho} \Big|_{\vartheta} + \vartheta \frac{\partial P}{\partial \vartheta} \Big|_{\rho}.$$

The specific entropy $s = s(\rho, \vartheta)$ is defined up to an additive constant by

$$\frac{\partial s}{\partial \vartheta} \Big|_{\rho} = \frac{1}{\vartheta} \frac{\partial e}{\partial \vartheta} \Big|_{\rho} \quad \text{and} \quad \frac{\partial s}{\partial \rho} \Big|_{\vartheta} = -\frac{1}{\rho^2} \frac{\partial P}{\partial \vartheta} \Big|_{\rho}.$$

An other important assumption on the entropy function is made,

the entropy s is a concave function of (ρ^{-1}, e) .

This property ensures in particular the non negativity of the so-called C_v coefficient given by

$$C_v = \frac{\partial e}{\partial \vartheta} \Big|_{\rho} = -\frac{1}{\vartheta^2} \frac{\partial^2 s}{\partial e^2} \Big|_{\rho}^{-1}.$$

If (ρ, ϑ) are smooth and bounded from below away from zero and if the velocity field is smooth, then the total energy balance can be replaced by the thermal energy balance

$$C_v \rho (\partial_t \vartheta + u \cdot \nabla \vartheta) - \operatorname{div}(\kappa(\vartheta) \nabla \vartheta) = \mathcal{S} : \nabla u - \vartheta \frac{\partial P(\rho, \vartheta)}{\partial \vartheta} \operatorname{div} u.$$

Furthermore, dividing by ϑ , we arrive at the entropy equation

$$\partial_t(\rho s) + \operatorname{div}(\rho s u) - \operatorname{div}\left(\frac{\kappa(\vartheta) \nabla \vartheta}{\vartheta}\right) = \frac{1}{\vartheta} \left(\mathcal{S} : \nabla u + \frac{\kappa |\nabla \vartheta|^2}{\vartheta} \right). \quad (10.2)$$

10.2 The entropy estimate through thermodynamical stability

The first difficulty in the Navier–Stokes–Fourier system is to obtain a L^2 estimate on ∇u , since, contrary to the barotropic case, it does not follow from the dissipation of energy.

Instead a first approach is to use the entropy. Coupling the equation on the total energy and the equation on s , we get

$$\begin{aligned} & \int \left(\frac{1}{2} \rho |u|^2 + \rho [e(\rho, \vartheta) - \vartheta_* s] \right) (t, \cdot) dx + \int_0^t \int \frac{\vartheta_*}{\vartheta} \left(\mathcal{S} : \nabla u + \frac{\kappa |\nabla \vartheta|^2}{\vartheta} \right) \\ &= \int \left(\frac{1}{2} \rho_0 |u_0|^2 + \rho_0 [e(\rho_0, \vartheta_0) - \vartheta_* s_0] \right) dx, \end{aligned} \quad (10.3)$$

for any constant temperature ϑ_* . This is exactly the type of modified energy estimate that we are looking for as it provides a control on $\kappa(\vartheta) |\nabla \vartheta|^2 / \vartheta^2$ and with any reasonable choice of stress tensor, a control on $|\nabla u|^2 / \vartheta$.

However it is useless unless one can bound

$$H_{\vartheta_*}(\rho, \vartheta) = \rho [e(\rho, \vartheta) - \vartheta_* s].$$

Observe that (using the mass conservation)

$$\frac{d}{dt} \int H_{\vartheta_*}(\rho, \vartheta) = \frac{d}{dt} \int \left[H_{\vartheta_*}(\rho, \vartheta) - \frac{\partial H_{\vartheta_*}(\rho_*, \vartheta_*)}{\partial \rho} (\rho - \rho_*) - H_{\vartheta_*}(\rho_*, \vartheta_*) \right].$$

Remark that, using the Maxwell equation and the definition of the entropy, we have

$$\frac{\partial H_{\vartheta_*}(\rho, \vartheta)}{\partial \vartheta} = \frac{\rho}{\vartheta} (\vartheta - \vartheta_*) \frac{\partial e(\rho, \vartheta)}{\partial \vartheta}, \quad \frac{\partial^2 H_{\vartheta_*}(\rho, \vartheta_*)}{\partial \rho^2} = \frac{1}{\rho} \frac{\partial P(\rho, \vartheta_*)}{\partial \rho}.$$

Those estimates were developed in the works by E. FEIREISL and collaborators, under the assumption of thermodynamic stability

$$\frac{\partial e(\rho, \vartheta)}{\partial \vartheta} > 0, \quad \frac{\partial P(\rho, \vartheta)}{\partial \rho} > 0. \quad (10.4)$$

The meaning of such condition is that both the specific heat at constant volume C_v and the compressibility of the fluid are positive. However as we have already seen, the latter condition is violated by several physical state law, such as the standard Van der Waals equation of state.

Under (10.4), one obtains that H_{ϑ_*} is increasing in ϑ for $\vartheta > \vartheta_*$ and decreasing for $\vartheta < \vartheta_*$, that is that H_{ϑ_*} has a minimum at $\vartheta = \vartheta_*$. The second part of (10.4) implies that H_{ϑ_*} is convex in ρ .

One chooses accordingly ρ_* as the minimum of $H_{\vartheta_*}(\rho, \vartheta_*)$ and decompose the quantity linked to $H_{\vartheta_*}(\rho, \vartheta)$ into two parts

$$\begin{aligned} & H_{\vartheta_*}(\rho, \vartheta) - \frac{\partial H_{\vartheta_*}(\rho_*, \vartheta_*)}{\partial \rho} (\rho - \rho_*) - H_{\vartheta_*}(\rho_*, \vartheta_*) \\ &= [H_{\vartheta_*}(\rho, \vartheta) - H_{\vartheta_*}(\rho, \vartheta_*)] \\ & \quad + [H_{\vartheta_*}(\rho, \vartheta_*) - \frac{\partial H_{\vartheta_*}(\rho_*, \vartheta_*)}{\partial \rho} (\rho - \rho_*) - H_{\vartheta_*}(\rho_*, \vartheta_*)] \geq 0 \end{aligned} \quad (10.5)$$

by the thermodynamic stability assumptions on $\partial e(\rho, \vartheta) / \partial \vartheta$ and $\partial P(\rho, \vartheta) / \partial \rho$. As a consequence under (10.4), one deduces that

$$\int_0^t \int \left(\bar{\mu} \frac{|\nabla u|^2}{\vartheta} + \kappa(\vartheta) \frac{\mu |\nabla \vartheta|^2}{\vartheta^2} \right) dx dt \leq \text{constant}. \quad (10.6)$$

10.3 The pressure laws covered by previous works.

In every previous work, the viscous stress tensor is assumed to be isotropic

$$\mathcal{S} = \mu (\nabla u + \nabla u^T) + \lambda \operatorname{div} u \operatorname{Id},$$

with coefficients μ , λ either constants or depending only on ϑ .

1) *The pressure law as a perturbation of the barotropic case.* It is due to E. FEIREISL who considered pressure laws under the form

$$P(\rho, \vartheta) = P_c(\rho) + \vartheta P_\vartheta(\rho),$$

where

$$\begin{aligned} P_c(0) &= 0, & P'_c(\rho) &\geq a_1 \rho^{\gamma-1} - b \text{ for } \rho > 0, \\ P_c(\rho) &\leq a_2 \rho^\gamma + b \text{ for all } \rho \geq 0, \\ P_\vartheta(0) &= 0, & P'_\vartheta(\rho) &\geq 0 \text{ for all } \rho \geq 0, \\ P_\vartheta(\rho) &\leq c(1 + \rho^\beta), \end{aligned} \tag{10.7}$$

and

$$\gamma > d/2, \quad \beta < \frac{\gamma}{2} \text{ for } d = 2, \quad \beta = \frac{\gamma}{3} \text{ for } d = 3$$

with constants $a_1 > 0$, a_2 , b and P_c , P_ϑ in $\mathcal{C}[0, +\infty) \cap \mathcal{C}^1(0, +\infty)$. In agreement with Maxwell law and the entropy definition, it implies the following form on the internal energy

$$e(\rho, \vartheta) = \int_{\rho_*}^{\rho} \frac{P_c(s)}{s^2} ds + Q(\vartheta),$$

where $Q'(\vartheta) = C_v(\vartheta)$ (specific heat at constant volume). The entropy is given by

$$s(\rho, \vartheta) = \int_{\rho_*}^{\rho} \frac{C_v(s)}{s} ds - S_\vartheta(\rho),$$

where S_ϑ is the thermal pressure potential given through $S_\vartheta(\rho) = \int_{\rho_*}^{\rho} P_\vartheta(s)/s^2 ds$.

The heat conductivity coefficient κ is assumed to satisfy

$$\kappa_1(\vartheta^\alpha + 1) \leq \kappa(\vartheta) \leq \kappa_2(\vartheta^\alpha + 1) \text{ for all } \vartheta \geq 0,$$

with constants $\kappa_1 > 0$ and $\alpha \geq 2$. The thermal energy $Q = Q(\vartheta) = \int_0^{\vartheta} C_v(z) dz$ has not yet been determined and is assumed to satisfy $\int_{z \in [0, +\infty)} C_v(z) > 0$ and $C_v(\vartheta) \leq c(1 + \vartheta^{\alpha/2-1})$. Because the energy and pressure satisfy (10.4), the estimate on H_ϑ gives a control on ρ^γ in $L^\infty(0, T; L^1(\Omega))$ and through (10.6) a control on ϑ in $L^2(0, T; L^6(\Omega))$ in dimension 3 and in $L^2(0, T; L^p(\Omega))$ for all $p < +\infty$ in dimension 2.

Because (10.6) does not provide an H_x^1 bound on u , E. FEIREISL combines it with a direct energy estimate (see below). Therefore one obtains the exact equivalent of

estimates (2.6) in this case. Using similar techniques, one then proves (2.9) for $0 < a < \min(1/d, 2d/\gamma - 1)$. The proof of compactness and existence follows the main steps shown in subsection 2.2.

2) *Pressure laws with large radiative contribution.* It is due to E. FEIREISL and A. NOVOTNY who consider pressure laws exhibiting both coercivity of type ρ^γ and ϑ^4 for large densities and temperatures namely

$$P(\rho, \vartheta) = \vartheta^{\gamma/(\gamma-1)} Q\left(\frac{\rho}{\vartheta^{1/(\gamma-1)}}\right) + \frac{a}{3} \vartheta^4 \text{ with } a > 0, \quad \gamma > 3/2,$$

with

$$Q \in C^1([0, +\infty)), \quad Q(0) = 0, \quad Q'(Z) > 0 \text{ for all } Z \geq 0,$$

and

$$\lim_{Z \rightarrow +\infty} \frac{Q(Z)}{Z^\gamma} = Q_\infty > 0.$$

In agreement to Maxwell law and the definition of entropy, it implies the following form on the internal energy and the entropy

$$e(\rho, \vartheta) = \frac{1}{(\gamma-1)} \frac{\vartheta^{\gamma/(\gamma-1)}}{\rho} Q\left(\frac{\rho}{\vartheta^{1/(\gamma-1)}}\right) + a \frac{\vartheta^4}{\rho},$$

and

$$s(\rho, \vartheta) = S\left(\frac{\rho}{\vartheta^{1/(\gamma-1)}}\right) + \frac{4a}{3} \frac{\vartheta^3}{\rho}.$$

They impose

$$0 < -S'(Z) = \frac{1}{\gamma-1} \frac{\gamma Q(Z) - Q'(Z)Z}{Z} < c < +\infty \text{ for all } Z > 0,$$

with $\lim_{Z \rightarrow +\infty} S(Z) = 0$ so that thermodynamical stability (10.4) holds. Therefore the energy provides uniform bounds in $L_t^\infty L_x^1$ for ϑ^4 and ρ^γ . One assumes in this case that the viscosities and heat conductivity satisfy

$$\mu, \lambda \in C^1([0, +\infty)) \text{ are Lipschitz and } \mu(1 + \vartheta) \leq \mu(\vartheta), \quad 0 \leq \lambda(\vartheta), \quad \mu_0 > 0,$$

and

$$\kappa \in C^1([0, +\infty)), \quad \kappa_0(1 + \vartheta^3) \leq \kappa(\vartheta) \leq \kappa_1(1 + \vartheta^3), \quad 0 < \kappa_0 \leq \kappa_1.$$

Almost everywhere convergence of the temperature is obtained using the radiation term. Extra integrability on $P(\rho, \vartheta)$ can be derived just as in the barotropic case. Finally the same procedure as in the barotropic case is followed to have compactness on the density, relying heavily on the monotonicity of the pressure $\partial P(\rho, \vartheta)/\partial \rho > 0$. This gives global existence of weak solutions (in a sense that we precise later).

With respect to these previous works, we focus here, as in the barotropic case, in removing the assumption of monotonicity on the pressure law, which is consistent with the discussion of such laws in subsection 2.4.

10.4 The direct entropy estimate

We explain here the general framework for our result on the Navier–Stokes–Fourier system. The estimates here closely follow the ones pioneered by P.–L. LIONS, and E. FEIREISL and A. NOVOTNY; just as in the barotropic case, our contribution is the new compactness argument not the energy estimates. With respect to the previous discussion, we only present them here in a more general context as in particular we will not need the monotonicity of P .

If one removes the monotonicity assumption on P then thermodynamic stability (10.4) does not hold anymore. Following P.–L. LIONS, it is however possible to obtain the estimate (10.6) directly by integrating the entropy equation (10.2)

$$\int_0^t \int \left(\mu \frac{|\nabla u|^2}{\vartheta} + \kappa(\vartheta) \frac{|\nabla \vartheta|^2}{\vartheta^2} \right) dx dt \leq C \int \rho s(t, x) dx.$$

Therefore the bound (10.6) holds under the general assumption that there exists C s.t.

$$s(\rho, \vartheta) \leq C e(\rho, \vartheta). \quad (10.8)$$

Recall that

$$e = m(\vartheta) + \int_{\rho^*}^{\rho} \frac{(P(\rho', \vartheta) - \vartheta \partial_{\vartheta} P(\rho', \vartheta))}{\rho'^2}(\rho', \vartheta) d\rho',$$

and

$$\partial_{\rho} s = -\frac{\partial_{\vartheta} P}{\rho^2}.$$

We also have that $\partial_{\vartheta} s = \partial_{\vartheta} e / \vartheta$, therefore as long as $m(\vartheta) \geq 0$ with

$$\int_{\vartheta^*}^{\vartheta} \frac{m'(s)}{s} ds \leq C(1 + m(\vartheta)),$$

and

$$-\partial_{\vartheta} P \leq C(P - \vartheta \partial_{\vartheta} P),$$

for some $C > 0$ then (10.8) is automatically satisfied and one obtains the bound (10.6). Moreover if $e(\vartheta, \rho) \geq \rho^{\gamma-1}/C$ then one also has that $\rho \in L_t^{\infty} L_x^{\gamma}$. Assuming now that

$$\kappa_1 (\vartheta^{\alpha} + 1) \leq \kappa(\vartheta) \leq \kappa_2 (\vartheta^{\alpha} + 1),$$

with $\alpha \geq 2$, one deduces from (10.6) that $\log \vartheta \in L_t^2 H_x^1$ and $\vartheta^{\alpha/2} \in L_t^2 H_x^1$ or by Sobolev embedding $\vartheta \in L_t^{\alpha} L_x^{\alpha/(1-2/d)}$. By a Hölder estimate, it is also possible to obtain a Sobolev-like, $L_t^{p_1} W_x^{1, p_2}$, bound on u

$$\begin{aligned} \int_0^T \left(\int |\nabla u|^{p_2} dx \right)^{p_1/p_2} dt &\leq \left(\int_0^T \int \frac{|\nabla u|^2}{\vartheta} dx dt \right)^{p_1/2} \\ &\left(\int_0^T \left(\int \vartheta^{p_2/(2-p_2)} dx \right)^{p_1(2-p_2)/(p_2(2-p_1))} dt \right)^{(2-p_1)/2} < \infty, \end{aligned}$$

provided that $p_2/(2 - p_2) = \alpha/(1 - 2/d)$ and $p_1/(2 - p_1) = \alpha$ or

$$p_1 = \frac{2\alpha}{1 + \alpha}, \quad p_2 = \frac{2\alpha d}{d(\alpha + 1) - 2}. \quad (10.9)$$

Unfortunately this Sobolev estimate does not allow to derive an equivalent of (2.9). Instead one requires a $L_t^2 H_x^1$ estimate on u (the critical point is in fact the L_t^2 with value in some Sobolev in x). Instead one can easily extend the argument by E. FEIREISL and A. NOVOTNY: For any $\phi(\rho)$, one can write

$$\frac{1}{2} \frac{d}{dt} \int \rho |u|^2 + \frac{d}{dt} \int \phi(\rho) + \int \mathcal{S} : \nabla u = \int (P(\vartheta, \rho) - \phi'(\rho) \rho + \phi(\rho)) \operatorname{div} u.$$

This leads to the assumption that there exists some ϕ s.t.

$$\begin{aligned} C^{-1} \rho^\gamma - C &\leq \phi(\rho) \leq C \rho^\gamma + C, \\ |P(\vartheta, \rho) - \phi'(\rho) \rho + \phi(\rho)| &\leq C \left(\rho^{\beta_1} + \vartheta^{\beta_2} + \sqrt{\rho e(\vartheta, \rho)} \right), \end{aligned} \quad (10.10)$$

with

$$\beta_1 \leq \frac{\gamma}{2}, \quad \beta_2 \leq \frac{\alpha}{2}. \quad (10.11)$$

Indeed, with (10.10), one has

$$\int_0^T \int \mathcal{S} : \nabla u \, dx \, dt \leq E(\rho^0, u^0, \vartheta^0) + C \int_0^T \int \left(\rho^{\beta_1} + \vartheta^{\beta_2} + \sqrt{\rho e(\vartheta, \rho)} \right) |\operatorname{div} u| \, dx \, dt.$$

By (2.2), this leads to

$$\int_0^T \int |\nabla u|^2 \, dx \, dt \leq C E(\rho^0, u^0, \vartheta^0) + C \|\nabla u\|_{L_{t,x}^2} \|\rho^{\beta_1} + \vartheta^{\beta_2}\|_{L_{t,x}^2}, \quad (10.12)$$

and the desired H^1 bound follows from (10.11). It is now possible to follow the same steps to obtain an equivalent of (2.9) if $\gamma > d/2$

$$\int_0^T \int \rho^{\gamma+a} \, dx \, dt \leq C(T, E(\rho^0, u^0, \vartheta^0)), \quad \text{for all } a < \max\left(\frac{1}{2}, \frac{2}{d}\gamma - 1\right). \quad (10.13)$$

Note here that the assumptions (10.10)-(10.11) are likely not optimal and will be improved in future works. They nevertheless already cover the two examples presented above.

10.5 Main result in the heat-conducting case

For convenience, we repeat here all the assumptions presented above and the concluding mathematical result which may be obtained. More precisely, we assume that

for some $C > 0$

$$\left\{ \begin{array}{l} P(\rho, \vartheta) \text{ such that } -\partial_\vartheta P \leq C(P - \vartheta \partial_\vartheta P) \text{ with } C > 0, \\ e(\rho, \vartheta) = m(\vartheta) + \int_{\rho^*}^\rho \frac{(P(\rho', \vartheta) - \vartheta \partial_\vartheta P(\rho', \vartheta))}{\rho'^2} d\rho' \geq \frac{\rho^{\gamma-1}}{C} + \frac{\vartheta^{\gamma\vartheta}}{C\rho}, \\ \text{with } m(\vartheta) \geq 0 \text{ and } \int_{\vartheta^*}^\vartheta \frac{m'(s)}{s} ds \leq C(1 + m(\vartheta)) \leq C(1 + \vartheta^{\alpha(\gamma+a-1)/2(\gamma+a)}), \\ \kappa_1(\vartheta^\alpha + 1) \leq \kappa(\vartheta) \leq \kappa_2(\vartheta^\alpha + 1), \quad \mu, \lambda \text{ constants and } \alpha \geq 2, \\ C^{-1}\rho^\gamma - C \leq \phi(\rho) \leq C\rho^\gamma + C, \\ |P(\rho, \vartheta) - \phi'(\rho)\rho + \phi(\rho)| \leq C\rho^{\beta_1} + C\vartheta^{\beta_2} + C\sqrt{\rho e(\rho, \vartheta)}, \\ |\partial_\vartheta P(\rho, \vartheta)| \leq C\rho^{\beta_3} + C\vartheta^{\beta_4} \end{array} \right. \quad (10.14)$$

for

$$\left\{ \begin{array}{l} \beta_1 \leq \frac{\gamma}{2}, \quad \beta_2 < \frac{\alpha}{2}, \quad \gamma > \max\left(\frac{3}{2}, \frac{d}{2}, \frac{3d}{d+2}\right), \\ \beta_3 < \frac{\gamma+a+1}{2}, \quad \beta_4 < \max\left(\frac{\alpha}{2}, \gamma\vartheta\left(\frac{1}{2} + \frac{1}{d}\right)\right), \\ \frac{2}{d}\mu + \lambda > 0, \quad \gamma\vartheta \geq 0 \end{array} \right. \quad (10.15)$$

where we recall that $a < \max(1/2, 2\gamma/d - 1)$. We also assume that either $\partial_\vartheta P(0, \vartheta)$ is strictly convex in ϑ or that the specific heat is positive (as is necessary for the physics) *i.e.*

$$C_v = \partial_\vartheta e(\rho, \vartheta) > 0, \quad \forall \rho, \vartheta, \quad (10.16)$$

and

$$|\partial_\rho P(\rho, \vartheta)| \leq C\rho^{\gamma-1}. \quad (10.17)$$

and that the initial data satisfies

$$\begin{aligned} \rho_0 \in L^\gamma(\Pi^d), \quad \vartheta_0 \in L^{\gamma\vartheta}(\Pi^d) \\ \text{with } \rho_0 \geq 0, \quad \vartheta_0 > 0 \text{ in } \Pi^d \quad \text{and } \int_{\Pi^d} \rho_0 = M_0 > 0, \end{aligned} \quad (10.18)$$

and

$$E_0 = \int_{\Pi^d} \left(\frac{1}{2} \frac{|(\rho u)_0|^2}{\rho_0} + \rho_0 e(\rho_0, \vartheta_0) \right) < +\infty. \quad (10.19)$$

Then we can prove the following result

Theorem 10.1 *Assume that (10.14)–(10.17) are satisfied and that the initial data satisfy (10.18) and (10.19). Then there exists*

$$\begin{aligned} \rho &\in L^\infty([0, T], L^\gamma(\Pi^d)) \cap L^{\gamma+a}([0, T] \times \Pi^d), \quad \forall a < \max\left(\frac{1}{2}, \frac{2}{d}\gamma - 1\right), \\ u &\in L^2([0, T], H^1(\Pi^d)) \cap L^\infty([0, T], L^2_\rho(\Pi^d)), \\ \vartheta &\in L^\alpha([0, T], L^{\alpha/(1-2/d)}(\Pi^d)), \quad \log \vartheta \in L^2([0, T], H^1(\Pi^d)), \end{aligned} \quad (10.20)$$

with $\vartheta > 0$ a.a. on $(0, T) \times \Pi^d$ that is solution in the sense of distributions to

$$\begin{aligned} \partial_t \rho + \operatorname{div}(\rho u) &= 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mu \Delta u - (\lambda + \mu) \nabla \operatorname{div} u + \nabla P(\rho, \vartheta) &= 0, \end{aligned} \quad (10.21)$$

while the equation on the temperature is satisfied in the following sense: In the sense of distribution, the entropy solves

$$\partial_t(\rho s(\rho, \vartheta)) + \operatorname{div}(\rho s(\rho, \vartheta) u) - \operatorname{div} \left(\frac{\kappa(\vartheta) \nabla \vartheta}{\vartheta} \right) \geq \frac{1}{\vartheta} \left(\mathcal{S} : \nabla u + \frac{\kappa |\nabla \vartheta|^2}{\vartheta} \right), \quad (10.22)$$

with $\mathcal{S} = \mu(\nabla u + \nabla u^T) + \lambda \operatorname{div} u \operatorname{Id}$ and recalling that

$$s(\rho, \vartheta) = - \int_{\rho^*}^{\rho} \frac{\partial_{\vartheta} P}{\rho'^2}(\rho', \vartheta) d\rho' + \int_0^{\vartheta} \frac{m'(\vartheta')}{\vartheta'} d\vartheta'.$$

This is supplemented by the total energy property

$$\int (\rho \frac{|u|^2}{2} + \rho e(\rho, \vartheta)) dx = \int (\rho^0 \frac{|u^0|^2}{2} + \rho^0 e(\rho^0, \vartheta^0)) dx. \quad (10.23)$$

and the initial data conditions satisfied by $(\rho, \rho u, \rho s)$ in a weak sense

$$\rho|_{t=0} = \rho_0, \quad \rho u|_{t=0} = (\rho u)_0, \quad \rho s|_{t=0^+} \geq \rho_0 s(\rho_0, \vartheta_0).$$

Remark 10.2 An example of pressure law included in Theorem 10.1 is a perturbation of the truncated virial expansion as described in subsection 2.4

$$P(\rho, \vartheta) = \rho^{\gamma} + \vartheta \sum_{n=0}^N B_n(\vartheta) \rho^n,$$

with $\gamma > N$. In that case we simply take $m(\vartheta) = 0$. The coefficients B_n can have any sign and we only require that

$$|\vartheta \partial_{\vartheta} B_n(\vartheta)| + |B_n(\vartheta)| + |\vartheta B_n(\vartheta) - \bar{B}_n| \leq C(\rho^{\gamma/2-n} + \vartheta^{\frac{\alpha}{2}(1-\frac{2n}{\gamma})}).$$

Then choosing $\phi(\rho) = \rho^{\gamma} + \sum_{n \leq N} \bar{B}_n \rho^n$, (10.10) together with (10.14) and (10.15) are satisfied. We will treat more general virial expansions in a future work. Note for a fixed ϑ then $P(\rho, \vartheta)$ is indeed increasing after a critical $\bar{\rho}_{\vartheta}$ which depends on ϑ and can be arbitrarily large where $\vartheta \gg 1$. This is the reason why P does not satisfy any of the classical monotonicity assumption such as (3.7) or (5.8) and why our new approach is needed.

Remark 10.3 The assumption (10.17) is more demanding in the Navier-Stokes-Fourier setting (though satisfied by the examples given earlier in this section). It would be more natural to have instead

$$|\partial_{\rho} P(\rho, \vartheta)| \leq \bar{P}(\vartheta) \rho^{\gamma-1},$$

for some unbounded function \bar{P} of ϑ . This is possible and require some modifications in the proof of Theorem 5.1, combined with appropriate estimates on ϑ . In general obtaining more optimal results for the Navier-Stokes-Fourier system seems to depend on adapting precisely the estimates and Theorem 5.1 to the specific model under consideration.

Remark 10.4 *With appropriate additional growth assumption, it is possible to show that the combination of (10.22) and (10.23) implies the usual energy equation (and not just an inequation as (10.22)). The reason for the formulation (10.22)-(10.23) is that it cannot be proved without those additional assumptions. We refer to [34] and [35].*

Remark 10.5 *In Theorem 10.1, the viscosity coefficients μ , λ are independent of the temperature ϑ . Instead several models use temperature dependent coefficients $\mu(\vartheta)$, $\lambda(\vartheta)$. To handle that case, the proof given below would have to be modified; the compactness of the temperature would have to be established first.*

Sketch of the proof. The proof of Theorem 10.1 follows the same steps as the proof of Theorem 3.1. For this reason we only sketch briefly the points which are similar and insist more on the differences induced by the presence of the temperature. The general method is taken from [35] and is adapted to take advantage of our new compactness estimates.

For simplicity we assume throughout the proof that

$$\gamma < 2 + \max\left(\frac{1}{2}, \frac{4}{d} - 1\right). \quad (10.24)$$

This assumption is not strictly necessary but removing it would add one or more additional stage of approximation. We explain in the limit from (10.33) to (10.34) at the end of the proof which additional steps would be required in the absence of (10.24).

As usual one starts from an approximate model

$$\begin{aligned} \partial_t \rho_k + \operatorname{div}(\rho_k u_k) &= \alpha_k \Delta \rho_k, \\ \partial_t(\rho_k u_k) + \operatorname{div}(\rho_k u_k \otimes u_k) - \mu \Delta u_k - (\lambda + \mu) \nabla \operatorname{div} u_k + \nabla P_{\varepsilon, \delta}(\rho_k, \vartheta_k) \\ &\quad + \alpha_k \nabla \rho_k \cdot \nabla u_k = 0, \\ \partial_t(\rho_k s_\varepsilon(\rho_k, \vartheta_k)) + \operatorname{div}(\rho_k s_\varepsilon(\rho_k, \vartheta_k) u_k) - \operatorname{div}\left(\frac{\kappa_\varepsilon(\vartheta_k)}{\vartheta_k} \nabla \vartheta_k\right) \\ &= \frac{1}{\vartheta_k} \left(\mathcal{S}_k : \nabla u_k + \frac{\kappa_\varepsilon(\vartheta_k)}{\vartheta_k} |\nabla \vartheta_k|^2 \right) \\ &\quad + \alpha_k \frac{\Delta \rho_k}{\vartheta_k} (\vartheta_k s_\varepsilon(\rho_k, \vartheta_k) - e_{\varepsilon, \delta}(\rho_k, \vartheta_k) - \frac{P_{\varepsilon, \delta}(\rho_k, \vartheta_k)}{\rho_k}), \end{aligned} \quad (10.25)$$

with again $\mathcal{S}_k = \mu(\nabla u_k + \nabla u_k^T) + \lambda \operatorname{div} u_k \operatorname{Id}$. Note that the last quantity in the entropy relation comes from the preservation of the total energy and the relation

between energy and entropy. Remark also that contrary to the barotropic case, we need here two steps of approximations in the pressure (and hence energy), indexed by ε and δ .

The first level of approximation is obtained by adding a barotropic correction

$$P_\delta(\rho, \vartheta) = P(\rho, \vartheta) + \delta \rho^2, \quad (10.26)$$

where the exponent 2 is crucial, *i.e.* it could not be any other for reasons that will be clear below. From the approximated pressure, one defines as usual the approximate energy

$$e_\delta(\rho, \vartheta) = m(\vartheta) + \int_{\rho^*}^{\rho} \frac{(P_\delta(\rho', \vartheta) - \vartheta \partial_\vartheta P_\delta(\rho', \vartheta))}{\rho'^2} d\rho' = e(\rho, \vartheta) + \delta \rho^2, \quad (10.27)$$

and the entropy also has the straightforward expression

$$s_\delta(\rho, \vartheta) = \int_0^\vartheta \frac{m'(s)}{s} ds - \int_{\rho^*}^{\rho} \frac{\partial_\vartheta P_\delta(\rho', \vartheta)}{\rho'^2} d\rho' = s(\rho, \vartheta). \quad (10.28)$$

In the second stage of approximation, indexed by ε , we first replace $\kappa(\vartheta)$ by $\kappa_\varepsilon(\vartheta)$. We assume that $\kappa_\varepsilon = \kappa + \tilde{\kappa}_\varepsilon$ with $\tilde{\kappa}_\varepsilon$ a positive function which vanishes for $2\varepsilon < \vartheta < 1/2\varepsilon$ and satisfies

$$\tilde{\kappa}_\varepsilon(\vartheta) \longrightarrow +\infty \text{ as } \vartheta \rightarrow \varepsilon \text{ or } \varepsilon^{-1}, \quad \int_\varepsilon^{\varepsilon^{-1}} \tilde{\kappa}_\varepsilon(\vartheta) d\vartheta < \infty. \quad (10.29)$$

This is slightly different from [35] but gives similar bounds on the temperature while being simpler to handle in our case. Then we define P_ε as an approximate pressure law of P_δ truncated for large densities, namely for $\bar{\rho}_\varepsilon = \varepsilon^{-1}$, we define

$$P_\varepsilon(\rho, \vartheta) = P(\rho, \vartheta) \quad \text{if } \rho \leq \bar{\rho}_\varepsilon, \quad P_\varepsilon(\rho, \vartheta) = P(\bar{\rho}_\varepsilon, \vartheta) \quad \text{if } \rho > \bar{\rho}_\varepsilon. \quad (10.30)$$

Note that since the temperature will be bounded by (10.29), it is not necessary to truncate it in (10.30). This leads to

$$P_{\varepsilon, \delta}(\rho, \vartheta) = P_\varepsilon(\rho, \vartheta) + \delta \rho^2.$$

The energy density is still defined from the pressure $P_{\varepsilon, \delta}$ or P_ε ,

$$\begin{aligned} e_\varepsilon(\rho, \vartheta) &= m(\vartheta) + \int_{\rho^*}^{\rho} \frac{(P_\varepsilon(\rho', \vartheta) - \vartheta \partial_\vartheta P_\varepsilon(\rho', \vartheta))}{\rho'^2} d\rho', \\ e_{\varepsilon, \delta}(\rho, \vartheta) &= m(\vartheta) + \int_{\rho^*}^{\rho} \frac{(P_{\varepsilon, \delta}(\rho', \vartheta) - \vartheta \partial_\vartheta P_{\varepsilon, \delta}(\rho', \vartheta))}{\rho'^2} d\rho' = e_\varepsilon(\rho, \vartheta) + \delta \rho^2, \end{aligned} \quad (10.31)$$

and so is the entropy which is actually independent of δ (just as s_δ is independent of δ in (10.28))

$$s_{\varepsilon, \delta}(\rho, \vartheta) = s_\varepsilon(\rho, \vartheta) = \int_0^\vartheta \frac{m'(s)}{s} ds - \int_{\rho^*}^{\rho} \frac{\partial_\vartheta P_\varepsilon(\rho', \vartheta)}{\rho'^2} d\rho'. \quad (10.32)$$

The first step is to pass to the limit $\alpha_k \rightarrow 0$ in the system (10.25) to obtain a global weak solution to

$$\begin{aligned}
\partial_t \rho + \operatorname{div}(\rho u) &= 0, \\
\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mu \Delta u - (\lambda + \mu) \nabla \operatorname{div} u + \nabla P_{\varepsilon, \delta}(\rho, \vartheta) &= 0, \\
\partial_t(\rho s_\varepsilon(\rho, \vartheta)) + \operatorname{div}(\rho s_\varepsilon(\rho, \vartheta) u) - \operatorname{div} \left(\frac{\kappa_\varepsilon(\vartheta) \nabla \vartheta}{\vartheta} \right) & \\
&\geq \frac{1}{\vartheta} \left(\mathcal{S} : \nabla u + \frac{\kappa_\varepsilon |\nabla \vartheta|^2}{\vartheta} \right). \tag{10.33}
\end{aligned}$$

Note that the entropy now satisfies only an inequality, however this is supplemented by the preservation of total energy

$$\int \rho \left(\frac{|u|^2}{2} + e_{\varepsilon, \delta}(\rho, \vartheta) \right) dx = \int \rho^0 \left(\frac{|u^0|^2}{2} + e_{\varepsilon, \delta}(\rho^0, \vartheta^0) \right) dx.$$

Let us remark that this identity occurs also for α_k fixed as mentioned in [35]. Then one considers a sequence $\varepsilon_k \rightarrow 0$ and the corresponding sequences ρ_k, u_k, ϑ_k of solution to (10.33). Passing again to the limit, we obtain global weak solutions to

$$\begin{aligned}
\partial_t \rho + \operatorname{div}(\rho u) &= 0, \\
\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mu \Delta u - (\lambda + \mu) \nabla \operatorname{div} u + \nabla P_\delta(\rho, \vartheta) &= 0, \\
\partial_t(\rho s(\rho, \vartheta)) + \operatorname{div}(\rho s_\varepsilon(\rho, \vartheta) u) - \operatorname{div} \left(\frac{\kappa(\vartheta) \nabla \vartheta}{\vartheta} \right) & \\
&\geq \frac{1}{\vartheta} \left(\mathcal{S} : \nabla u + \frac{\kappa |\nabla \vartheta|^2}{\vartheta} \right), \tag{10.34}
\end{aligned}$$

still with the energy equality

$$\int \rho \left(\frac{|u|^2}{2} + e_\delta(\rho, \vartheta) \right) dx = \int \rho^0 \left(\frac{|u^0|^2}{2} + e_\delta(\rho^0, \vartheta^0) \right) dx.$$

Finally, we take a sequence $\delta_k \rightarrow 0$ and obtain global weak solutions as announced by Theorem 10.1.

The three limits are somewhat similar with slightly different *a priori* estimates and in the case of (10.25) to (10.33) additional difficulties in obtaining those a priori estimates. This limit uses point i) of Theorem 5.1 and the other two point ii). For this reason, we sketch the limit (10.25) to (10.33) separately and both limits, (10.33) to (10.34) and (10.34) to (10.21)-(10.22) at the same time.

The limit from (10.25) to (10.33). We use the a priori estimates described above, in particular in subsection 10.4. The entropy estimate though requires more care for System (10.25). We emphasize here that most of the complications in this limit were already present in [35] and are solved here in a very similar manner. The novel

features of our work are mostly present in the other two limits, from (10.33) to (10.34) and from (10.34) to the limit system.

However because we handle pressure terms that are more general, we can unfortunately not simply use the result of [35] to skip this first limit. Instead one has to check carefully that indeed at this level, estimates work the same.

First of all, given assumption (10.29), standard parabolic estimates imply that any solution to (10.25) is either constant in x or satisfies

$$\varepsilon \leq \vartheta_k(t, x) \leq \frac{1}{\varepsilon}. \quad (10.35)$$

Remark that since $\int_{\varepsilon}^{\varepsilon^{-1}} \tilde{\kappa}_{\varepsilon} < \infty$, we do not have strict inequalities in (10.35). In addition, the conservation of energy yields

$$\sup_{t,k} \int_{\Pi^d} \rho_k^2(t, x) dx < \infty. \quad (10.36)$$

Observe that

$$\begin{aligned} & \frac{\Delta \rho_k}{\vartheta_k} \left(\vartheta_k s_{\varepsilon} - e_{\varepsilon, \delta} - \frac{P_{\varepsilon, \delta}}{\rho_k} \right) \\ &= \operatorname{div} \left[\left(\vartheta_k s_{\varepsilon, \delta} - e_{\varepsilon, \delta} - \frac{P_{\varepsilon, \delta}}{\rho_k} \right) \frac{\nabla \rho_k}{\vartheta_k} \right] + \partial_{\rho} P_{\varepsilon, \delta} \frac{|\nabla \rho_k|^2}{\rho_k \vartheta_k} \\ & \quad - \left(e_{\varepsilon, \delta} + \rho \partial_{\rho} e_{\varepsilon, \delta} \right) \frac{\nabla \rho_k \cdot \nabla \vartheta_k}{\vartheta_k^2} \end{aligned} \quad (10.37)$$

where the functions $P_{\varepsilon, \delta}$, $e_{\varepsilon, \delta}$ and s_{ε} are all taken on the points $\rho_k(t, x)$, $\vartheta_k(t, x)$. The identity (10.37) is obtained through the general relations between s , e and P and in particular

$$\frac{\partial}{\partial \rho} \left(\vartheta s - e - \frac{P}{\rho} \right) = -\frac{1}{\rho} \frac{\partial P}{\partial \rho}$$

and

$$\frac{\partial}{\partial \vartheta} \left(s - \frac{e}{\vartheta} - \frac{P}{\rho \vartheta} \right) = \frac{1}{\vartheta^2} \left(e + \rho \frac{\partial e}{\partial \rho} \right).$$

Integrating in space and time the entropy equation in (10.25), we therefore get

$$\begin{aligned} & \int_0^t \int_{\Pi^d} \frac{1}{\vartheta_k} \left(\mathcal{S}_k : \nabla u_k + \frac{\kappa_{\varepsilon}(\vartheta_k)}{\vartheta_k} |\nabla \vartheta_k|^2 \right) + \alpha_k \int_0^t \int_{\Pi^d} \partial_{\rho} P_{\varepsilon, \delta} \frac{|\nabla \rho_k|^2}{\rho_k \vartheta_k} \\ & \leq \alpha_k \int_0^t \int_{\Pi^d} \left(e_{\varepsilon, \delta} + \rho_k \partial_{\rho} e_{\varepsilon, \delta} \right) \frac{\nabla \rho_k \cdot \nabla \vartheta_k}{\vartheta_k^2} + \int_{\Pi^d} \rho_k s_{\varepsilon}(t) - \int_{\Pi^d} (\rho_k s_{\varepsilon})(0). \end{aligned} \quad (10.38)$$

Therefore for α_k fixed we have extra terms in the right-hand side with respect to the case of System (10.33) or (10.34) as described in subsection 10.4, namely

$$\alpha_k \int_{\Pi^d} \left(e_{\varepsilon, \delta} + \rho_k \partial_{\rho} e_{\varepsilon, \delta} \right) \frac{\nabla \rho_k \cdot \nabla \vartheta_k}{\vartheta_k^2}$$

and an additional term on the left-hand side

$$\alpha_k \int_{\Pi^d} \partial_\rho P_{\varepsilon, \delta} \frac{|\nabla \rho_k|^2}{\vartheta_k}.$$

On the other hand by the bounds (10.35) on ϑ_k and the truncation in P_ε imply that s_ε is bounded by some constant depending on ε , hence trivially in that case

$$\int_{\Pi^d} \rho_k s_\varepsilon(\rho_k, \vartheta_k) dx \leq C_\varepsilon.$$

From the definition (10.30) of P_ε and the definition of $P_{\varepsilon, \delta}$, we obtain that there exists $\bar{\rho}_\varepsilon$ s.t.

$$\partial_\rho P_{\varepsilon, \delta}(\rho, \vartheta) > \delta \rho, \quad \forall \rho > \bar{\rho}_\varepsilon, \quad \forall \varepsilon \leq \vartheta_k(t, x) \leq \frac{1}{\varepsilon}. \quad (10.39)$$

From the bounds (10.35) on ϑ_k , we obtain that $P_{\varepsilon, \eta}$ satisfies the monotonicity assumption (5.8) and moreover

$$\alpha_k \int_{\Pi^d} \partial_\rho P_{\varepsilon, k} \frac{|\nabla \rho_k|^2}{\vartheta_k} \geq \delta \varepsilon \int \rho_k |\nabla \rho_k|^2 - C \alpha_k \int_{\rho_k \leq \bar{\rho}_\varepsilon} |\nabla \rho_k|^2.$$

On the other hand, the energy bound (10.36) implies that $\int \rho_k \log \rho_k$ is uniformly bounded. The continuity equation then shows that

$$\begin{aligned} \alpha_k \int_0^T \int_{\Pi^d} \frac{|\nabla \rho_k|^2}{\rho_k} dx dt &\leq C + \int_0^T \int_{\Pi^d} \rho_k |\operatorname{div} u_k| dx dt \\ &\leq C + C_\varepsilon \left(\int_0^T \int_{\Pi^d} \frac{\mathcal{S}_k : \nabla u_k}{\vartheta_k} dx dt \right)^{1/2}, \end{aligned}$$

by using that ρ_k is uniformly in L^2 again from (10.36). Therefore we obtain the straightforward estimate

$$\begin{aligned} \alpha_k \int_0^t \int_{\Pi^d} \partial_\rho P_{\varepsilon, k} \frac{|\nabla \rho_k|^2}{\vartheta_k} &\geq \alpha_k \delta \varepsilon \int_0^t \int_{\Pi^d} \rho_k |\nabla \rho_k|^2 \\ &\quad - C - C_\varepsilon \left(\int_0^t \int_{\Pi^d} \frac{\mathcal{S}_k : \nabla u_k}{\vartheta_k} \right)^{1/2}. \end{aligned} \quad (10.40)$$

Turning to the last term in the entropy estimate, we first observe that

$$e_{\varepsilon, \delta} + \rho_k \partial_\rho e_{\varepsilon, \delta} = e_\varepsilon + \rho_k \partial_\rho e_\varepsilon$$

because the barotropic part, $\delta \rho$, in $e_{\varepsilon, \delta}$ (see (10.31)) cancels. This is the reason why the barotropic part in P_δ has to be exactly ρ^2 ; with any other exponent, we would not have the above cancellation (see again the book by E. FEIREISL and A. NOVOTNY

[35] where it has first been observed). Remarking that e_ε and $\rho \partial_\rho e_\varepsilon$ are bounded still from the truncation of P_ε , one obtains that

$$\begin{aligned} \alpha_k \int_{\Pi^d} (e_{\varepsilon,\delta} + \rho_k \partial_\rho e_{\varepsilon,\delta}) \frac{\nabla \rho_k \cdot \nabla \vartheta_k}{\vartheta_k^2} \\ \leq \alpha_k C_\varepsilon \int_{\Pi^d} \nabla \rho_k \cdot \nabla \vartheta_k \\ \leq \alpha_k \frac{\delta \varepsilon}{2} \int_{\Pi^d} |\nabla \rho_k|^2 + C_{\varepsilon,\delta} \alpha_k \int_{\Pi^d} |\nabla \vartheta_k|^2. \end{aligned} \quad (10.41)$$

Combining (10.40)-(10.41) with (10.38) concludes the bound on the entropy for α_k small enough. Let us note for later that this bound also implies that $\partial_t(\rho_k s_\varepsilon(\rho_k, \vartheta_k))$ is bounded in $L_t^1 W_x^{-1,1}$ uniformly in k .

The rest of the a priori estimates described in subsection 10.4 follow in a straightforward manner. We hence summarize the uniform bound here

$$\begin{aligned} \sup_{t,k} \int_{\Pi^d} (\rho_k^2 + \rho_k |u_k|^2 + \vartheta_k^{\gamma_\vartheta}) dx < \infty, \quad \varepsilon \leq \vartheta_k \leq \frac{1}{\varepsilon}, \\ \sup_k \int_0^T \int_{\Pi^d} (\rho_k^{2+a_\varepsilon} + |\nabla u_k|^2) dx dt < \infty, \quad \forall a_\varepsilon < \max\left(\frac{1}{2}, \frac{4}{d} - 1\right), \\ \sup_k \int_0^T \int_{\Pi^d} (|\nabla \vartheta_k^{\alpha/2}|^2 + |\nabla \log \vartheta_k|^2) dx dt < \infty, \\ \sup_k \|\partial_t(\rho_k s_\varepsilon(\rho_k, \vartheta_k))\|_{L_t^1 W_x^{-1,1}} < \infty. \end{aligned} \quad (10.42)$$

Combining those bounds with the continuity and momentum equations shows that (5.4), (5.5), (5.6), and (5.7) are satisfied. The pressure law also satisfies (5.8). Moreover taking the divergence of the momentum equation in (10.25) and inverting the Laplacian, one obtains

$$\begin{aligned} (\lambda + 2\mu) \operatorname{div} u_k = P_{\varepsilon_k}(\rho_k, \vartheta_k) + \Delta^{-1} \operatorname{div} (\partial_t(\rho_k u_k) + \operatorname{div}(\rho_k u_k \otimes u_k)) \\ + \alpha_k \Delta^{-1} \operatorname{div} (\nabla \rho_k \cdot \nabla u_k). \end{aligned}$$

This is exactly the identity (5.2) with $P_k(\rho, t, x) = P_{\varepsilon,\delta}(\rho, \vartheta_k(t, x))$ and $\mu_k = \lambda + 2\mu$ which satisfies (5.3). Observe that as a consequence

$$|P_k(\rho, t, x) - P_k(\rho, t, y)| \leq \max_{s \in [\vartheta_k(t,x), \vartheta_k(t,y)]} |\partial_\vartheta P_{\varepsilon,\delta}(\rho, s)| |\vartheta(t, x) - \vartheta(t, y)|.$$

Since ϑ_k belongs uniformly to an appropriate Sobolev space, that means that the estimate (5.3) is also satisfied. We may hence apply the variant of Theorem 5.1–point i, described in the remark 5.3. This yields the compactness of ρ_k in L^1 .

Since $u_k \in L_t^2 H_x^1$, we get the compactness on $\sqrt{\rho_k} u_k$ in $L_t^2 L_x^2$ from the momentum equation. The passage to the limit in the continuity equation is therefore as usual. We also have compactness in space for u_k and ϑ_k respectively from the viscosity and conductivity.

The next step is to obtain the compactness of the temperature. If there is a radiative part in P , that is if $\partial_\vartheta P(0, \vartheta)$ is non vanishing and convex, then it is possible to obtain directly the compactness of ϑ_k through standard diffusion estimates on the entropy equation. Otherwise it relies in a delicate manner on the entropy inequation in (10.33) but follows the now classical approach and for this reason we only sketch the procedure:

- Observe that $\rho_k s_\varepsilon(\rho_k, \vartheta_k)$ in some $L^p_{t,x}$ with $p > 1$ from (10.42) and the fact that s_ε is bounded.
- Observe that $(\partial_t(\rho_k s_\varepsilon(\rho_k, \vartheta_k)))_+$ is uniformly in $L^1_t W_x^{-1,1}$, again from (10.42).
- Obtain the *a.e.* convergence of $\rho_k s_\varepsilon(\rho_k, \vartheta_k)(t, x)$. From the first point and the compactness in space of ϑ_k and ρ_k , we deduce that $\rho_k s_\varepsilon(\rho_k, \vartheta_k)$ is compact in space. From the second point, we deduce that it is compact in time. Therefore after extraction $\rho_k s_\varepsilon(\rho_k, \vartheta_k)$ converges strongly in $L^1_{t,x}$ and after possibly further extraction, one has that there exists $l(t, x)$ s.t.

$$\rho_k(t, x) s_\varepsilon(\rho_k(t, x), \vartheta_k(t, x)) \longrightarrow l(t, x), \quad \text{for a.e. } t, x.$$

- Obtain the ρ a.e. convergence of $\vartheta_k(t, x)$. Extracting again, we have from the compactness of $\rho_k(t, x)$, that $\rho_k(t, x)$ converges *a.e.* to $\rho(t, x)$. This shows that $l(t, x) = \rho(t, x) \tilde{l}(t, x)$ and moreover that for *a.e.* t, x s.t. $\rho(t, x) \neq 0$, one has that $s_\varepsilon(\rho(t, x), \vartheta_k(t, x))$ converges to $\tilde{l}(t, x)$. By the assumption on the specific heat (10.16), s_ε is invertible in ϑ_k (P_ε and so s_ε is not truncated in ϑ only in ρ). Therefore this implies that $\vartheta_k(t, x)$ converges also ρ a.e.
- The dominated convergence theorem then implies that any function $F(\rho_k, \vartheta_k)$, with F smooth and $F(0, \vartheta) = 0$, converges to $F(\rho, \vartheta)$.

This allows us to pass to the limit in every term of the momentum equation, including the barotropic term $\delta \rho_k^2$ using the extra-integrability $\rho_k \in L^{2+a_\delta}$ in (10.42). We can also simply pass to the limit in the entropy $s_\varepsilon(\rho_k, \vartheta_k)$.

Define $\bar{\kappa}_\varepsilon$ by $\bar{\kappa}'_\varepsilon = \kappa_\varepsilon$. By (10.29), we know that $\bar{\kappa}_\varepsilon(\vartheta)$ is bounded on $[\varepsilon, \varepsilon^{-1}]$. This lets us pass to the limit in $\kappa_\varepsilon(\vartheta_k) \nabla \vartheta_k = \nabla \bar{\kappa}_\varepsilon(\vartheta_k)$.

But it is not possible to pass to the limit in the r.h.s. of the entropy equation (10.25)₃. For instance the r.h.s. contains $\mu |\nabla u_k|^2$ which does not in general converge to $\mu |\nabla u|^2$, as this would require the compactness of ∇u_k in L^2 . Instead one uses convexity to prove that

$$w - \lim \frac{1}{\vartheta_k} \left(\mathcal{S}_k : \nabla u_k + \frac{\kappa_\varepsilon |\nabla \vartheta_k|^2}{\vartheta_k} \right) \geq \frac{1}{\vartheta} \left(\mathcal{S} : \nabla u + \frac{\kappa_\varepsilon |\nabla \vartheta|^2}{\vartheta} \right).$$

This leads to the inequality in the entropy equation in (10.33)₃. Note we can also write an equality with a positive (and a priori unknown) measure. But it still remains to pass to the limit in

$$\alpha_k \frac{\Delta \rho_k}{\vartheta_k} \left(\vartheta_k s_\varepsilon - e_{\varepsilon, \delta} - \frac{P_{\varepsilon, \delta}}{\rho_k} \right).$$

By relation (10.37), the bounds on s_ε , $e_{\varepsilon,\delta}$, $P_{\varepsilon,\delta}$ and the a priori estimates in (10.42), this term will converge to 0 if we can prove that

$$\alpha_k \int_0^T \int_{\Pi^d} \frac{|\nabla \rho_k|^2}{\rho_k} dx dt \longrightarrow 0.$$

On the other hand from the continuity equation, one has that

$$\begin{aligned} & \int_{\Pi^d} \rho_k \log \rho_k(t, x) dx - \int_{\Pi^d} \rho_k^0 \log \rho_k^0(x) dx + \int_0^t \int_{\Pi^d} \rho_k \operatorname{div} u_k dx dt \\ &= -\alpha_k \int_0^T \int_{\Pi^d} \frac{|\nabla \rho_k|^2}{\rho_k} dx dt. \end{aligned}$$

By the compactness of ρ_k , we can pass to the limit in every term in the l.h.s. so

$$\begin{aligned} & \int_{\Pi^d} \rho \log \rho(t, x) dx - \int_{\Pi^d} \rho^0 \log \rho^0(x) dx + \int_0^t \int_{\Pi^d} \rho \operatorname{div} u dx dt \\ &= -\lim \alpha_k \int_0^T \int_{\Pi^d} \frac{|\nabla \rho_k|^2}{\rho_k} dx dt. \end{aligned}$$

On the other hand, ρ solves the continuity equation without any diffusion and since it belongs to L^2 by (10.42), one also has that

$$\int_{\Pi^d} \rho \log \rho(t, x) dx - \int_{\Pi^d} \rho^0 \log \rho^0(x) dx + \int_0^t \int_{\Pi^d} \rho \operatorname{div} u dx dt = 0,$$

which has for consequence the required property

$$\lim \alpha_k \int_0^T \int_{\Pi^d} \frac{|\nabla \rho_k|^2}{\rho_k} dx dt = 0.$$

We again refer to [34, 35] for the details of the procedure. Finally we conclude the proof of this limit by noticing that we may pass to the strong limit in all the terms of the energy conservation. Therefore we indeed obtain the equality in (10.23).

The limits (10.33) to (10.34) and (10.34) to the final system. First we consider a sequence $\varepsilon_k \rightarrow 0$ and corresponding sequences of solutions ρ_k , u_k , ϑ_k to (10.33). Some of the a priori estimates for this sequence of solutions to (10.33) are obtained by simply keeping only the estimates in (10.42) uniform in ε . The others, such as the entropy estimate, have to be derived again. But one proceeds exactly as described in subsection 10.4, leading to

$$\varepsilon \rightarrow 0 \left\{ \begin{array}{l} \sup_{t,k} \int_{\Pi^d} (\rho_k^2 + \rho_k |u_k|^2 + \vartheta_k^{\gamma_\vartheta}) dx < \infty, \\ \sup_k \int_0^T \int_{\Pi^d} (\rho_k^{2+a_\delta} + |\nabla u_k|^2) dx dt < \infty, \quad \forall a_\delta < \max\left(\frac{1}{2}, \frac{4}{d} - 1\right), \\ \sup_k \int_0^T \int_{\Pi^d} (|\nabla \vartheta_k^{\alpha/2}|^2 + |\nabla \log \vartheta_k|^2) dx dt < \infty. \end{array} \right. \quad (10.43)$$

Similarly for (10.34), we obtain

$$\delta \rightarrow 0 \left\{ \begin{array}{l} \sup_{t,k} \int_{\Pi^d} (\rho_k^\gamma + \rho_k |u_k|^2 + \vartheta_k^{\gamma\vartheta}) dx < \infty, \\ \sup_k \int_0^T \int_{\Pi^d} (\rho_k^{\gamma+a} + |\nabla u_k|^2) dx dt < \infty, \quad \forall a < \max\left(\frac{1}{2}, \frac{2}{d}\gamma - 1\right), \\ \sup_k \int_0^T \int_{\Pi^d} (|\nabla \vartheta_k^{\alpha/2}|^2 + |\nabla \log \vartheta_k|^2) dx dt < \infty. \end{array} \right. \quad (10.44)$$

The bounds (10.44) will of course imply (10.20) after passage to the limit.

As before combining those bounds with the continuity and momentum equations shows that (5.4), (5.5), (5.6), and (5.7) are satisfied with $p = 2 + a_\delta$ for System (10.33) and $p = \gamma + a$ for System (10.34). In particular remark here that from (10.17) we have the bound $P \leq C \rho^\gamma$ and hence the uniform bound $P_\varepsilon \leq C \rho^\gamma$ which is controlled by the a priori estimates (10.43) provided $\gamma \leq 2 + a_\delta$; this is exactly the assumption (10.24).

By (10.17), the pressure law satisfies (5.9) for both limits with $\tilde{\gamma} = \max(\gamma, 2)$. We can again apply Theorem 5.1, point ii in this case, through the variant 5.3. We again need that $p > \tilde{\gamma}$ that is assumption (10.24).

Therefore we obtain the compactness of ρ_k and from the momentum equation, the compactness of $\sqrt{\rho_k} u_k$. As in the limit of (10.25) to (10.33), the next step is to obtain the compactness of ϑ_x . We follow the same procedure which however now requires more work on the first two points:

- Bound $\rho_k s_{\varepsilon_k}(\rho_k, \vartheta_k)$ or $\rho_k s(\rho_k, \vartheta_k)$ uniformly in some $L^p_{t,x}$ with $p > 1$. This is not immediate anymore since s is not bounded and s_{ε_k} is not uniformly bounded. One uses the expression of the entropy in (10.32) and the bounds on $|\partial_\vartheta P|$ and m in (10.14) to find

$$\rho_k s_{\varepsilon_k}(\rho_k, \vartheta_k) \leq C (\rho_k^{\beta_3} + \vartheta_k^{\beta_4} + \rho_k \vartheta_k^{\alpha(\gamma+a-1)/(\gamma+a)}), \quad (10.45)$$

and similarly for $\rho_k s$. Then the a priori estimates on $\rho_k \in L^{\gamma+a}$ or $\rho_k \in L^{2+a_\delta}$ and $\vartheta_k \in L^{\alpha}_{t,x} \cap L^{\infty}_{t,x} L^{\gamma\vartheta}$ are enough provided $\beta_3 < \gamma + a$ and $\beta_4 < \alpha$ or $\beta < \gamma\vartheta$; both are ensured by the stronger condition in (10.15).

- Bound $(\partial_t(\rho_k s_{\varepsilon_k}(\rho_k, \vartheta_k)))_+$, $(\partial_t(\rho_k s(\rho_k, \vartheta_k)))_+$ uniformly in $L^1_t W_x^{-1,1}$. This now also requires additional work. One uses the entropy inequation in (10.33). The right-hand side is bounded in $L^1_{t,x}$ (just by the entropy dissipation) and thus one only has to bound $\rho_k s_{\varepsilon_k} u_k$ in $L^1_{t,x}$. For instance

$$\int_0^T \int_{\Pi^d} \rho_k^{\beta_3} u_k \leq \int_0^T \int_{\Pi^d} (\rho_k |u_k|^2 + \rho_k^{2\beta_3-1}),$$

which is bounded provided $2\beta_3 - 1 < \gamma + a$ and $2\beta_3 - 1 < 2 + a_\delta$ leading to the assumption on β_3 in (10.15). As for $\int \vartheta_k^{\beta_4} u_k$, we have by Sobolev embedding that $u_k \in L^2_t L^{2d/(d-2)}$ and therefore need that $\vartheta_k^{\beta_4} \in L^2_t L^{2d/(d+2)}$. Given that

$\vartheta_k \in L_{t,x}^\alpha \cap L_t^\infty L_x^{\gamma\vartheta}$, this is ensured by the condition on β_4 (note that the condition could in fact be improved by using that $\vartheta_k \in L_t^\alpha L_x^{\alpha/(1-2/d)}$). The last term is treated in a similar manner.

To conclude to pass to the limit in every term of the momentum equation, we use the integrability properties on ρ_k, ϑ_k (10.14)₄, (10.14)₅ and (10.14)₆ to provide an $L_t^p L_x^p$ integrability with $p > 1$ of the pressure. Note that from the previous analysis either $P(0, \vartheta) = 0$ and we may use the last point above to pass to the limit in any truncation of $P_{\varepsilon,\delta}(\rho_k, \vartheta_k)$ or $P_\delta(\rho_k, \vartheta_k)$; or if $P(0, \vartheta) \neq 0$ then we have compactness on the whole sequence ϑ_k and not only far from the vacuum. We also point out that the truncature of the pressure $P_{\varepsilon,\delta}$ or P_δ again require assumptions (10.24) or (10.15)

Finally the passage to the limit in the entropy inequation is handled exactly as for the limit of System (10.25) to (10.33); it is even simpler in fact as the term in α_k is not present.

Let us briefly come back to the assumption (10.24). As seen above, it is used to control the pressure term in the limit from System (10.33) to (10.34) when the uniform bound on $\rho_k \in L^{2+a\delta}$ is only provided by the added barotropic term $\delta \rho^2$.

If (10.24) does not hold, the simplest (if tedious) way to solve the problem is limit to add intermediary steps with barotropic corrections $\delta_n \rho^{\gamma_n}$ one after the other. Since there is always a gain of integrability, it is possible to choose the γ_n increasing simply ensuring that

$$\gamma_{n+1} < \gamma_n + \max\left(\frac{1}{2}, \frac{2}{d}\gamma_n - 1\right).$$

Once γ_n is large enough so that

$$\gamma < \gamma_n + \max\left(\frac{1}{2}, \frac{2}{d}\gamma_n - 1\right),$$

one can derive the limit system. The passage to the limit in each intermediary step is handled exactly as above.

11 Models occurring in other contexts

Macroscopic models in various biological settings involve a density ρ that is transported by a velocity vector field u with source term, such as

$$\partial_t \rho + \operatorname{div}(\rho u) = \rho G(P(\rho), c)$$

for some functions G and P . The function G may include birth and death terms and it could also depend on other quantities such as nutrients concentration denoted c , for example Oxygen in cancer modeling. This concentration is typically governed by a parabolic equation with right-hand side modeling the consumption of the resource(s)

$$\partial_t c - \Delta c = -\rho H(P(\rho), c) \tag{11.1}$$

with

$$\partial_P H \leq 0, \quad \partial_c H \geq 0, \quad H(P, 0) = 0.$$

The velocity field is described through a constitutive law for instance

$$-\nu \Delta \Psi + \alpha \Psi = P(\rho) - S, \quad u = -\nabla \Psi \quad (11.2)$$

where S is a given source term.

There have been several studies of such systems with applications to crowd motion, traffic jams, cancerology using specific reformulations: gradient flow or kinetic descriptions and appropriate choices for f and G .

For instance the special case $c = 0$, $\alpha = 0$, $G \equiv 0$ has been studied by B. MAURY, A. ROUDNEFF-CHUPIN and F. SANTAMBROGIO (see [52]) through the framework of optimal transportation reformulating the problem as a gradient flow in the Wasserstein space of measures. Other examples concern the reformulation through a kinetic formulation. For instance, very recently, B. PERTHAME and N. VAUCHELET (see [55]) have studied the case $c = 0$ in the whole space with the pressure law $P(\rho) = (\gamma + 1)\rho^\gamma/\gamma$ with $\gamma > 1$, G satisfying

$$G \in \mathcal{C}^1(\mathbb{R}), \quad G'(\cdot) \leq -\eta < 0, \quad G(P_M) = 0 \text{ for some } P_M > 0$$

and $\alpha, \nu > 0$.

The main result of this last paper is the "stiff pressure law" limit, namely the limit $\gamma \rightarrow +\infty$, leading to a free boundary model which generalizes the classical Hele-Shaw equation. Such kind of limit has also been performed for the compressible Navier-Stokes equations by P.-L. LIONS and N. MASMOUDI (see [50]) with $P(\rho) = a\rho^\gamma$ with $G \equiv 0$.

Recently C. PERRIN and E. ZATORSKA (see [54]) have studied the singular limit $\varepsilon \rightarrow 0$ for a singular pressure law $P(\rho) = \varepsilon\rho^\gamma/(1-\rho)^\beta$ with $\gamma, \beta > 3$. The advantage of such pressure law is that $0 \leq \rho \leq 1$ for a fixed ε which is important for some applications as mentioned by B. MAURY in his review paper [51].

Pressure laws which blows-up for a critical density are of course the exact analogous of the Van der Waals equation of state for compressible fluid dynamics. They are also encountered in other setting such as crowd motion, granular flow, sedimentation problems.

An other possibility to describe the velocity field is to consider the Brinkman equation instead of Eq. (11.2), namely

$$-\nu \Delta u + \alpha u + \nabla P(\rho) = S,$$

or with a Stokes viscosity term

$$-\nu \Delta u - (\lambda + \nu) \nabla \operatorname{div} u + \alpha u + \nabla P(\rho) = S.$$

This type of correction accounts for flow through medium where the grains of the media are porous themselves and has been justified in [2]. If the velocity is irrotational then this model is exactly reduced to Eq. (11.2). Note however that

viscoelastic models for tumor growth may allow for instance to observe a lemon-like shape tumor, whereas with a Newtonian model an ovoid is obtained. Even limited, this difference can eventually lead to bigger ones as the outer rim is composed of proliferating cells with exponential growth. The kind of shape obtained for instance in [14] is observed in *in-vitro* experiments.

What our method can bring:

We do not try to state a theorem here given the large variety of possible models. Instead we give a few elements for which the method introduced here could prove crucial

- More complex pressure laws, attractive and repulsive, could be considered. This would be the exact equivalent of Theorem 3.1. Note that biological systems frequently exhibit preferred ranged of densities for instance with attractive interactions for low densities and repulsive at higher ones.
- More importantly the transition from attractive to repulsive interactions may depend on the concentration c of nutrients or other bio-chemicals. This is similar to the dependence on the temperature in the state laws for the Navier-Stokes-Fourier system. For example if the pressure blows-up at some thresholds, enforcing a maximal density, then this threshold and the maximal density will depend in general on c . Because c is not necessarily uniformly bounded, the range of attractive interactions (where $G(P(\rho), c)$ is decreasing in ρ) is not compactly supported and classical approaches may fail.
- There can be several nutrients or bio-chemicals. That means that in general one has several c_1, c_2, \dots with several equations (11.1) (or a vector-valued one if the diffusion speeds are the same). If chemotaxis is considered, some of those bio-chemicals may be attractive while other are repulsive. This may lead to a complicated pattern of interactions which again cannot be handled by classical approaches.
- Many of this models are posed in porous media which are inherently anisotropic. In biology for instance the tissue or the porous matrix is heterogeneous. Therefore the equation for u should read

$$-\operatorname{div}(A(t, x)D(u)) + \alpha(t, x)u + \nabla P(\rho) = S$$

for some A and α and with in general a non-monotone pressure law. Our new approach could for instance help to enrich the model mathematically studied recently in [28].

12 Appendix: Notations

For the reader convenience, we repeat and summarize here some of our main notations.

Physical quantities.

- $\rho(t, x)$, or $\rho_k(t, x)$ denotes the density of the fluid.
- $u(t, x)$, or $u_k(t, x)$ denotes the velocity field of the fluid.
- $P(\cdot)$, or $P_k(\cdot)$ denotes the pressure law.
- $e(\rho)$ is the internal energy density.
In the barotropic case, $e(\rho) = \int_{\rho_{ref}}^{\rho} P(s)/s^2 ds$.
- $E(\rho, u) = \int \rho (|u|^2/2 + e(\rho))$ is the total energy of the fluid.
- μ , λ and μ_k denote various viscosity coefficients or combination thereof.
- \mathcal{S} denotes the viscous stress tensor.
In the simplest isotropic case: $\mathcal{S} = 2\mu D(u) + \lambda \operatorname{div} u \operatorname{Id}$.
- \mathcal{D} is the diffusion term related to the viscous stress tensor by $\mathcal{D}u = \operatorname{div} \mathcal{S}$.

In the Navier–Stokes–Fourier case:

we have the additional notations, appearing only in section 10.

- $\vartheta(t, x)$ is the temperature field of the fluid.
- $s(\rho, \vartheta)$ is the entropy of the fluid.
- $\kappa(\vartheta)$ is the heat conductivity coefficient.
- C_v is the specific heat of the fluid.

Technical notations.

- d is the dimension of space.
- k as an index always denotes the index of a sequence.
- h and h_0 are scaling parameters used to measure oscillations of certain quantities such as the density.
- K_h is a convolution kernel on Π^d ,
 $K_h(x) = (h + |x|)^{-a}$ for x small enough and with $a > d$.
- \bar{K}_h is equal to $K_h / \|K_h\|_{L^1}$.
- $\mathcal{K}_{h_0} = \int_{h_0}^1 \bar{K}_h(x) \frac{dh}{h}$ is the weighted average of K_h . Note that $\|\mathcal{K}\|_{L^1} \sim |\log h_0|$.
- w_0 , w_1 and w_a are the weights and $w_{i,h} = \bar{K}_h \star w_i$ their regularization with $i = 0, 1, a$.
- C is a constant whose exact value may change from one line to another but which is always independent of k , h or other scaling parameters.
- $\varepsilon(h)$ is a smooth function with $\varepsilon(0) = 0$.
- θ is an exponent whose exact value may change as for C but in $(0, 1)$.
- The exponent p is most of the time such that $\rho \in L^p_{t,x}$.
- q and r are other exponents for L^p type spaces that are used when needed.

- I, II, \dots and $A, B, D, E \dots$ are notations for some intermediary quantities used in the proofs. Their definitions may change from one proof to another.
- x, y, w, z are typically variables of integration over the space domain.
- $\delta\rho_k(x, y) = \rho_k(x) - \rho_k(y)$ the difference of densities.
- $\bar{\rho}_k(x, y) = \rho_k(x) + \rho_k(y)$ the sum of densities.
- $D\rho u_k = \Delta^{-1} \operatorname{div}(\partial_t(\rho_k u_k) + \operatorname{div}(\rho_k u_k \otimes u_k))$ denotes the effective flux.
- The individual weights $w(t, x) = w_0, w_1, w_a$ satisfy Eq. (7.2) or

$$\partial_t w + u_k \cdot \nabla w = -Dw + \alpha_k \Delta_x w,$$

where $D = D_0, D_1, D_a$ are the penalizations in (7.3), (7.4), (7.5).

- The weights w_0 or w_a may be convolved to give $w_h = \bar{K}_h \star w_0, w_{a,h} = \bar{K}_h \star w_a$.
- The weights are then added or multiplied to obtain the composed $W(t, x, y) = W_0, W_1, W_2, W_a$ with

$$\begin{aligned} W_0 &= w_0(x) + w_0(y), & W_1 &= w_1(x) + w_1(y), \\ W_2 &= w_1(x)w_1(y), & W_a &= w_a(x) + w_a(y). \end{aligned}$$

The main properties of the weights are given in Prop. 7.2.

13 Appendix: Besov spaces and Littlewood-Paley decomposition

We only recall some basic definitions and properties of Besov spaces for use in Lemma 6.3. We start with the classical Littlewood-Paley decomposition and refer to the readers for instance to [6], [1] and [8] for details and applications to fluid mechanic. Choose any family $\Psi_k \in \mathcal{S}(\Pi^d)$ s.t.

- Its Fourier transform $\hat{\Psi}_k$ is positive and compactly supported in the annulus $\{2^{k-1} \leq |\xi| \leq 2^{k+1}\}$.
- It leads to a decomposition of the identity in the sense that there exists $\hat{\Phi}$ with $\hat{\Phi}$ compactly supported in $\{|\xi| \leq 2\}$ s.t. for any ξ

$$1 = \hat{\Phi}(\xi) + \sum_{k \geq 1} \hat{\Psi}_k(\xi).$$

- The family is localized in Π^d in the sense that for all $s > 0$

$$\sup_k \|\Psi_k\|_{L^1} < \infty, \quad \sup_k 2^{ks} \int_{\Pi^d} |z|^s |\Psi_k(z)| dz < \infty.$$

Note that in \mathbb{R}^d , one usually takes $\Psi_k(x) = 2^{kd} \Psi(2^k x)$ but in the torus, it can be advantageous to use a more general family. It is still necessary to take it smooth enough for the third assumption to be satisfied (it is for instance the difference between the Dirichlet and Fejer kernels).

For simplicity, we then denote $\Psi_0 = \Phi$ for $k = 0$ and for $k \geq 1$, $\Psi_k(x) = 2^{kd} \Psi(2^{-k} x)$. For any $f \in \mathcal{S}'(\mathbb{R}^d)$, we also write $f_k = \Psi_k \star f$ and then obtain the decomposition

$$f = \sum_{k=0}^{\infty} f_k. \quad (13.1)$$

From this decomposition one may easily define the Besov spaces

Definition 13.1 *The Besov space $B_{p,q}^s$ is the space of all $f \in L_{loc}^1 \cap \mathcal{S}'(\mathbb{R}^d)$ for which*

$$\|f\|_{B_{p,q}^s} = \|2^{sk} \|f_k\|_{L_x^p}\|_{l_k^q} = \left(\sum_{k=0}^{\infty} 2^{skq} \|f_k\|_{L_x^p}^q \right)^{1/q} < \infty.$$

The main properties of the Littlewood-Paley decomposition that we use in this article can be summarized as

Proposition 13.2 *For any $1 < p < \infty$ and any s , there exists $C > 0$ s.t. for any $f \in L_{loc}^1 \cap \mathcal{S}'(\mathbb{R}^d)$*

$$\begin{aligned} \frac{2^{sk}}{C} \|f_k\|_{L^p} &\leq \|\Delta^{s/2} f_k\|_{L^p} \leq C 2^{sk} \|f_k\|_{L^p}, \\ C^{-1} \left\| \left(\sum_{k=0}^{\infty} 2^{2ks} |f_k|^2 \right)^{1/2} \right\|_{L^p} &\leq \|f\|_{W^{s,p}} \leq C \left\| \left(\sum_{k=0}^{\infty} 2^{2ks} |f_k|^2 \right)^{1/2} \right\|_{L^p}. \end{aligned}$$

And as a consequence for $1 < p \leq 2$

$$C^{-1} \|f\|_{B_{p,2}^s} \leq \|f\|_{W^{s,p}} \leq C \|f\|_{B_{p,p}^s}.$$

In particular a consequence of Prop. 13.2 is the following bound on truncated Besov norm

Lemma 13.3 *For any $1 < p \leq 2$, there exists $C > 0$ s.t. for any $f \in L_{loc}^1 \cap \mathcal{S}'(\mathbb{R}^d)$ and any $K \in \mathbb{N}$*

$$\sum_{k=0}^K 2^{sk} \|f_k\|_{L_x^p} \leq C \sqrt{K} \|f\|_{W^{s,p}}$$

Proof By a simple Cauchy-Schwartz estimate

$$\sum_{k=0}^K 2^{s k} \|f_k\|_{L_x^p} \leq \sqrt{K} \left(\sum_{k=0}^{\infty} 2^{2 s k} \|f_k\|_{L_x^p}^2 \right)^{1/2} = \sqrt{K} \|f\|_{B_{p,2}^s},$$

which concludes by Prop. 13.2. \square

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